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LORD KELVIN, G.C.V.O. D.C.L. LL.D. F.R.S. &c.

JOHN JOLY, M.A. D.Sc. F.R.S. F.G.S.

AND

WILLIAM FRANCIS, F.L.S.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit.* lib. i. cap. 1. Not.
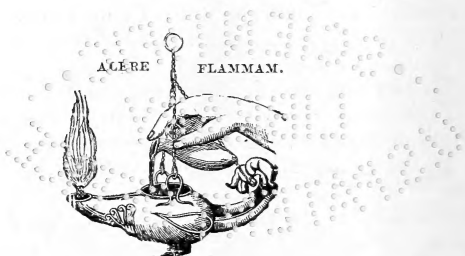
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"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

———

———" Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

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J. B. Pinelli ad Mazonium.

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ACÉRE FLAMMAM.

CONTENTS OF VOL. XI.

(SIXTH SERIES).

NUMBER LXI.—JANUARY 1906.

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PLATES.

I. *Deep Sea Ship-Waves* *. (Continued from *Phil. Mag.,*
June 1905.) *By* Lord KELVIN.

§ 65. REFERRING to § 63, we must, for the present, as
time presses, leave detailed interpretation of the
curves of fig. 17: merely remarking that, according to § 44,
if $\delta = 0$, (which means that J is an integer), the disturbance, d,
is infinitely great; of which the dynamical meaning is clear in
(70) of § 39.

§ 66. Let us now find the depression of the water at
distance x from the origin, when the disturbance is due to a
single forcive, expressed by the formula †

$$\Pi(x) = g\frac{kb^2}{x^2 + b^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (95),$$

travelling uniformly with any velocity v. If this forcive
were applied steadily to the surface of water at rest it would
produce a steady depression $\frac{1}{g}\Pi(x)$, as we are taking the
density of the water, unity. Thus the forcive $\Pi(x)$ would
shape the water to an infinitely long trough, of cross-section
shown in fig. 25, representing $z = kb^2/(x^2 + b^2)$ on the scale of
$k = 10$ cms. and $b = 1$ cm.

Taking $\frac{1}{g}\int_0^x dx$ of (95) we find $\tan^{-1}(x/b) \cdot bk$. Hence the

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* Communicated by the Author; having been read before the Royal
Society of Edinburgh, July 17, 1905.

† What is denoted by x in this and following expressions, is the
$(x - vt)$ of §§ 36 .... 40; the origin of co-ordinates being now fixed
relatively to the travelling forcive.

$$z = \frac{kb^2}{x^2 + b^2}$$

$k = 10$ cms.
$b = 1$ cm.
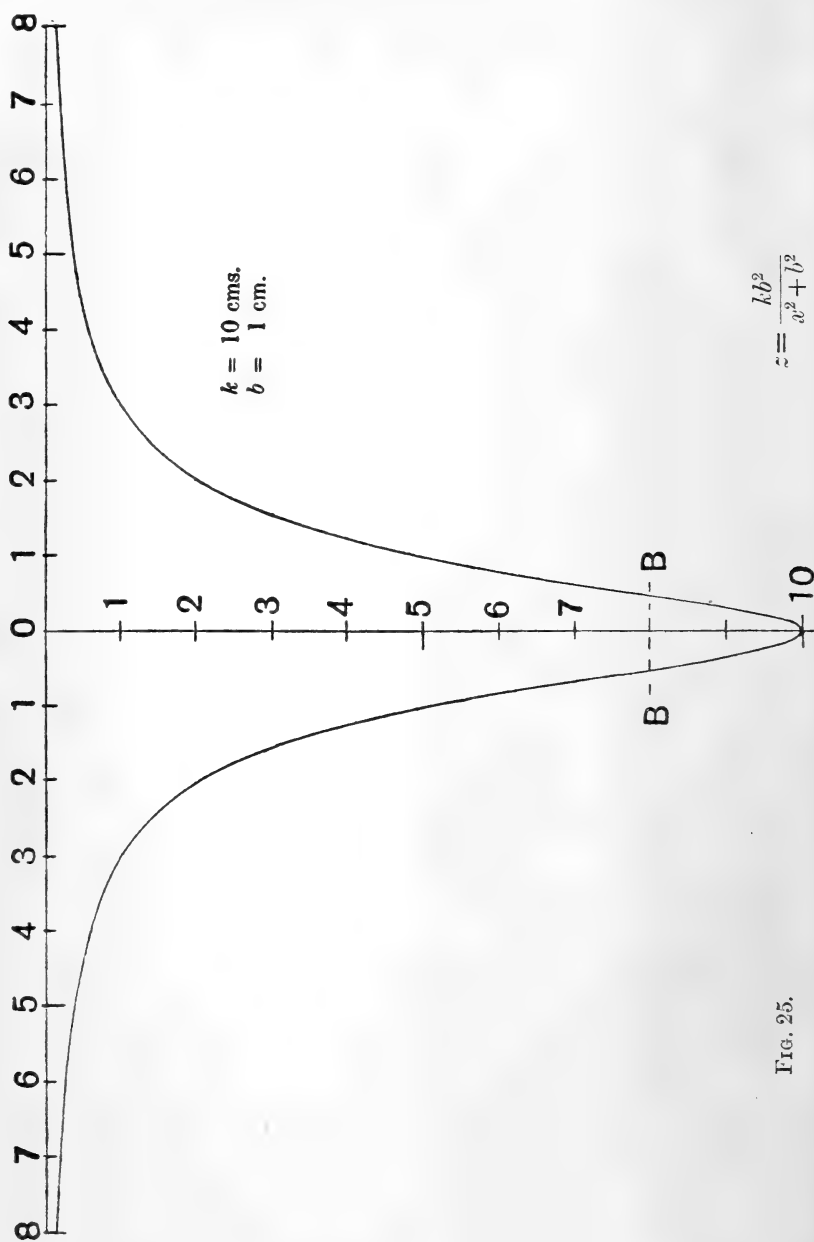
Fig. 25.

area of fig. 25 is $2 \tan^{-1} 8 \cdot bk$, or $\frac{166}{180} \cdot \pi bk$, and the total
area of the diagram extended to infinity on each side is
πbk. Hence the area of fig. 25 is $\frac{166}{180}$, or ·92, of the total
area. This total area, πbk, I call, for brevity, the forcive-
area ; and πb, I call the mean breadth of the forcive-area.
The breadth of the forcive where $z = \cdot 8k$ (as shown by the
dotted line B B in the diagram) is b.

§ 67. Now let the forcive be suddenly set in motion, and
kept moving uniformly with any velocity v in the rightward
direction of our diagrams. This will produce a great com-
motion, settling ultimately into more and more nearly steady
motion through greater and greater distances from O. The
investigation of §§ 1–10 (Phil. Mag. June 1904), and
particularly the results described in §§ 5, 6, and illustrated
in figs. 2, 3, show that in our present case the commotion,
however violent, even if including *splashes* *, divides itself
into two parts which travel away in the two directions from O,
ultimately at wave-speed increasing in proportion to square
root of distance (according to the law of falling bodies); and
leaving in their rears, through ever broadening spaces, what
would be more and more nearly absolute quiescence if the
forcive were suddenly to cease after having acted for any
time, long or short.

§ 68. But if the forcive continues acting, and travelling
rightwards with constant speed, v, according to § 67, the
travelling away of the two parts of the initial commotion in
the two directions from O (itself merely a point of reference,
moving uniformly rightwards), leaves the water, as shown by
fig. 26, in a state of more and more nearly quite steady
motion through an ever broadening space on the rear side of
O, and through a small space in advance of O ; provided
certain moderating conditions are fulfilled in respect to
k, b, v.

§ 69. To illustrate and prove § 68 ; first suppose v infinitely
small. The water will be infinitely little disturbed from the
static forcive-curve shown in fig. 25, and described in § 66.
Small enough velocities will make very small disturbance
with any finite value of k/b.

§ 70. But now go to the other extreme and let v be very
great. It is clear, on dynamical principles without calcula-
tion, that v may be *great enough to make but very little*

---

* However sudden and great the commotion is, the motion of the
liquid is, and continues to be, irrotational throughout.

Lord Kelvin *on*

$$\mathrm{d}\,(x) - \tfrac{1}{2}h\sin\frac{2\pi x}{\lambda}.$$
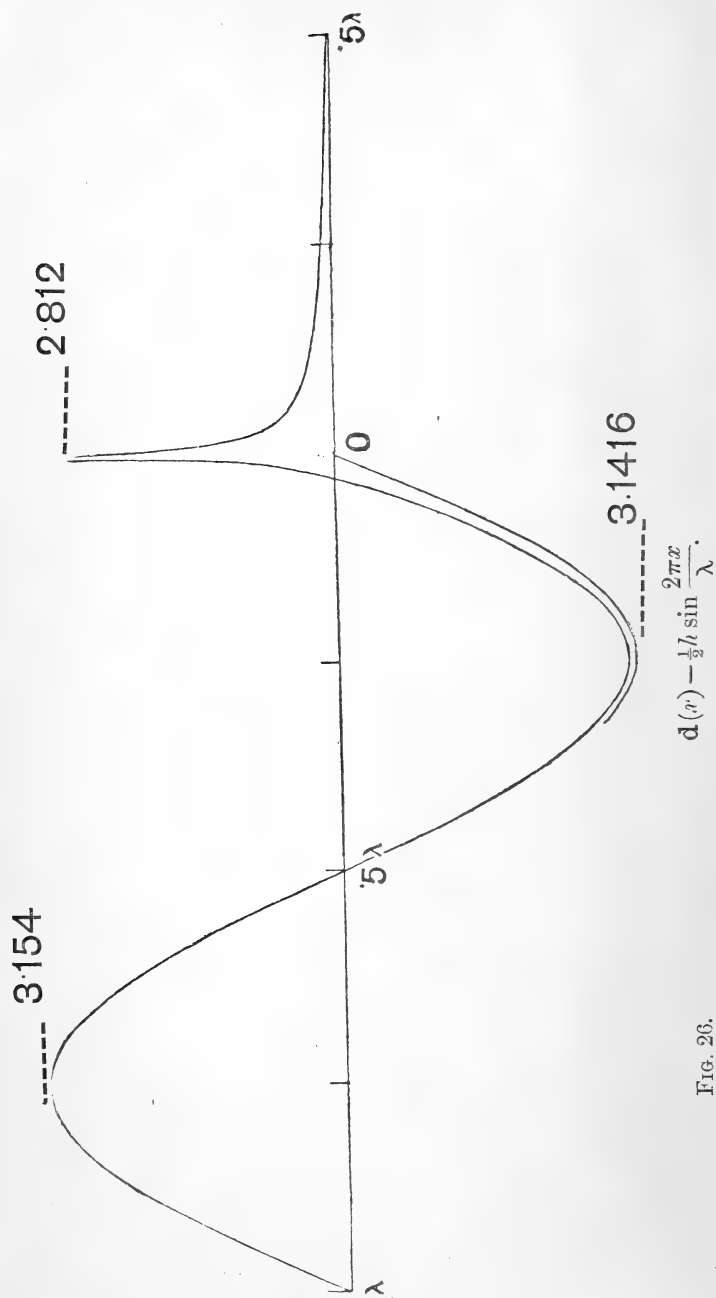
Fig. 26.

disturbance of the water-surface, however steep be the static forcive curve. A "skipping stone" and a ricochetting cannon shot, illustrate the application of the same dynamical principle in three-dimensional hydrokinetics. By mathematical calculation (§ 79 below) we shall see that, when v is great enough, we have

$$h \doteq 4\pi \frac{\mathrm{A}}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (97),$$

where h denotes the height of crests above mean water-level in the train of sinusoidal free waves left in the rear of the travelling forcive; A denotes the area of the forcive-curve (fig. 25); being given in § 66 by the equation

$$\mathrm{A} = \pi b k \quad . \quad . \quad . \quad . \quad . \quad . \quad (98):$$

and λ, given [§ 39, (71)] by

$$\lambda = 2\pi v^2 / g \quad . \quad . \quad . \quad . \quad . \quad (99),$$

denotes the wave-length of free waves travelling with velocity v.

§ 71. A very important theorem in respect to ship-waves is expressed by (97). Without calculation we see that, *if* λ *is very great* in comparison with πb, (the "mean breadth" of the forcive-curve according to § 66), h must be simply proportional to A, for different forcives travelling at the same speed. This we see because, for the same value of b, h/k is the same, and because superposition of different forcives within any breadth small in comparison with λ, gives for h the sum of the values which they would give separately. Farther without calculation, we can see, by imagining altered the scale of our diagrams, that $h\lambda/\mathrm{A}$ must be constant. But without calculation I do not see how we could find the factor 4π of (97), as in § 79 below.

§ 72. The effect of the condition prescribed in § 71 is illustrated and explained by considering cases in which it is not fulfilled. For example, let two forcives be superposed with their middles at distance $\frac{1}{2}\lambda$; they will give $h = 0$, that is to say no train of waves. The displaced water surface for this case is represented in fig. 27. Or let their distance be $\frac{1}{3}\lambda$ or $\frac{2}{3}\lambda$; the two will give the same value of h as that given by one only. Or let the two be at distance λ; they will make h twice as great as one forcive makes it.

§ 73. In figs. 26, 27, 29, 30, representing results of the calculations of §§ 78, 79 below, the abscissas are all marked according to wave-length. The scale of ordinates corre-

sponds, in each of figs. 26, 27, 29, to $k = 243 \cdot 89$, and $\pi b = 1 \cdot 0251 \cdot 10^{-3} \cdot \lambda$. This makes by (98) and (97) $A = \frac{1}{4}\lambda$, and $h = \pi$. Fig. 30 represents the curve of fig. 29 at the maximum, in the neighbourhood of O, on a greatly magnified scale : about 1720 times for the abscissas, and 39 times for the ordinates.

§ 74. Fig. 26 shows, on the right-hand side, the water slightly heaped up in front of the travelling forcive, which is a distribution of *downward* pressure whose middle is at O. On the left side of O, we see the water surface not differing perceptibly from a curve of sines beyond half a wave-length rearwards from O. A small portion of a wave-length of true curve of sines in the diagram shows how little the water's surface differs from the curve of sines at even so small a distance from O as a quarter wave-length.

It must be remembered that in reality the water surface is everywhere very nearly level; and in considering, as we shall have to do later, the work done by the forcive, we must interpret properly the enormous exaggeration of slopes shown in the diagrams. It is interesting to remark that the static *depression*, k, which the forcive if at rest would produce, is about 87 times the *elevation* actually produced above O by the forcive, travelling at the speed at which free waves, of the wave-length shown in the diagrams, travel. It is interesting also to remark that the limitation to very small slopes is not binding on the static forcive curve. Thus for example, a distribution of static pressure, everywhere perpendicular to the free surface, producing static depression exactly agreeing with fig. 25, would, if caused to travel at a speed for which the free-wave-length is very large in comparison with b, produce a disturbance, represented by fig. 26 with waves of moderate slopes : and, as said in § 69 above, would produce no disturbance at all if the speed of travelling were infinitely great.

§ 75. Fig. 27 is interesting as showing the waveless disturbance produced by two equal and similar forcives with their middles at distance equal to half the wave-length. This disturbance is essentially symmetrical in front and rear of the middle between the two forcives. By dynamical considerations of the equilibrium of downward pressures, we see that the area of fig. 27 (portion above line of abscissas being reckoned as negative) must be exactly equal to 2 A, the sum of the areas of the two forcives, representing their integral amount of downward pressure. This area, being $2\pi bk$, with the numerical data of § 73, is numerically $\frac{1}{2}\lambda$; that is to say a rectangle whose length is $\frac{1}{2}\lambda$, and breadth

$$\mathrm{d}\left(x-\frac{\lambda}{4}\right)+\mathrm{d}\left(x+\frac{\lambda}{4}\right).$$
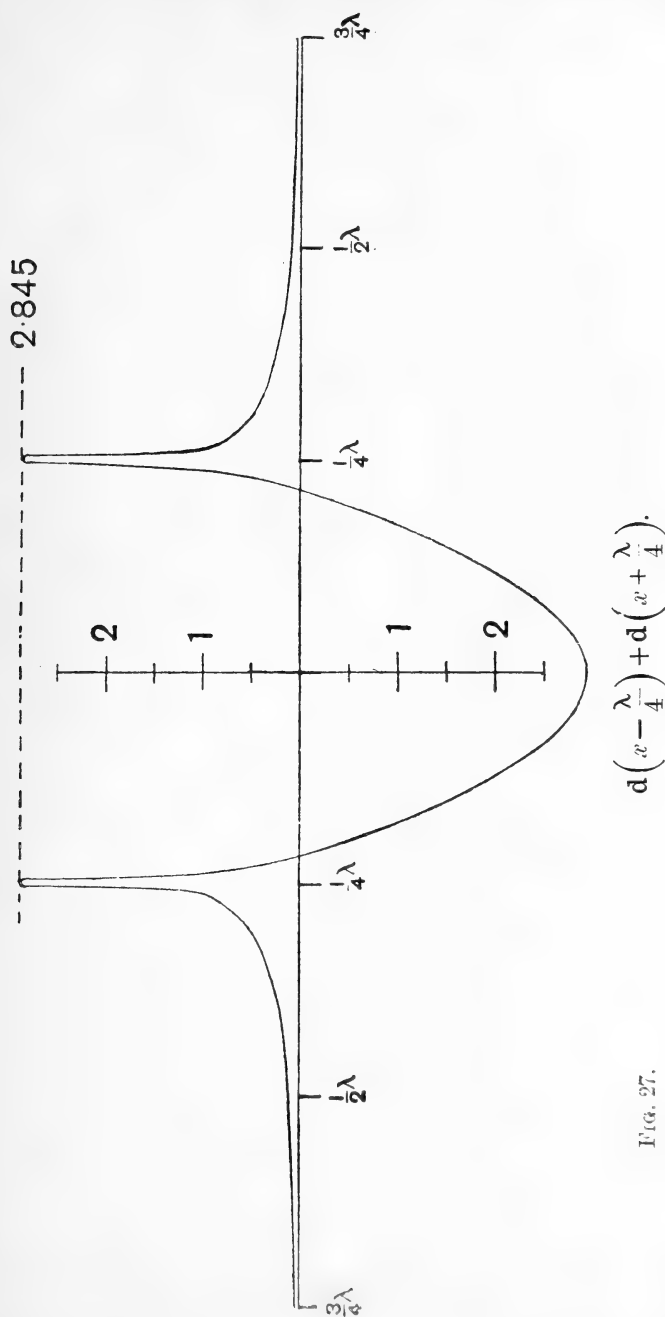
Fig. 27.

the unit of our vertical scale. Approximate mensuration, with a very rough estimate of the area beyond the range of the diagram, continued to infinity on the two sides, verifies this conclusion.

§ 76. Fig. 28 is designed on the same plan as fig. 27 but with eleven half-wave-lengths as the distance between the two forcives instead of one half-wave-length. Like fig. 27, it is symmetrical on the two sides of the middle of the diagram; but, instead of being waveless, as is fig. 27, it shows four and a half waves, all very approximately sinusoidal, with two depressional halves of waves at their two ends, and elevations coming asymptotically to zero beyond the two ends of the diagram. The curve represented by fig. 26 is very accurately the right-hand extreme of fig. 28 : and the same figure, turned right to left, is the left-hand extreme of fig. 28. If we commence with the water wholly at rest, and start the forcives at the proper speed, with force gradually (or somewhat suddenly) increasing up to the prescribed amount, the motion produced will be that represented by fig. 28, with, superimposed upon it, a disturbance quickly disappearing in ever lengthening waves of diminishing amplitude, travelling away in both directions from our field. If now, with the regular régime represented by fig. 28, we suddenly cease to apply the forcives, we have left a free procession of four and a half very approximately sinusoidal waves, between a front and a rear deviating from sinusoidality as shown in the diagram. From the instant of being left free, the front of this procession and its rear will rapidly become modified ; while for three periods the central part of the procession will have travelled three wave-lengths, with very little deviation from sinusoidality. But, after four or five periods from the instant of being left free, the whole procession will have got into confusion. After twenty or thirty or forty periods, the water will be sensibly quiescent, not only through the space where the procession was, but through a considerable part of the space over which it would have travelled if its front and rear had been kept guarded by the continued action of the two travelling forcives. At no time after the cessation of the forcives can we reasonably or conveniently assign a " group velocity " to the group or procession of waves with which we are concerned. A prevalent idea is, I believe, that such a group of deep sea waves could be regarded as travelling with half the " wave-velocity " of waves of the length given in the original group. In § 30 above, reasons are given for accepting the theory of " group velocity " only for the case of mutually supporting
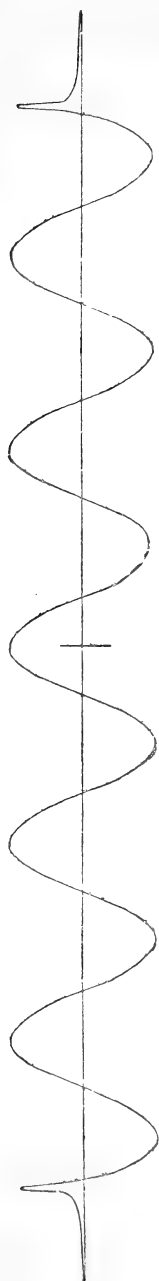
$$\mathrm{d}\left(x - \frac{11\lambda}{4}\right) + \mathrm{d}\left(x + \frac{11\lambda}{4}\right).$$

Scale of ordinates reduced to one quarter.

Fig. 28.

groups, given by Stokes in his Smith's Prize examination paper, published in the *Cambridge University Calendar* for 1876 : and for rejecting it for the case of any single group of waves. In reality the front of a group, left to itself, travels with accelerated velocity exceeding the velocity of periodic waves of the given wave-length, instead of with half that velocity.

§ 77. Fig. 29 shows *the steady motion*, symmetrical in front and rear of a single travelling forcive, which is a solution of our problem ; but it is an unstable solution (as probably are the solutions of the problem of § 45 above, shown in figs. 13, 14, 15). If any large finite portion of the water is given in motion according to fig. 29, say, for example, 50 waves preceding O (the forcive) and 50 waves following O, the front of the whole procession, to the right of O, will become dissipated into non-periodic waves travelling rightwards and leftwards with increasing wave-lengths and increasing velocities ; and the approximately steady periodic portion of it will shrink backwards relatively to the forcive. Thus before the forcive has travelled fifty wave-lengths, the periodic waves in front of it are all gone : but there is still irregular disturbance both before and behind it. After the forcive has travelled a hundred wave-lengths, the whole motion in advance of it, and the motion for perhaps 30 wave-lengths or more in its rear, will have settled to nearly the condition represented by fig. 26, in which there is a small regular elevation in advance of the forcive, and a regular train of approximately sinusoidal waves in its rear ; these waves being of double the wave-height given originally. This motion, as said above in § 68, will go on, leaving behind the forcive a train of steady periodic waves, increasing in number ; and behind these an irregular train of waves, shorter and shorter, and less and less high the farther rearward we look for them (see R in fig. 10 of §§ 26, 27 above). It is an interesting, but not at all an easy problem, to investigate the extreme rear (with practically motionless water behind it) of the train of waves in the wake of a forcive travelling uniformly for ever. I hope to return to this subject when we come to consider the work done by the travelling forcive.

§ 78. Pass now to the investigation of the formulas by the calculation of which figs. 26, 27, 28, 29, 30 have been drawn, and the theorem of (97) proved. Go back to the problem of § 41 above : but instead of taking $e = \cdot 9$, as in §§ 46–61,

Fig. 29.

$10^{-1}\lambda$

$\cdot05$

$\cdot1$

$\cdot15$

$d(x) - d(0).$
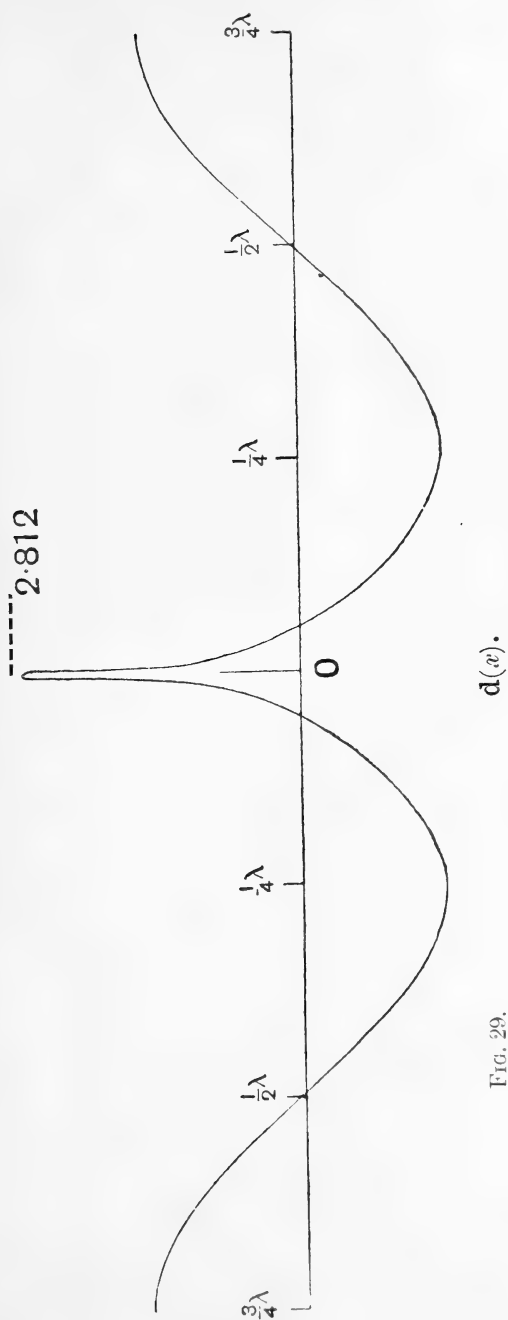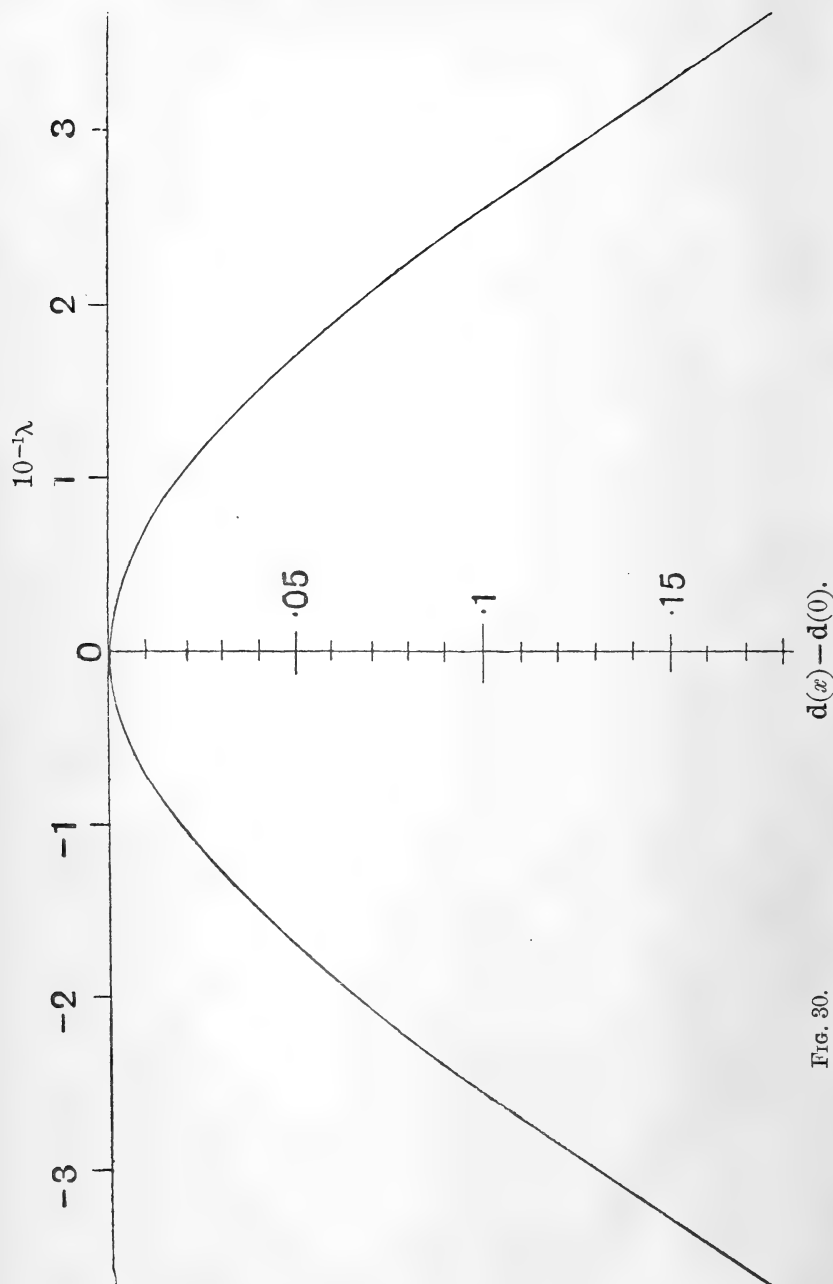
Fig. 30.

take $e = 1 - 10^{-4}$; and $c = 1/(2j+1)$. By (86) and (87) of § 45 we have the following solution

$$d = \mathscr{I} + \mathscr{G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

where

$$\mathscr{I} = e^{j+\frac{1}{2}} \left\{ \cdot \tfrac{1}{2} \sin (j + \tfrac{1}{2}) \theta \tan^{-1} \frac{2\sqrt{e} \sin \tfrac{1}{2}\theta}{1 - e} \right.$$

$$\left. - \tfrac{1}{4} \cos (j + \tfrac{1}{2}) \theta \log \frac{1 + 2\sqrt{e} \cos \tfrac{1}{2}\theta + e}{1 - 2\sqrt{e} \cos \tfrac{1}{2}\theta + e} \right\} \quad . \quad (101)$$

and

$$\mathscr{G} = \tfrac{1}{2} \cdot \frac{1}{2j+1} + \frac{e \cos \theta}{2j-1} + \frac{e^2 \cos 2\theta}{2j-3} + \ldots\ldots + e^j \cos j\theta \quad . \quad (102).$$

Fig. 29 has been calculated by putting $\theta = \dfrac{x}{\lambda} \cdot \dfrac{360°}{j + \frac{1}{2}}$, and taking $j = 20$. The explanation is that, as we shall see by (78) of § 43 above, (100), (101), (102), express the water disturbance due to an infinite row of forcives at consecutive distances each equal to $(20\frac{1}{2}) \lambda$; the expression for each forcive being

$$\frac{cba/2\pi}{b^2 + (x - na)^2} \quad . \quad . \quad . \quad . \quad . \quad (103),$$

where n is zero or any positive or negative integer ; and by (79) we have

$$b = \frac{20 \cdot 5 \cdot 10^{-4} \cdot \lambda}{2\pi} \quad . \quad . \quad . \quad . \quad (104).$$

Thus we see that the pressure at O due to each of the forcives next to O, on the two sides, is $1/\{1 + (2\pi \cdot 10^4)^2\}$ of the pressure due to the forcive whose centre is O. Thus we see that the pressures due to all the forcives, except the last mentioned, may be neglected through several wavelengths on each side of O : and we conclude that (100), (101), (102) express, to a very high degree of approximation, the disturbance produced in the water by the single travelling forcive whose centre is at O.

§ 79. To prove (97) take $\theta = 180°$ in (100), (101), (102); we thus find

$$(-1)^j d(180°) = e^j \left\{ \frac{1}{2} \sqrt{e} \tan^{-1} \frac{2\sqrt{e}}{1-e} + 1 - \frac{e^{-1}}{3} + \frac{e^{-2}}{5} \ldots\ldots \right.$$

$$\left. + \tfrac{1}{2}(-e)^{-j} \cdot \frac{1}{2j+1} \right\} . \quad (105).$$

Instead now of taking $e = 1 - 10^{-4}$, as we took in our calculations for $\mathbf{d}(0)$, let us take $e = 1$. This reduces (105) to

$$(-1)^j \mathbf{d}(180°) = \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} \cdots$$
$$+ \frac{(-1)^{j-1}}{2j-1} + \frac{1}{2}\frac{(-1)^j}{2j+1} \cdot \cdot \cdot (106).$$

Lastly take j an infinitely great odd or even integer, and we find

$$\mathbf{d}(180°) = (-1)^j \cdot \frac{\pi}{2} \cdot \cdot \cdot \cdot \cdot (107).$$

Now fig. 26 is, as we have seen, found by superimposing on the motion represented by fig. 29 an infinite train of periodic waves represented by $-\frac{1}{2}h \cdot \sin\frac{2\pi x}{\lambda}$, and therefore $h = \pi$, which proves (97).

§ 80. To pass now from the two-dimensional problem of canal-ship-waves to the three-dimensional problem of sea-ship-waves, we shall use a synthetic method given by Rayleigh at the end of his paper on "The form of standing waves on the surface of running water," communicated to the London Mathematical Society in December 1883 [*]. In an infinite plane expanse of water, consider two or more forcives, such as that represented by (95) of § 66, with their horizontal medial generating lines in different directions through one point O, travelling with uniform velocity, v, in any direction. The superposition of these forcives, and of the disturbances of the water which they produce, each calculated by an application of (100), (101), (102), gives us the solution of a three-dimensional wave problem; which becomes the ship-wave-problem if we make the constituents infinitely small and infinitely numerous. Rayleigh took each constituent forcive as confined to an infinitely narrow space, and combated the consequent troublesome infinity by introducing a resistance to be annulled in interpretation of results for points not infinitely near to O. I escape from the trouble in the two-dimensional system of waves, by taking (95) to express the distribution of pressure in the forcive, and making b as small as we please. Thus, as indicated in §§ 79, 73, 76, by taking $b = 10\frac{1}{4}\lambda/(10^4 \cdot \pi)$ we calculated a finite value for $\mathbf{d}(0)$. But for values of x, considerably

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[*] Proc. L.M.S., 1883; republished in Rayleigh's Scientific Papers, vol. ii. art. 109.

greater than half a wave-length, we were able to simplify the calculations by taking $b=0$.

§ 81. For the three-dimensional system let, in fig. 31, ψ be the inclination to O X of the rearward wave-normal of one of the constituent systems of waves. This is also the inclination to O Y of the medial line of the travelling forcive to which that set of waves is due. Take now for the forcive obtained by the superposition of an infinite number of

Fig. 31.
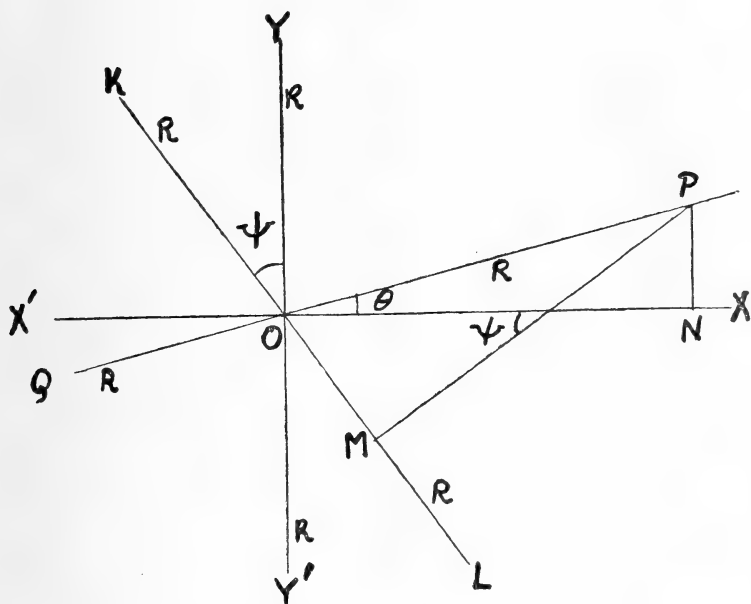
constituents, as described in § 80,

$$\frac{1}{g}\,\Pi\,(x,\,y) = \int_0^\pi d\psi\; \frac{b^2 k}{[(x\cos\psi + y\sin\psi)^2 + b^2]}\quad\cdot\;(108),$$

where k may be a function of ψ, and b is the same for all values of ψ.

For the case of a circular forcive system we must take k constant ; and we find

$$\frac{1}{g}\,\Pi(r) = \frac{\pi b k}{\sqrt{(r^2 + b^2)}}\quad\text{where } r^2 = x^2 + y^2.\quad\cdot\quad\cdot\quad(109).$$

§ 82. Let now the forcive, whether circular or not, be kept travelling in the direction of x negative [*] with velocity v : and let λ denote the corresponding free wave-length given by the formula $2\pi v^2/g$. This is the wave-length of the constituent train of waves corresponding to $\psi = 0$. For the ψ-constituent, the component velocity perpendicular to the front is $v \cos \psi$, and the wave-length is $\lambda \cos^2 \psi$. Looking now to fig. 26, with $\lambda \cos^2 \psi$ instead of λ ; and to fig. 31 ; and to equations (97), (98) ; we see that the portion of the depression at (x, y) due to the constituent of forcive shown under the integral in (108) is

$$\frac{4\pi^2 bk \, d\psi}{\cdot \lambda \cos^2 \psi} \sin \frac{2\pi (x \cos \psi + y \sin \psi)}{\lambda \cos^2 \psi} \quad . \quad . \quad . \quad (110),$$

provided $x \cos \psi + y \sin \psi$ is considerably greater than $\frac{1}{2}\lambda \cos^2 \psi$. Hence for the depression at (x, y) due to the whole travelling forcive, we have

$$\mathbf{d}(x, y) = 4\pi^2 b \int_{-\left(\frac{\pi}{2}-\theta\right)}^{\frac{\pi}{2}} \frac{kd\psi}{\lambda \cos^2 \psi} \sin \frac{2\pi (x \cos \psi + y \sin \psi)}{\lambda \cos^2 \psi} \quad . \quad (111).$$

§ 83. The reason for choosing the limits $-\left(\frac{\pi}{2}-\theta\right)$ to $\frac{\pi}{2}$ is that each constituent forcive gives a train of sinusoidal waves in its rear, and no perceptible disturbance in its front at distances from it exceeding half a wave-length. Look now to fig. 31, and consider the infinite number of medial lines of the forcives included in the integrals (108), (111) ; all as lines passing through O. Four examples, Q P, Y' Y, L K, X X' of these lines are shown in the diagram : corresponding respectively to $\psi = -\left(\frac{\pi}{2}-\theta\right)$, $\psi = 0$, $\psi =$ any positive acute angle, $\psi = \frac{\pi}{2}$. On each of the first three of these lines R R indicates the rear. The fourth, X X', is in the direction of the motion, and has neither front nor rear. The integral (111) must include all, and only all, the medial lines which have rears towards P. Hence Q P is one limit of ψ in (111) because it passes through P ; X X' is the other limit because it has neither front or rear. Thus all the lines included in the integral, lie in the obtuse angle

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[*] This is opposite to the direction of the motion of the forcive in fig. 26.

P O X′. Thus the integral (111) expresses the depression at P(x, y) due to the joint action of all the constituent forcives, because none except those whose medial lines lie in the angle P O X′, contribute anything to the disturbance of the water at P.

§ 84. For interpreting and approximately evaluating the definite integral, we may conveniently put

$$r = \sqrt{x^2 + y^2}, \quad \text{and} \quad u = \frac{\cos(\psi - \theta)}{\cos^2 \psi} \quad . \quad . \quad . \quad (112),$$

and write (111) as follows:

$$\mathbf{d}(x, y) = 4\pi^2 b \int_{-\left(\frac{\pi}{2} - \theta\right)}^{\frac{\pi}{2}} \frac{k\, d\psi}{\lambda \cos^2 \psi} \sin \frac{2\pi r u}{\lambda} \quad . \quad . \quad (113).$$

Now if we suppose r/λ very great, there will be exceedingly rapid transitions between equal positive and negative values of sin $(2\pi r u/\lambda)$, *which will cause cancelling of all portions of the integral except those, if any there are, for which* du/dψ *vanishes.* We shall see presently that *there are* two such values, ψ_1, ψ_2, both real if tan $\theta < \sqrt{\frac{1}{8}}$; u being a maximum (u_1) for one of them, and a minimum (u_2) for the other; and that, when θ has any value between $\tan^{-1}\sqrt{\frac{1}{8}}$ and $2\pi - \tan^{-1}\sqrt{\frac{1}{8}}$, the values of ψ_1, ψ_2 are both imaginary. Consideration of this last-mentioned case shows that, in the whole area of sea in advance of two lines through the centre of the travelling forcive inclined at equal angles of $\tan^{-1}\sqrt{\frac{1}{8}}$, (or 19° 28′) on each side of the mid-wake, there is no perceptible disturbance at distances of much more than a half wave-length from the centre of the forcive. The main disturbance by ship-waves, therefore, lies in the rearward angular space between these two lines. It is illustrated by fig. 32, as we now proceed to prove by the proper interpretation of (113). Expanding the argument of the sin in (113) by Taylor's theorem for values of ψ differing from ψ_1 by small fractions of a radian, we find

$$\frac{2\pi r u}{\lambda} \doteqdot \frac{2\pi r}{\lambda} \left[u_1 + \frac{1}{2} \left(\frac{d^2 u}{d\psi^2} \right)_1 (\psi - \psi_1)^2 \right] = \alpha_1 - q_1^2 \quad . \quad (114),$$

where

$$\alpha_1 = \frac{2\pi r u_1}{\lambda}, \quad \text{and} \quad q_1 = (\psi - \psi_1) \sqrt{\frac{\pi r}{\lambda} \left(-\frac{d^2 u}{d\psi^2} \right)_1} \quad . \quad (115).$$

Lord Kelvin *on*
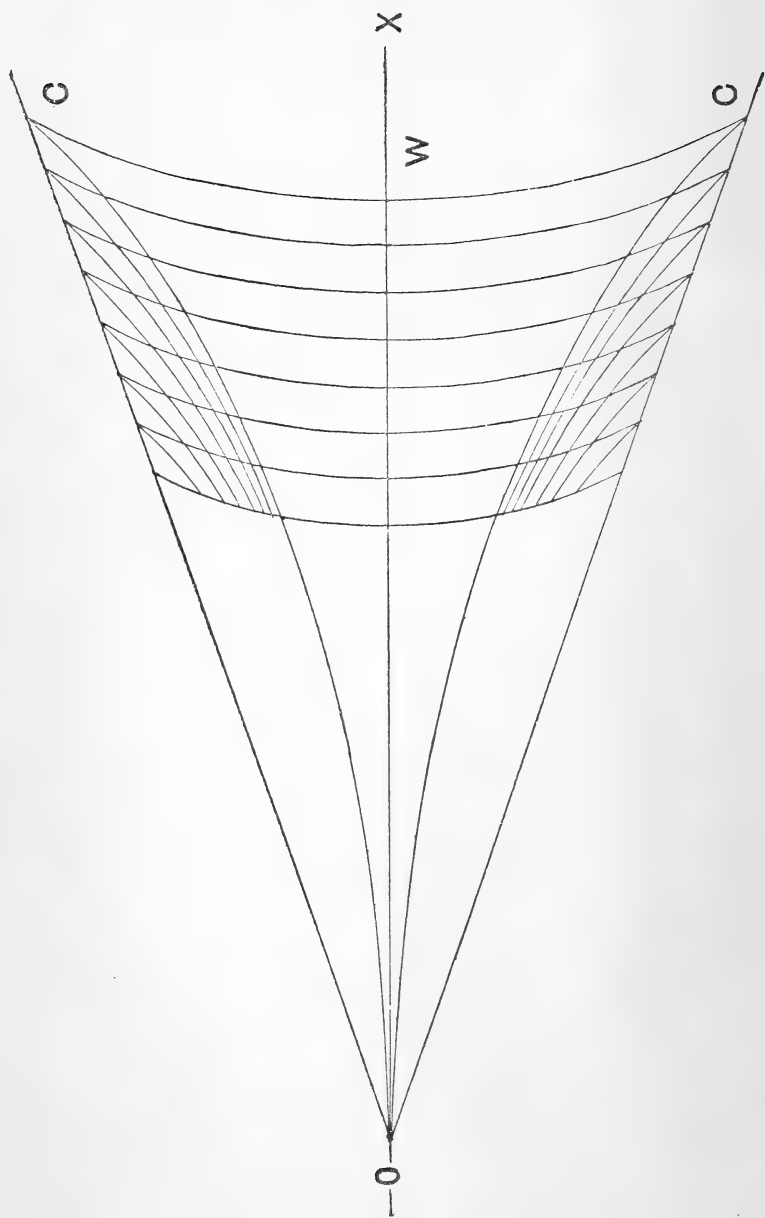
FIG. 32.—Isophasal Curves.

From the second of (115) we find $d\psi = dq_1/(\beta_1\sqrt{\pi})$, where

$$\beta_1 = \sqrt{\frac{r}{\lambda}\left(-\frac{d^2u}{d\psi^2}\right)_1} \quad . \quad . \quad . \quad . \quad . \quad (116).$$

Dealing similarly in respect to ψ_2 and values of ψ differing but little from it, we take $+q_2^2$ instead of the $-q_1^2$ of (114), and $(d^2u/d\psi^2)_2$ instead of the $-(d^2u/d\psi^2)_1$ of (115); because u_1 is the maximum and u_2 the minimum. Calling k_1, k_2 the values of k corresponding to ψ_1, ψ_2, and using these expressions properly in (113), we find, for the depression of the water at (x, y),

$$\mathsf{d}(x, y) = \frac{4b\pi^{3/2}}{\lambda}\left[\frac{k_1}{\beta_1\cos^2\psi_1}\int_{-\infty}^{\infty}dq_1\sin(\alpha_1 - q_1^2)\right.$$
$$\left. + \frac{k_2}{\beta_2\cos^2\psi_2}\int_{-\infty}^{\infty}dq_2\sin(\alpha_2 + q_2^2)\right] . (117).$$

The limits ∞, $-\infty$ are assigned to the integrations relatively to q_1 and q_2 because the greatness of r/λ in (115) and corresponding formula relative to ψ_2 makes q_1 and q_2 each very great, (positive or negative,) for moderate properly small positive or negative values of $\psi - \psi_1$ and $\psi - \psi_2$. Now as discovered by Euler or Laplace (see *Gregory's Examples*, p. 479), we have

$$\int_{-\infty}^{\infty}dq\sin q^2 = \int_{-\infty}^{\infty}dq\cos q^2 = \sqrt{\pi/2},$$

and using these in (117) we find

$$\mathsf{d}(x, y) = \frac{2\sqrt{2}\pi^2 b}{\lambda}\left[\frac{k_1(\sin\alpha_1 - \cos\alpha_1)}{\beta_1\cos^2\psi_1} + \frac{k_2(\sin\alpha_2 + \cos\alpha_2)}{\beta_2\cos^2\psi_2}\right]. (118).$$

Substituting for α_1, α_2 values by (115) we find

$$\mathsf{d}(x, y) = \frac{4\pi^2 b}{\lambda}\left[\frac{k_1}{\beta_1\cos^2\psi_1}\sin\frac{2\pi}{\lambda}\left(ru_1 - \frac{\lambda}{8}\right)\right.$$
$$\left. + \frac{k_2}{\beta_2\cos^2\psi_2}\sin\frac{2\pi}{\lambda}\left(ru_2 + \frac{\lambda}{8}\right)\right] \quad (118)'.$$

§ 85. To determine the quantities denoted by β_1, β_2 in (116) .... (118)', we write (112) as follows :—

$$ru = (x + yt)\sqrt{1 + t^2}, \text{ where } t = \tan\psi. \quad . \quad . \quad (119).$$

C 2

Hence, by differentiation on the supposition of x, y, r constant, we find

$$r\frac{du}{d\psi} = \left[xt + y(1 + 2t^2)\right]\sqrt{1 + t^2} \quad \text{. . . (120)}.$$

$$r\frac{d^2u}{d\psi^2} = \left[x(1 + 2t^2) + yt(5 + 6t^2)\right]\sqrt{1 + t^2} \quad \text{. (121)}.$$

By (120) we find for t_m, which makes u a maximum or minimum,

$$xt_m + y(1 + 2t_m^2) = 0 \quad \text{. . . . . (122)};$$

a quadratic equation which, when $(y/x)^2 < \frac{1}{8}$, has real roots as follows,—

$$t_1 = -\frac{x}{4y} - \sqrt{\left[\left(\frac{x}{4y}\right)^2 - \frac{1}{2}\right]}, \; t_2 = -\frac{x}{4y} + \sqrt{\left[\left(\frac{x}{4y}\right)^2 - \frac{1}{2}\right]} \; \text{(123)}.$$

And substituting t_m, (either of these,) for t in (121) we find

$$r\left(\frac{d^2u}{d\psi^2}\right)_m = \left[x(1 - t_m^2) + 2yt_m\right]\sqrt{1 + t_m^2} \quad \text{. . (124)},$$

or with simplification by (119),

$$r\left(\frac{d^2u}{d\psi^2}\right)_m = 2ru_m - x(1 + t_m^2)^{3/2} \quad \text{. . . (124)}'.$$

Eliminating t_m^2 from the first factor of (124) by (122) we find

$$r\left(\frac{d^2u}{d\psi^2}\right)_m = \left[\frac{3x}{2} + t_m\left(2y + \frac{x^2}{2y}\right)\right].\sqrt{1 + t_m^2} \quad \text{. (124)}'',$$

which, with $m = 1$, and $m = 2$, gives β_1 and β_2 by (116).

§ 86. Using (123) we see that $(d^2u/d\psi^2)_m$ vanishes when $x = y\sqrt{8}$, and that it is negative for t_1, and positive for t_2, when $x > y\sqrt{8}$. Hence t_1 makes $d_2u/d\psi^2$ negative. Therefore u_1 is the maximum; and t_2 makes it positive. Therefore u_2 is the minimum; and (119) gives for these maximum and minimum values

$$ru_1 = (x + yt_1)\sqrt{1 + t_1^2}, \quad ru_2 = (x + yt_2)\sqrt{1 + t_2^2}. \; \text{(125)}.$$

By (122), (123) we see that when $y/x = 0$, we have $-t_1 = +\infty$, and $-t_2 = 0$. If we increase y from 0 to $+x/\sqrt{8}$, $-t_1$ falls continuously from ∞ to $\sqrt{\frac{1}{2}}$, and $-t_2$ rises continuously from 0 to $\sqrt{\frac{1}{2}}$. Thus $-t_1$ and $-t_2$ become, each of them, $\sqrt{\frac{1}{2}}$; which is the tangent of 35° 16′.

§ 87. *Geometrical digression on a system of autotomic, monoparametric co-ordinates* *. §§ 87–90.

In (119) put

$$ru = a \quad \cdots \cdots \quad (126)$$

where a denotes the parameter O W of the curve O C C, fig. 32, which we are about to describe ; being the curve given intrinsically by (119) and (122) with suffix '*m*' omitted from t. In the present paper these curves may be called isophasals, because the argument of the sine in (130) below is the same for all points on any one of them.

Solving (119) and (122) for x and y, we find

$$x = a\,\frac{1 + 2t^2}{(1 + t^2)^{3/2}}, \quad y = a\,\frac{-t}{(1 + t^2)^{3/2}} \quad \cdot \quad (127).$$

The largest of the eight curves shown in fig. 32 has been described according to values of x, y calculated from these two equations, by giving to $-t$ values tan 0°, tan 10°, tan 20°, ... tan 90°. The seven other isophasals partially shown in fig. 32, all similar to the largest, have been drawn to correspond to seven equidifferent smaller values, 19λ, 18λ.... 13λ, of the parameter a, if we make the largest equal to 20λ.

§ 88. It is seen in the diagram that every two of these isophasals cut one another in two points, at equal distances on the two sides of O W. If we continue the system down to parameter 0, every point within the angle Č O C is the intersection of two and only two of the curves given by (127), with two different values of the parameter a. If we are to complete each curve algebraically, we must duplicate our diagram by an equal and similar pattern on the left of O : and the doubled pattern, thus obtained, would show a system of waves, equal and similar in the front and rear, which (§ 77 above) is possible but unstable. We are, however, at present only concerned with the stable ship-waves contained in the angle $\pm$ 19° 28′ on the two sides of the mid-wake ; and we leave the algebraic extension with only the remark that all points in the angle C O C of the diagram, and the opposite angle leftward of O, can be specified by real values of the parameter a : while imaginary values of it would specify real points in the two obtuse angles.

* Of this kind of co-ordinates in a plane, we have a well-known case in the elliptic co-ordinates consisting of confocal ellipses and hyperbolas.

§ 89. By differentiation of (127), we find

$$\frac{dx}{dy} = -t = -\tan \psi \quad . \quad . \quad . \quad . \quad . \quad (128);$$

which proves that $\tan^{-1} t$ is the angle measured anti-clockwise from O Y to the tangent to the curve at any point (x, y), in the lower half of the diagram. Elimination of t between the two equations of (127) gives, as the cartesian equation of our curve,

$$(x^2 + y^2)^3 + a^2(8y^4 - 20x^2y^2 - x^4) + 16a^4y^2 = 0 \quad . \quad (129).$$

But the implicit equations (127) are much more convenient for all our uses. It is interesting to verify (129) for the case $-t = \pm \sqrt{\frac{1}{2}}$ in (127), corresponding to either of the two cusps shown in the diagram.

§ 90. Going back now to § 86 and the continuous variations considered in it, we see that $-t_1$ and $-t_2$ are respectively the tangents of the inclinations, reckoned from O Y clockwise, of portions of the long arc O C and of the short arc W C, in the upper half of the diagram. Thus, if we carry a point from O to C in the long arc, and from C to W in the short arc, we have the change of inclinations to O Y represented continuously by the decrease of $\tan^{-1}(-t_1)$ from 90° to 35° 16′, while y increases from 0 to $x\sqrt{8}$; and the farther decrease of $\tan^{-1}(-t_2)$ from 35° 16′ to 0°, while y diminishes from $x\sqrt{8}$ to 0 again. The inclination to O Y of the two branches meeting in the cusp, C, is 35° 16′ (or $\tan^{-1}\sqrt{\frac{1}{2}}$). For any point in the short arc C W C of the curve u or $\cos(\psi - \theta)/\cos^2\psi$, is a minimum. In each of the long arcs u is a maximum. At every point of the curve the value of u, whether minimum or maximum, is a/r. Hence for different points of the curve, u is inversely proportional to the radius vector from O.

§ 91. Going back to (118)′ we now see that for all points on any one of our curves, ru_1 and ru_2 have both the same value, being the parameter O W of the curve. The first part of (118)′ is one constituent of the depression at any point on either of the long arcs ; and the second part of (118)′ is one constituent of the depression at any point on the short arc. Taking for example the largest of the curves shown in fig. 32, we now see that for any point of either of its long arcs, the second constituent of the depression of the water is to be calculated from the second part of (118)′ ; while for any point of its short arc, the second constituent of the depression is to be calculated from the first part of (118)′.

§ 92. Explaining quite similarly the determination of $\mathbf{d}(x, y)$ for every point of each of the smaller curves which we see in the diagram cutting the longer arcs of the largest curve, we arrive at the following conclusions as *the complete solution of our problem.*

The whole system of standing waves in the wake of the travelling forcive is given by the superposition of constituents calculated according to (127), with greater and smaller values of the parameter a with infinitely small successive differences. Hence, what we see in looking at the waves from above is exactly a system of crossing hills and valleys, with ridges and beds of hollows, all shaped according to the isophasal curves shown in fig. 32. Looking at any one of the short arc-ridges and following it through the cusps, we find it becoming the middle line of a valley in each of the long arcs of the curve. And following a short arc mid-valley through the cusps, we find, in the continuation of the curve, two long ridges. Every ridge, long or short, is furrowed by valleys. All the curved ridges and valleys are parts of one continuous system of curves, illustrated by fig. 32 and expressed by the algebraic equation (129).

With these explanations we may write (118)′ as follows :

$$\mathbf{d}(x, y) = \frac{4\pi^2 bk \sec^2 \psi}{\lambda\beta} \sin \frac{2\pi}{\lambda}\left(ru - \frac{\lambda}{8}\right) \quad . \quad . \text{ (130),}$$

where

$$\beta = \sqrt{\frac{r}{\lambda}\left(\pm \frac{d^2 u}{dx^2}\right)} \quad . \quad . \quad . \quad . \quad . \text{ (131).}$$

§ 93. An important, perhaps the most important, feature of the wave-system which we actually see on the two sides of the mid-wake of a steamer travelling through smooth water at sea, or of a duckling * swimming as fast as it can in a pond, is the steepness of the waves in two lines which we know to be inclined at 19° 28′ to the mid-wake. The theory of this feature is expressed by the coefficient of the sine in (130), and is well illustrated by the calculation of $\sqrt{\frac{a}{\lambda}} \cdot \frac{\sec^2 \psi}{\beta}$ for eleven points of any one of the curves of fig. 32, the results of which are shown in column 6 of the following table. They express the depression below, and elevation

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* In the case of even the highest speed attained by a duckling, this angle is perhaps perceptibly greater than 19° 28′, because of the dynamic effect of the capillary surface tension of water. See ' Baltimore Lectures,' p. 593 (letter to Professor Tait, of date 23rd Aug. 1871) and pp. 600, 601 (letter to William Froude, reprinted from ' Nature ' of 26th Oct. 1871).

above mid-level, due to one constituent of the system of crossing hills and valleys described in § 92. Column 1 is $-\psi$. Columns 2, 3 are x/a and y/a, calculated from (127). Column 4 is u, calculated by (126) from columns 2, 3. Column 5 is $\frac{r}{a} \cdot d^2u/d\psi^2$, calculated from (124) and columns 2, 4. Column 6 is $\sqrt{\dfrac{a}{\lambda} \cdot \dfrac{\sec^2 \psi}{\beta}}$, calculated from columns 1, 6. u, being, as we have seen, a maximum for values of $-\psi$ from 0 to 35° 16′, and a minimum for values from this to 90°, we see that the proper suffix in columns 4, 6, for the first four lines of each column is 1, and for the last six lines is 2.

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
$-\psi$	$\dfrac{x}{a}$	$\dfrac{y}{a}$	u	$\dfrac{r}{a}\dfrac{d^2u}{d\psi^2}$	$\sqrt{\dfrac{a}{\lambda}\cdot\dfrac{\sec^2\psi}{\beta}}$
0°	1·0000	0·0000	1·00000	1·00000	1·0000
10°	1·0145	·1685	·97239	·93782	1·0647
20°	1·0497	·3201	·91587	·73497	1·3210
30°	1·0825	·3750	·87290	·33333	2·3094
35° 16′	1·0887	·3849	·86602	0·00000	∞
40°	1·0826	·3773	·87225	− ·40830	2·6660
50°	1·0201	·3166	·93624	− 1·84070	1·7839
60°	·8750	·2165	1·10941	− 5·00003	1·7888
70°	·6441	·1100	1·53041	−14·0987	2·2793
80°	·3421	·0297	2·91222	−63·3341	4·1672
90°	0·0000	0·0000	∞	− ∞	∞

§ 94. In (130), k is generally a function of ψ; but if the forcive is circular, (§ 81 above) k is a constant, and for points on one of the isophasal curves ($a =$ constant) the only variable coefficients of the sine are $\sec^2 \psi$, and β^{-1}. But for different isophasal curves the coefficient in (130) expressing the magnitude of the range above and below mean level, varies inversely as $\sqrt{a}$. For mid-wake ($\psi = 0$) a is simply the distance from the forcive : and we conclude, not merely for our point-forcive, but for a great ship, that the waves at a very large number of wave-lengths right astern, are smaller in height inversely as the square root of the distance from the forcive or from the middle of the ship.

§ 95. The infinity for $\psi = \pm 35°$ 16′ represents a feature analogous to a caustic in optics. There is in nature no infinity for either case, if the source is finite and distributed, not infinitely intense and confined to an infinitely small space. According to the methods followed in §§ 1–72 above, we

have in every case a finite intensity of source, or of forcive, except in § 80 where we have supposed b infinitely small, in comparison with λ, we avoid the infinity shown in column 6: and can, by great labour, calculate a table of mitigated numbers, rising to a very large maximum at $\psi = \pm 35° \ 16'$; but not to infinity ; and so arrive mathematically at an expression for the very high waves seen on the two bounding lines of the wave-disturbance, inclined at $19° \ 28'$ to the mid-wake. But it is interesting to remember that we see in reality a considerable number of white-capped waves (would-be infinities) before the well-known large glassy waves which form so interesting a feature of the wave-disturbances.

§§ 80–95 of the present paper is merely a working out of the simple problem of purely gravitational waves with no surface-tension on the principle given by Rayleigh [*] in 1883 for the much more complex problem of capillary waves in front, in which surface-tension is the chief constituent of the forcive, and waves in the rear, in which the chief constituent of the forcive is gravitational.

In all the work arithmetical, algebraic, graphic of §§ 32–95 above, I have had much valuable assistance from Mr. J. de Graaff Hunter ; who has just now been appointed to a post in the National Physical Laboratory.

II. *On an Expeditious Practical Method of Harmonic Analysis* [†]. *By* THOMAS R. LYLE, *M.A.*, *Professor of Natural Philosophy in the University of Melbourne.*

[Plate I.]

1. FOURIER has shown that if any function $f(t)$ $(=y$ say) of a variable t be such that

$$f(t) = f(t+\tau) = f(t+2\tau) = \ \&c.,$$

where τ is a constant, that is, if $f(t)$ be periodic in t, of period τ, then $f(t)$ can be expressed as the sum of a constant and a series of terms called harmonics, each of the form

$$a_p \sin p \, (\omega t - \theta_p),$$

where p has the values 1, 2, 3, 4, &c., and

$$\omega = 2\pi/\tau.$$

The number p is called the order of the harmonic, a_p its amplitude, and θ_p its phase.

If, in addition, $f(t)$ be such that

$$f(t) = -f(t + \tau/2),$$

then it is easy to see, by substituting $t + \tau/2$ for t, *i. e.* $\omega t + \pi$ for ωt in

$$y = a_0 + \Sigma a_p \sin p\,(\omega t - \theta_p),$$

that in order for y_t to be $= -y_{t+\tau/2}$,

$$a_0 = 0, \quad a_2 = 0, \quad a_4 = 0, \&c.$$

Hence in this case the constant term vanishes and the harmonics, of which $f(t)$ is the sum, are all of odd order. When such is the case, $f(t)$ is called an odd periodic function. This is the type generally met with in alternating electric-current investigations.

2. If we define the nth component (C_n say) of a periodic function $f(t)$ of period τ as the periodic function which is the sum of those harmonics of $f(t)$ whose orders are $n, 3n, 5n, 7n, \&c.$, then

$$2n\,C_n = f(t) - f\left(t + \frac{\tau}{2n}\right) + f\left(t + 2\,\frac{\tau}{2n}\right) - \ldots$$

$$- f\left(t + \overline{2n-1}\,\frac{\tau}{2n}\right). \quad \ldots \ldots \ldots \quad \text{(I.)}$$

For if we represent the expression on the right of the above equation by $\psi(t)$, we find by substituting successively for t, $t + \tau/2n$, and $t + \tau/n$ in it, that

$$\psi(t) = -\dot{\psi}\left(t + \frac{\tau}{2n}\right) = \psi\left(t + \frac{\tau}{n}\right).$$

Hence $\psi(t)$ is an odd periodic function of period τ/n, that is to say, if

$$f(t) = a_0 - \Sigma a_p \sin p\,(\omega t - \theta_p),$$

where $p = 1, 2, 3, 4, \&c.$, then $\psi(t)$ is of the form

$$\psi(t) = \Sigma b_q \sin qn\,(\omega t - \beta_q),$$

where $q = 1, 3, 5, 7, 9, \&c.$

In evaluating $\psi(t)$, therefore, only those harmonics whose arguments are $n\omega t, 3n\omega t, 5n\omega t, \&c.$, need be considered. Neglecting all other harmonics in the different f functions that make up $\psi(t)$, we find that the remainders in the $2n$ terms

$$f(t), \quad -f\left(t + \frac{\tau}{2n}\right), \quad f\left(t + 2\,\frac{\tau}{2n}\right), \quad \&c.,$$

are all equal, and that each remainder is the nth component of $f(t)$; hence

$$\psi(t) = 2n\,C_n.$$

3. If $f(t)$ itself contain only odd harmonics as in the case of alternate-current periodic functions, then

$$f(t) = -f\left(t + \frac{\tau}{2}\right),$$

and equation I., § 2, reduces to

$$n\,C_n = f(t) - f\left(t + \frac{\tau}{2n}\right) + \ldots + f\left(t + \overline{n-1}\,\frac{\tau}{2n}\right). \quad \text{(II.)}$$

The operation on $f(t)$ mathematically represented on the right-hand side of equations I. or II. is practically performed on alternate-current waves by the wave-tracer and analyser * designed by the author. In the simplest case, when $n=1$, the wave-tracer gives the first component of the periodic quantity operated on, which in the case of alternating electric currents is the full wave. By the movement of two pairs of brushes n can be made 3, or 5, or 7, in which cases the analyser will give the 3rd, 5th, or 7th components of the wave respectively.

Now, in practical investigations with this apparatus on alternating-current waves whose harmonic expressions were required, it was found much better to obtain by its means only the full wave-trace, and then by an arithmetical process identical with the action of the analyser and indicated by equation II. above, to obtain the 3rd and higher components of the wave, and thence to deduce its harmonics.

This method of harmonic analysis was drawn attention to in the paper already quoted, and though based on a different formula to that of Wedmore†, is practically similar to his. It is more suitable, however, for waves containing only odd harmonics; and as I have had considerable experience in its use during the last two years and have found it both expeditious and accurate, it is possible that a short account may be of value to those interested in alternating-current work.

4. In wave graphs it is more convenient to use angular abscissæ x where

$$x = \omega t = 2\pi t/\tau.$$

Making this substitution in the equation $y = f(t)$, it becomes $y = g(x)$ say, where

$$g(x) = g(x + 2\pi),$$

---

and if $f(t)$ is an odd periodic function as in the case of alternate-current waves which we are now considering,

$$g(x) = -g(x+\pi) = g(x+2\pi).$$

Substituting $g(x)$ for $f(t)$ in equation II., it becomes

$$nC_n = g(x) - g(x+\pi/n) + g(x+2\pi/n) - \ldots \ldots$$
$$+ g(x + \overline{n-1}\,\pi/n),$$

from which we conclude that, if

$$y_0, \ y_1, \ y_2, \ \ldots \ldots y_{n-1}$$

be n equi-spaced ordinates that exactly include half the wave, *i. e.* ordinates corresponding to the abscissæ

$$x, \ x+\pi/n, \ x+2\pi/n, \ \ldots \ldots x+\overline{n-1}\,\pi/n$$

respectively, and called e.s. ordinates in the sequel ; and if

$$N_0, \ N_1, \ N_2, \ldots \ldots N_{n-1}$$

be the ordinates of the nth component C_n whose abscissæ are the same as those of

$$y_0, y_1, y_2, \ldots \ldots y_{n-1}$$

respectively, then

$$y_0 - y_1 + y_2 - \ldots + y_{n-1} = nN_0 = -nN_1 = nN_2 = \quad = nN_{n-1}$$

when n is an odd number, and

$$y_0 - y_1 + y_2 - \ldots \ldots - y_{n-1} = 0$$

when n is an even number, as we are now considering odd periodic functions only.

Thus from n e.s. ordinates of the original half-wave we obtain only one ordinate per half-wave of C_n, so that in order to obtain m e.s. ordinates per half-wave of C_n it is necessary to have mn e.s. ordinates of the original half-wave.

For instance, to obtain 3 e.s. ordinates of C_n we must measure $3n$ e.s. ordinates of $g(x)$. Let these be

$$y_0, \ y_1, \ y_2, \ y_3, \ \ldots \ldots y_{3n-1},$$

and let the corresponding ordinates of C_n be

$$N_0, \ N_1, \ N_2, \ N_3, \ \ldots \ldots N_{3n-1},$$

then

$$y_0 - y_3 + y_6 - \ldots + y_{3n-3} = nN_0 = -nN_3 = nN_6 = \quad = nN_{3n-3},$$
$$y_1 - y_4 + y_7 - \ldots + y_{3n-2} = nN_1 = -nN_4 = nN_7 = \quad = nN_{3n-2},$$
$$y_2 - y_5 + y_8 = \ldots + y_{3n-1} = nN_2 = -nN_5 = nN_8 = \quad = nN_{3n-1}.$$

Subtracting now the ordinates of C_n so obtained from the corresponding y ordinates, we obtain a new set of $3n$ e.s. ordinates which are those of the original half-wave with its nth component removed.

5. In practice it will generally be sufficient to determine the 1st, 3rd, 5th, 7th, and 9th harmonics ($H_1 H_3 H_5 H_7 H_9$ say). This can be done with considerable accuracy when 15 e.s. ordinates of the original half-wave are given.

Thus if these be
$$y_0, \ y_1, \ y_2, \ \ldots \ldots \ y_{14},$$
corresponding to the angular abscissæ
$$x_0, \ x_1, \ x_2, \ \ldots \ldots \ x_{14},$$
where
$$x_1 - x_0 = x_2 - x_1 = \ \ldots \ldots = x_{14} - x_{13} = \pi/15,$$
and if z_0, z_1, z_2, z_3, z_4 be 5 e.s. ordinates of the half-wave of C_3, then
$$3z_0 = y_0 - y_5 + y_{10} = -3z_5 = 3z_{10},$$
$$3z_1 = y_1 - y_6 + y_{11} = -3z_6 = 3z_{11},$$
$$\cdots \cdots \quad \cdots \cdots$$
$$3z_4 = y_4 - y_9 + y_{14} = -3z_9 = 3z_{14},$$
and if u_0, u_1, u_2 be 3 e.s. ordinates of the half-wave of C_5, then
$$5u_0 = y_0 - y_3 + y_6 - y_9 + y_{12} = -5u_3 = 5u_6 = -5u_9 = 5u_{12},$$
$$5u_1 = y_1 - y_4 + y_7 - y_{10} + y_{13} = -5u_4 = 5u_7 = -5u_{10} = 5u_{13},$$
$$5u_2 = y_2 - y_5 + y_8 - y_{11} + y_{14} = -5u_5 = 5u_8 = -5u_{11} = 5u_{14},$$

the figures subscribed to each ordinate indicating the abscissa to which it corresponds.

Now the full wave
$$C_1 = H_1 + H_3 + H_5 + H_7 + H_9 + \ \&c.,$$
and
$$C_3 = \qquad H_3 + H_9 + H_{15},$$
$$C_5 = \qquad H_5 + H_{15},$$

so that if H_{15} be neglected, and the sums of the corresponding ordinates of C_3 and C_5 be subtracted from those of C_1, the fifteen remainders are ordinates of
$$H_1 + H_7 + \ \ldots,$$
i.e. of H_1, if we neglect H_7.

If H_{15} cannot be neglected it can at once be removed from C_5 before subtracting from C_1, for as it is (*q. p.*) the 3rd

component of C_5, of which we have 3 e.s. ordinates u_0, u_1, u_2, its three corresponding ordinates are $i_0, -i_0, i_0$, where

$$3i_0 = u_0 - u_1 + u_2.$$

Hence H_5 will be completely given by

$$c_0, \quad c_1, \quad c_2, \quad \text{where}$$
$$c_0 = u_0 - i_0, \quad c_1 = u_1 + i_0, \quad c_2 = u_2 - i_0.$$

H_{15} can now be taken from C_3, thus

$$z_0 - i_0, \quad z_1 + i_0, \quad z_2 - i_0, \quad z_3 + i_0, \quad z_4 - i_0$$

are the 5 e.s. ordinates of $H_3 + H_9$.

In order to determine H_3 and H_9 it will now be necessary to plot the 5 ordinates of $H_3 + H_9$, measure off 6 e.s. ordinates from the smooth curve drawn through them, and from these determine their first component, that is 2 e.s. ordinates of H_9. These will completely determine H_9 if H_{27} &c. be neglected, and by subtracting them from the corresponding ordinates of $H_3 + H_9$ 6 e.s. ordinates of H_3 are obtained.

If H_7 cannot be neglected it will be necessary (if the original wave-trace is not available) to plot the 15 ordinates of $H_1 + H_7$ obtained above, and from the smooth curve drawn through them to measure off 14 e.s. ordinates. From these, 2 e.s. ordinates of the half-wave of H_7, which determine H_7, can be obtained. By subtracting these from the corresponding ones of $H_1 + H_7$, 14 corrected ordinates of H_1 are obtained.

6. It now remains to determine the amplitudes and phases of the harmonics of C_1 from their ordinates which we have obtained. It is easy to show that

$$\frac{2}{n} \left\{ \sin^2 \theta + \sin^2 \left(\theta + \frac{\pi}{n} \right) + \sin^2 \left(\theta + \frac{2\pi}{n} \right) + \ldots \ldots \right.$$
$$\left. + \sin^2 \left(\theta + \overline{n-1} \, \frac{\pi}{n} \right) \right\} = 1,$$

from which we conclude that the square root of twice the mean of the squares of n e.s. ordinates of half a sine wave is equal to its amplitude.

Hence, with the help of a table of squares or of the quarter squares given in most sets of tables, the amplitudes of H_1, H_3, &c. can be quickly determined.

[The rule that the amplitude is equal to $\pi/2 \times$ mean of the ordinates is only sufficiently accurate when a large number of ordinates is taken.]

If $a_0, a_1, a_2, \ldots a_{14}$ be the ordinates we have found for H_2 corresponding to the angular abscissæ $x_0, x_1, x_2, \ldots x_{14}$,

respectively, and if h_1, a be the amplitude and phase of H_1, or in other words if

$$H_1 = h_1 \sin(\omega t - a),$$

then any of the equations

$$\sin(x_0 - \alpha) = a_0/h_1,$$
$$\sin(x_1 - \alpha) = a_1/h_1,$$
$$\sin(x_2 - \alpha) = a_2/h_1, \&c.$$

would determine α, provided the ordinates a_0, a_1, a_2, &c. are exactly those of a sine wave.

In practice, however, small upper harmonics will invariably be left in a_0, a_1, a_2 &c. [it may not have been thought worth while to remove H_7], and though their amplitudes may be negligibly small, yet they might cause considerable error in the value of α when determined from only one of the above equations. Hence it is advisable to obtain four values of α from the first four ordinates on the rising side of the wave and four from the last four ordinates on the falling side, and take the mean of the eight. In this way we can to a great extent eliminate any error that might arise through an harmonic even as low as the seventh not having been removed.

In a similar way the phases of H_3, H_5, &c. can be determined, but it must be remembered that if, for instance,

$$H_3 = h_3 \sin 3 (\omega t - \beta),$$

and if b_0, b_1, ... b_4 are the ordinates of H_3 corresponding to the abscissæ x_0, x_1, ... x_4, then

$$\sin 3 (x_0 - \beta) = b_0/h_3, \&c. ;$$

similarly, if

$$H_5 = h_5 \sin 5 (\omega t - \gamma),$$

with ordinates c_0, c_1, c_2, then

$$\sin 5 (x_0 - \gamma) = c_0/h_5.$$

7. The wave to be analysed may be given in either of two ways. We may have the complete trace of it obtained by the author's wave-tracer by the photographic method, or by any form of oscillograph that gives a trace of the wave form; or we may have the values of a definite number only of ordinates per half-wave, such as would be obtained by the author's wave-tracer by the galvanometer and scale method.

From the wave-trace the complete harmonic expression can theoretically be obtained, but the impossibility of accurately measuring on the photograph, without elaborate apparatus, the different ordinates required leads to great inaccuracy in the result.

From a given number of e.s. ordinates only an approximate analysis can be obtained, more approximate, of course, as the number of ordinates is greater. When, however, each individual ordinate has been obtained with the accuracy of which the galvanometer and scale method is susceptible, the analysis obtained from fifteen such ordinates is much more reliable, as far as the harmonics up to the 9th are concerned, than that determined from any photographic trace.

I will therefore illustrate the method by applying it in full detail to the analysis of the wave whose 15 e.s. ordinates are given in row 5 of Table I. (Pl. I.). Every figure necessary in the calculation will be given.

The first row of figures in Table I. are the abscissæ x_0, x_1, &c., to which the given ordinates correspond. Space for three rows of figures is left, and then the 15 given ordinates are written down. These are divided into three sets of five each, and the numbers of the middle set are subtracted in order from the sums of first and last set, giving five numbers which are the corresponding ordinates of $3C_3$. Space for two or more rows is left, and the given ordinates are now written down as in the table, in two rows of six each and one row of three, in order. The columns formed are added and the last three of the sums are subtracted from the first three, giving three ordinates of $5C_5$. The first of these minus the second, plus the third, gives one ordinate of $15C_{15}$, whose other ordinates are got by alternating the sign. Subtracting $5C_{15}$ from $5C_5$ we obtain $5H_5$. Having obtained C_{15} we now subtract $3C_{15}$ from $3C_3$ and obtain $3(H_3 + H_9)$.

Above the given ordinates write those of C_3 with signs changed (row 4), and above these write those of H_5 with signs changed (row 3). Add rows 3, 4, and 5 to get row 2, in which are the ordinates of $H_1 + H_7 + H_{11}$ &c. Neglecting H_7, H_{11}, &c., as is done in the analysis in Table I., we may consider the figures in row 2 as the ordinates of H_1, and neglecting H_9 we may consider the figures in row 11 as the ordinates of $3H_3$.

The first 15 numbers under Amp. H_1 are the quarter squares of the ordinates of H_1. Twice the sum of these is divided by 15, the number of ordinates, and the quotient is found to be the quarter square of 987. Hence h_1, the amplitude of H_1, is 987. Similarly for the amplitudes of H_3 and H_5.

Under the heading "Phase of H_1" in the first column under sines, are the quotients got by dividing the first four ordinates on the rising side of H_1 and the last four on the falling side of H_1 by h_1; in the second column under angles are the corresponding angles, and in the third column are

the eight values of $12° - \alpha$ deduced. The mean of these $2° \, 2'$ when subtracted from $12°$ gives the crossing-point or phase of H_1 as $9° \, 58'$. Similarly for the phases of H_3 and H_5. It will be noticed that at the crossing-point determined for H_3, H_3 crosses down, which is expressed analytically by writing its amplitude negative.

8. It will be noticed in the determination of the phase of H_1 in Table I., that the eight values of $12° - \alpha$ differ considerably from each other, indicating the presence in what we there take for H_1 of a considerable upper harmonic, probably H_7. In order to determine H_7, fourteen e.s. ordinates of the half-wave are required. If the wave-trace were given these could be measured off from it, but if, as in the case we are considering, only 15 original ordinates are given, it is necessary to plot the 15 ordinates of $H_1 + H_7$ obtained in Table I., and from the smooth curve drawn through them to measure off 14 e.s. ordinates. This has been done and the values obtained are given in row 4, Table II., as well as the calculation necessary for the determination of H_7 and its elimination from $H_1 + H_7$.

What is called the amplitude of H_1 in Table I. is really $\sqrt{2}$ R.M.S. $(H_1 + H_7)$. To get amp. H_1 it is better to remove the effect of H_7 by treating it as a correction, thus avoiding error that might be introduced in the plotting. This is easily done, since

$$\text{M.S } (H_1 + H_7) = \frac{h_1{}^2}{2} + \frac{h_7{}^2}{2} \, ;$$

hence $\quad h_1 \text{ (corrected)} = \sqrt{\text{Amp. } (H_1 + H_7)^2 - h_7{}^2},$
$$= \sqrt{h_1{}^2 \text{ (uncorrected)} - h_7{}^2}$$

In Table II. the corrected crossing-point of H_1 is determined, and it is seen to differ in phase only by 2 minutes from the value obtained in Table I.

The differences between the four values of $3 \, (24° - \beta)$ when determining the crossing-point of H_3 in Table I. point to the presence of a ninth harmonic, which exists as a third component in $H_3 + H_9$. H_9 can, if desired, be determined by plotting the five ordinates obtained in Table I., measuring off from the curve six e.s. ordinates, and proceeding as before. It will be found that

$$H_9 = 3 \sin 9 \, (\omega t - 13°).$$

9. In Table III. is given most of the work required for the determination of the first six harmonics of a complete wave that contains harmonics both of odd and even orders. Twenty-four e.s. ordinates of the full wave are taken. This

number is specially suitable, as it enables us to determine directly C_1, C_2, C_3, C_4, and C_6. To determine C_5, replotting will have to be resorted to if the full wave-trace be not available.

At the top of Table III. are written the 24 given ordinates under their corresponding abscissæ. From these ordinates the constant term of $f(t)$ has been removed. This can be done by aid of the formula

$$f(t) + f(t + \tau/n) + f(t + 2\tau/n) + \ldots + f(t + \overline{n-1}\tau/n)$$
$$= n\left[a_0 + a_n \sin n(\omega t - \theta_n) + a_{2n} \sin 2n(\omega t - \theta_{2n})\right.$$
$$\left. + a_{3n} \sin 3n(\omega t - \theta_{3n}) + \&c.\right], \quad \ldots \quad \text{(III.)}$$

which can be easily established by the method used in § 2.

From this formula we see that the mean of n e.s. ordinates embracing one period of a periodic function is equal to its constant term, if its nth, $2n$th, &c. harmonics are neglected.

Returning to Table III., we add the second twelve ordinates with their signs changed to the first twelve, in order, and obtain 12 e.s. ordinates of $2C_1$, *i.e.* of $2[H_1 + H_3 + H_5 + \&c.]$. (See equation I., § 2.)

Subtracting these from twice the given ordinates, those of $2[H_2 + H_4 + H_6 + \&c.]$ are left, and the remainder of the work proceeds as in Table I.

$2[H_2 + H_4 + H_6 + \&c.]$ could be obtained directly from the 24 given ordinates by adding the second 12 to the first 12 of them, in order. (See formula III., § 9.)

The amplitudes and phases of the different harmonics were determined as in Table I., but the figures necessary in their calculation are not given.

The following are interesting applications of the above method to more general harmonic analysis.

To obtain the harmonic expression for the odd periodic function whose graph for half a period is the sides of an isosceles triangle of altitude h. (See fig. 1.)

Taking 0 and π as the abscissæ of the extremities of the base, relative values of any number of e.s. ordinates can be written down, and any component at once obtained. Thus, 30 e.s. ordinates would be 0, 1, 2, 3, . . 14, 15, 14, . . . 2, 1, and these correspond to an altitude 15.

It will be found that all the components (*i. e.* 3rd, 5th, &c. in this case) are the sides of isosceles triangles passing through the origin, and that the altitudes are

$$-h/3^2, \quad h/5^2, \quad -h/7^2, \quad \&c. \text{ respectively. (See fig. 1.)}$$

(The same can be quickly arrived at geometrically.)

Fig. 1.
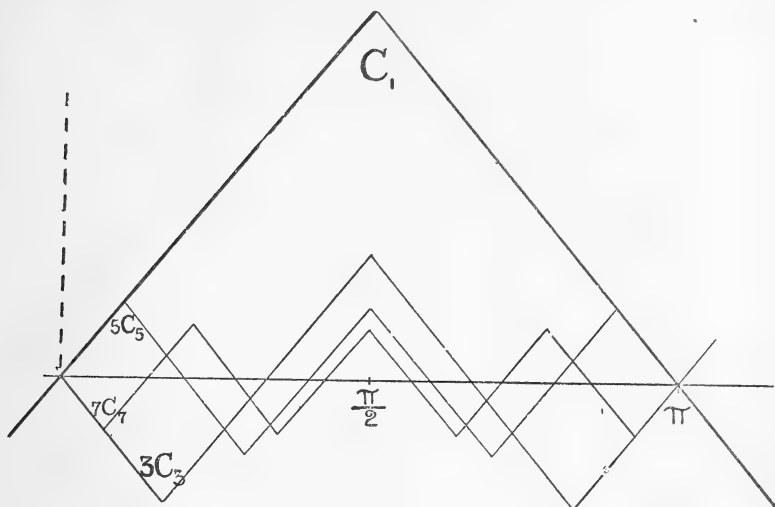
Hence, if the full wave or C_1 be represented by

$$C_1 = a_1 \sin(\omega t - \theta_1) + a_3 \sin 3(\omega t - \theta_3) + a_5 \sin 5(\omega t - \theta_5) + \&c.,$$

its third component C_3 is

$$= -\frac{1}{3^2} [a_1 \sin(3\omega t - \theta_1) + a_3 \sin 3(3\omega t - \theta_3)$$
$$+ a_5 \sin 5(3\omega t - \theta_5) + \&c.],$$

and its fifth component C_5 is

$$\frac{1}{5^2} [a_1 \sin(5\omega t - \theta_1) + a_3 \sin 3(5\omega t - \theta_3) + a_5 \sin 5(5\omega t - \theta_5) + \&c.],$$

and so on; but by definition C_3 and C_5 are also given by

$$C_3 = a_3 \sin 3(\omega t - \theta_3) + a_9 \sin 9(\omega t - \theta_9) + \dots$$
$$C_5 = a_5 \sin 5(\omega t - \theta_5) + a_{15} \sin 15(\omega t - \theta_{15}) + \dots$$

Hence, identifying the expressions for the same components, we find that

$$a_1 = -3^2 a_3 = 5^2 a_5 = -7^2 a_7 = \&c.,$$
$$\theta_1 = 3\theta_3 = 5\theta_5 = 7\theta_7 = \&c.,$$

so that

$$C_1 = a_1 \left[\sin(\omega t - \theta_1) - \frac{\sin(3\omega t - \theta_1)}{3^2} + \frac{\sin(5\omega t - \theta_1)}{5^2} - \&c. \right].$$

D 2

But $C_1 = 0$ when $\omega t = 0$, therefore $\theta_1 = 0$; and $C_1 = h$ when $\omega t = \pi/2$; therefore

$$h = a_1 [1 + 1/3^2 + 1/5^2 + 1/7^2 + \&c.] = a_1 \pi^2/8.$$

Hence $a_1 = 8h/\pi^2$, and the Fourier series required is

$$C_1 = \frac{8h}{\pi^2}\Big[\sin \omega t - \frac{\sin 3\omega t}{3^2} + \frac{\sin 5\omega t}{5^2} - \&c.\Big]$$

$$= h\mathrm{N}(\omega t) \text{ say.}$$

If the left extremity of the base were at a distance α from the origin instead of coinciding with it, then

$$C_1 = h\mathrm{N}(\omega t - a).$$

Let us call the function $h\mathrm{N}(\omega t - \alpha)$ an isosceles function, and the series of isosceles triangles which is the graph of $h\mathrm{N}(\omega t - \alpha)$ an isosceles wave specified by h its altitude, and α its phase.

11. Any wave containing only odd harmonics whose form is polygonal, with n vertices per half-wave, can be resolved into n isosceles waves of the same period, and hence can be analytically represented by a sum of n isosceles functions.

A vertex may or may not occur where the wave crosses the axis of abscissæ. In the latter case the base angles of the polygon will be equal.

For the sake of definiteness let us consider the case when the polygon has 4 vertices per half-wave, and let it be specified by m_1, m_2, m_3, m_4, m_5 ($m_5 = -m_1$), the tangents of the angles its sides, taken in the positive direction, make with the axis of x, and by the abscissæ x_{12}, x_{23}, x_{34}, x_{45}, of its vertices.

In the first place let us determine the form of the wave got by adding to the above the isosceles wave $\mathrm{I} = h\mathrm{N}(\omega t - \alpha)$ specified in the above manner by M, $-\mathrm{M}$, and X, so that $h = \mathrm{M}\pi/2$ and $\alpha + \pi/2 = \mathrm{X}$. In general the new wave will have 5 vertices, the abscissa of the one introduced being X, while the abscissæ of the others are unchanged, and if X lie say between x_{23} and x_{34}, and the tangents of the slopes of the sides of the new polygon be n_1, n_2, n_3, $n_3{}'$, n_4, $-n_1$, then, remembering that the equations of the different sides are of the form

$$y = mx + \kappa,$$

we see that

$$n_1 = m_1 + \mathrm{M}, \quad n_2 = m_2 + \mathrm{M}, \quad n_3 = m_5 + \mathrm{M},$$
$$n_3{}' = m_3 - \mathrm{M}, \quad n_4 = m_4 - \mathrm{M}.$$

Hence
$$n_1 - n_2 = m_1 - m_2,$$
$$n_2 - n_3 = m_2 - m_3,$$
$$n_3 - n_3' = 2\,\mathrm{M},$$
$$n_3' - n_4 = m_3 - m_4,$$
$$n_4 - n_5 = m_4 - m_5.$$

Thus, if we call $m_1 - m_2$ the function of the vertex x_{12}, we see that by addition of I to the given wave a new vertex is introduced whose function is equal to that of $I(=2\mathrm{M})$, while the functions of all the other vertices are unchanged. It is easy to see that the function of the vertex of $-I$ is $-2\mathrm{M}$, so that if I be subtracted from the given wave a new vertex is introduced whose function is $-2\mathrm{M}$.

If the abscissa X of the vertex of I correspond with that of one of the vertices of the given polygon x_{23}, say, then no new vertex will be introduced by the addition (or subtraction) of I, and
$$n_1 - n_2 = m_1 - m_2,$$
$$n_2 - n_3 = m_2 - m_3 + 2\mathrm{M},$$
$$n_3 - n_4 = m_3 - m_4,$$
$$n_4 - n_5 = m_4 - m_5.$$

If, in addition, $2\mathrm{M} = m_3 - m_2$, then $n_2 - n_3 = 0$ and the vertex or break at x_{23} is removed. Thus, by subtracting from the polygonal wave an isosceles wave whose vertex has the same function and abscissa as a vertex of the given polygon, this vertex of the polygonal wave is removed, while the functions of its remaining vertices are unchanged. To remove each vertex therefore a definite isosceles wave is required, and since, when all vertices are removed the axis of abscissæ or $y=0$ remains, we see that the sum of the several isosceles waves required to extinguish the given wave is equal to the latter.

In the general case, therefore, of a polygonal wave with n vertices, the vertex m_r, m_{r+1}, $x_{r, r+1}$ will be removed by subtracting the isosceles wave $I = h\mathrm{N}(\omega t - \alpha)$, where
$$m_r - m_{r+1} = 2\mathrm{M} = 4h/\pi,$$
$$x_{r, r+1} = \mathrm{X} = \alpha + \pi/2\ ;$$

and the complete wave will be fully represented by the sum of the n isosceles functions given by
$$\frac{\pi}{4} \sum_{r=1}^{r=n+1} \left\{ (m_r - m_{r+1})\,\mathrm{N}(\omega t - x_{r, r+1} + \pi/2) \right\},$$

remembering that $m_{n+1} = -m_1$.

The following are examples of the preceding method :—

12 (*a*). Wave-form a trapezium with equal base angles. This is the sum of two equal isosceles waves.

Take the left extremity of the base of the trapezium as the origin of the abscissæ, and let it be specified by

$$m_1 = m, \quad m_2 = 0, \quad m_3 = -m, \quad a_{12} = \mu, \quad a_{23} = \pi - \mu,$$

so that its altitude $t = \mu m$.

By §11 the expression for the wave is

$$\frac{\pi m}{4} \left[N(\omega t - \mu + \pi/2) + N(\omega t + \mu - \pi/2) \right]$$

$$= \frac{2m}{\pi} \left[\sin(\omega t - \mu + \pi/2) - \frac{\sin 3(\omega t - \mu + \pi/2)}{3^2} + \&c. \right.$$

$$\left. + \sin(\omega t + \mu - \pi/2) - \frac{\sin 3(\omega t + \mu - \pi/2)}{3^2} + \&c. \right]$$

$$= \frac{4t}{\mu \pi} \left[\sin \mu \, \sin \omega t + \frac{\sin 3\mu \, \sin 3\omega t}{3^2} \right.$$

$$\left. + \frac{\sin 5\mu \, \sin 5\omega t}{5^2} + \&c. \right],$$

which is Fourier's expansion for a wave of this form.

12 (*b*). Wave-form a triangle. This is the difference of two isosceles waves when the vertex of one lies on a side of the other.

Take the left extremity of the base of the triangle as the origin of abscissæ, and let it be specified by

$$m_1 = m, \quad m_2 = -n, \quad m_3 = -m, \quad a_{12} = \mu, \quad a_{23} = \pi,$$

so that its altitude $h = \mu m = (\pi - \mu) n$, and m, n are the tangents of its base angles.

By §11 its expansion in isosceles functions is

$$\frac{\pi}{4} \left[(m+n) N(\omega t - \mu + \pi/2) + (m-n) N(\omega t - \pi/2) \right]$$

$$= \frac{\pi m}{4} \left[N(\omega t - \mu + \pi/2) + N(\omega t - \pi/2) \right]$$

$$+ \frac{\pi n}{4} \left[N(\omega t - \mu + \pi/2) - N(\omega t - \pi/2) \right],$$

which is

$$= \frac{4m}{\pi} \left[\sin \frac{\mu}{2} \sin(\omega t - \mu/2) + \frac{\sin \frac{3\mu}{2} \sin 3(\omega t - \mu/2)}{3^2} + \&c. \right]$$

$$+ \frac{4n}{\pi} \left[\cos \frac{\mu}{2} \cos(\omega t - \mu/2) + \frac{\cos \frac{3\mu}{2} \cos 3(\omega t - \mu/2)}{3^2} + \&c. \right]$$

where $m\mu = n(\pi - \mu) = h$.

When $\mu = 120°$ the above expression for the triangle reduces to

$$3\sqrt{7}\,\frac{h}{\pi^2}\left\{\sin(\omega t - \beta) - \frac{\sin(5\omega t - \beta)}{5^2} + \frac{\sin(7\omega t - \beta)}{7^2} - \&c.\right\}$$

$$+ \frac{4h}{3\pi^2}\left\{\cos 3\omega t + \frac{\cos 9\omega t}{3^2} + \frac{\cos 15\omega t}{5^2} + \&c.\right\},$$

where
$$\tan\beta = \frac{1}{3\sqrt{3}}.$$

12 (c). Wave-form a polygon with n vertices per half-wave and such that the functions of its vertices are all equal and also the projections of its sides on the axis of x.

Let
$$q = m_1 - m_2 = m_2 - m_3 = m_3 - m_4 = \&c. = m_n + m_1,$$

and let the abscissæ of its vertices be

$$\alpha, \quad \alpha + \pi/n, \quad \alpha + 2\pi n, \ldots \alpha + (n-1)\pi/n\,;$$

then by § 11 the expression for the wave is

$$\frac{\pi q}{4}\,\Sigma\,.\,\mathrm{N}(\omega t + \pi/2 - [\alpha + r\,\pi/n]),$$

where r has all values from 0 to $n-1$.

Substituting for the N functions their equivalent harmonic series, summing the terms that have the same arguments and remembering that $nq = 2m_1$, the expression for the polygonal wave under consideration becomes

$$\frac{4m_1}{n\pi}\left\{\frac{1}{\sin\dfrac{\pi}{2n}}\sin(\omega t - \alpha + \pi/2n) + \frac{1}{3^2\sin\dfrac{3\pi}{2n}}\sin 3(\omega t - \alpha + \pi/2n)\right.$$

$$+ \frac{1}{5^2\sin\dfrac{5\pi}{2n}}\sin 5(\omega t - \alpha + \pi/2n) + \&c.\Bigg\}.$$

12 (d). If in example (c) n become infinite, the polygon becomes a smooth curve satisfying the following conditions :—

$$\frac{d^2y}{dx^2} = \text{const.},$$

$$\frac{dy}{dx} = m_1 \text{ when } x = 0, \text{ and } = -m_1 \text{ when } x = \pi,$$

$$y = 0 \text{ when } x = 0 \text{ and when } x = \pi.$$

This curve is the parabola

$$y = m_1 x - \frac{m_1}{\pi} x^2,$$

whose axis is $x = \pi/2$ and vertex $x = \pi/2$, $y = m_1\pi/4$; and the harmonic expression for the wave of which it is the type is obtained by making $n = \infty$ and $\alpha = 0$ in the expression in (c), and is

$$\frac{8m_1}{\pi^2} \left\{ \sin \omega t + \frac{\sin 3\omega t}{3^3} + \frac{\sin 5\omega t}{5^3} + \&c. \right\}.$$

13. To find the harmonic expression for the complete periodic function whose graph for one period is made up of the sides of two equal and similar triangles ABC and A'BC' so placed that A' and C' lie in AB and CB produced respectively. Take A as origin, then the abscissæ of B and A' will be π and 2π respectively; and let the abscissa of $C = \mu$, hence that of $C' = 2\pi - \mu$.

By geometrical construction the different components of this wave can be easily obtained if we remember formula I. § 1.

Thus, to get the 2nd component we cut the wave in four portions by ordinates at $\pi/2$, π, $3\pi/2$, 2π, invert the second and fourth portions, superpose them and the third portion on the first, add the corresponding ordinates, divide each sum by four, and the plot of the results will be a half-wave, which gives the 2nd component.

It will be found for the wave under consideration that all the components are, in general, trapeziums of the type treated in § 12(a); and if the trapezium which is the rth component be specified as in § 12(a) by m_r and μ_r measured on the original scale of abscissæ, it will be found that

$$m_r = \pm \frac{\tan A + \tan B}{2r}$$

(*i. e.*, that $2r$ times the function of its vertex is equal to $\pm$ the function of the vertex of the triangle); and that

$$\sin r\mu_r = \pm \sin r\mu,$$

the same signs being taken together.

It is to be noted that as the base of the rth component is π/r, the altitudes of its isosceles elements are each

$$= \frac{\pi}{2r} M = \frac{\pi}{4r} m_r;$$

its expression in isosceles functions is

$$\frac{\pi m_r}{4r} \left\{ \mathrm{N}r\left(\omega t - \mu_r + \frac{\pi}{2r}\right) + \mathrm{N}r\left(\omega t + \mu_r - \frac{\pi}{2r}\right) \right\};$$

and its harmonic expression is

$$\frac{4m_r}{r\pi} \left\{ \sin r\mu_r \, \sin r\omega t + \frac{\sin 3r\mu_r \, \sin 3r\omega t}{3^2} + \&c. \right\}.$$

Hence the rth component

$$C_r = \frac{2(\tan \mathrm{A} + \tan \mathrm{B})}{\pi r^2} \left\{ \sin r\mu \, \sin r\omega t + \frac{\sin 3r\mu \, \sin 3r\omega t}{3^2} \right.$$
$$\left. + \frac{\sin 5r\mu \, \sin 5r\omega t}{5^2} + \&c. \right\}.$$

If h be the altitude of either of the given triangles, then

$$h = \mu \tan \mathrm{A} = (\pi - \mu) \tan \mathrm{B},$$

and the development for the complete wave is

$$f(t) = \frac{2h}{\mu(\pi - \mu)} \left\{ \sin \mu \, \sin \omega t + \frac{\sin 2\mu \, \sin 2\omega t}{2^2} \right.$$
$$\left. + \frac{\sin 3\mu \, \sin 3\omega t}{3^2} + \&c. \right\}.$$

III. *On the Faraday and Kerr Effects in the Infra-red Spectrum. By* L. R. INGERSOLL [*].

Introduction.

THE aim of the present work is a study of electromagnetic rotatory dispersion, particularly in the infra-red spectrum. Because of the important bearing of the phenomenon of magnetic rotation of the plane of polarization of light in the field of electro-optics, the subject has been investigated in all its various aspects by many observers during the last half-century, and the effects of the different factors which govern the rotation, such as strength of magnetic field, angle of incidence, and temperature, carefully determined. The most important factor of all, however, if one considers the place it must have in any explanation of the phenomenon—the dependence of the magnetic rotation on wave-length, or magnetic rotatory dispersion—has been studied only over a very limited range of spectrum. Thus while the rotatory dispersion in

the visible spectrum may be considered as accurately known for many typical substances, measurements in the ultra-violet and infra-red have been scanty, and, especially in the latter, the broader and more important field, far from satisfactory.

As a reason for this lack of attention to the infra-red, it may be urged that the problem of measuring rotations in this part of the spectrum is one of rather peculiar difficulty, necessitating as it does magnetic and electrical complications in spectrobolometric work which requires the highest order of sensibility. But even if an accuracy equal to that of visual measurements cannot be expected, the extension of the work into this field is nevertheless a matter to which considerable interest attaches ; for it is there that evidence must be sought in support of rotatory dispersion formulæ, which cannot as yet be considered as generally established, because of the narrow range of spectrum only over which they have been tested. The effect of absorption-bands on rotation must also be considered in this region, and whether or not their action may be accounted for by a type of dispersion formula analogous to the Ketteler-Helmholtz. It is in the infra-red, too, that one must seek for an explanation of the anomalous character of the rotatory dispersion shown by magnetic metals, and question whether the rotation continues to increase with increasing wave-length throughout the infra-red spectrum, and therefore perhaps be very considerable for electromagnetic waves metres long.

It was in the hope of throwing a little more light on these and similar questions, that the present work was attempted. The two distinct cases which arise will be treated separately. They are (1) Faraday rotation, or rotation suffered by the plane of polarization of light transmitted through some substance in a magnetic field; and (2) the Kerr effect, or rotation due to reflexion at the polished pole-face of a magnet, or at a paramagnetic mirror in a magnetic field.

PART I.

FARADAY ROTATION.

That radiant heat as well as light may show Faraday rotation was first proved by Wartmann [*], and measurements of the rotation for a number of different substances were later made by De la Provostaye and Desains [†] and by

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[*] *Compt. Rend.* xxii. p. 745 (1846).
[†] *Ann. Chim. Phys.* [3] xxvii. p. 232 (1849).

L. Grunmach *. None of these experimenters, however, made any attempt to obtain radiations of uniform and known wave-length for their work; and G. Moreau † seems to be the only one who has heretofore done this and so obtained measurements of magnetic rotatory dispersion in the infra-red. He determined the rotation in a tube of carbon bisulphide, placed in a powerful magnetizing solenoid, for wave-lengths ranging from 0.79μ to 1.42μ. A beam of sunlight was passed in turn through a polarizer, carbon-bisulphide tube, and analyser, and its intensity after dispersion by a prism of flint glass was measured by a thermopile placed in the spectrum. The method of observation, in brief, consisted in determining the azimuth of the plane of polarization of the light emerging from the carbon bisulphide by two observations of the intensity of radiation for two positions of the analyser $90°$ different in azimuth. By repeating this when the current was passing in the solenoid, the rotation due to the magnetic field could readily be calculated. Measurements by ordinary polarimetric means allowed the calculation of the field-strengths by the use of known data, and the effect of this field at the galvanometer was annulled by placing near by a small secondary spool in shunt with the main solenoid. He found rotations amounting in general to several degrees, being at $\lambda = 0.79 \mu$ equal to 52 per cent. of the rotation for sodium light, and at $\lambda = 1.42 \mu$, 32 per cent., with an estimated maximum error averaging about 10 per cent.

In the present work the magnetic rotation has been measured for thirty wave-lengths between $\lambda = 0.58 \mu$ and 4.3μ; carbon bisulphide, the substance generally used in magnetic polarimetry, being chosen for the test, because of its high rotatory power, combined with transparency in the infra-red. A powerful electromagnet was used, instead of a solenoid, to furnish the magnetic field; for the gain in field-strength allowed the use of a shorter carbon-bisulphide tube without sensible loss of rotation, and with the great added advantage of less absorption of radiation in the infra-red. To simplify the problem, all other conditions upon which the rotation depends, such as field-strength, length of carbon-bisulphide tube, and temperature,—save the wave-length of the light used,—were kept as nearly as possible the same throughout all the measurements, and hence need not be accurately known, since, being constant, they could in no way affect the form of the dispersion-curve, the determination of which was the desired end of the experiment.

* Wied. *Ann.* xiv. p. 85 (1881).
† *Ann. Chim. Phys.* [7] i. p. 227 (1894).

Dispersion Formulæ.

It is not the intention to enter into a discussion of the various theories of magneto-optic phenomena which have been proposed by Maxwell, Lorentz, Goldhammer, Drude, and others, but rather to test, over a much wider range of spectrum than has heretofore been possible, the various rotatory-dispersion formulæ deduced on the basis of each, and thus perhaps to furnish more evidence in favour of some one of them, or a modification.

Among such formulæ may be mentioned that due to Maxwell and formulated by Moreau [*],

$$\delta = \frac{n^2}{\lambda^2} \left\{ n - \lambda \frac{dn}{d\lambda} \right\} \left(k + k' \frac{n^2}{\lambda^2} \right),$$

and the very similar one of Van Schaik [†],

$$\delta = An^2 \left\{ n - \lambda \frac{dn}{d\lambda} \right\} \left(\frac{C_1}{\lambda^2} + \frac{C_2}{\lambda^4} \right),$$

also that due to Joubin [‡],

$$\delta = \frac{C}{\lambda} \left\{ n - C'\lambda \frac{dn}{d\lambda} \right\}.$$

To these must be added the formulæ given by Drude [§],

$$\delta = n \left\{ \frac{a}{\lambda^2} + \frac{b}{\lambda^2 - \lambda_1^2} \right\},$$

deduced on the hypothesis of molecular currents ; and

$$\delta = \frac{1}{n} \left\{ \frac{a'}{\lambda^2} + \frac{b'\lambda^2}{(\lambda^2 - \lambda_1^2)^2} \right\},$$

deduced on the hypothesis of the Hall effect, λ_1 being in each case the wave-length of an ultra-violet absorption-band.

With the data hitherto obtained, it is impossible to decide definitely in favour of any one of these, for over the limited range of spectrum over which they have been satisfactorily tested—barely half a micron—it is possible, by properly choosing the constants, to make any one of them fit the facts sufficiently well. Moreau's work in the infra-red points in a general way to the formula of Joubin, although this is hardly as satisfactory as the others in the visible; but, as will be seen later, the points he obtained do not lie at all on a regular curve, and it is possible that his experimental error is larger than estimated.

* *Ann. Chim. Phys.* [7] i. p. 341 (1894).
† *Arch. Néer.* xvii. (1882) and xxi. (1886).
‡ *Ann. Chim. Phys.* [6] xvi. p. 78 (1889).
§ *Lehrbuch der Optik.* pp. 396 & 403.

Method.

The method adopted has some points in common both with the one due to De la Provostaye and Desains * and that used by Moreau *. A beam of light after passing in turn through a polarizer, rotating substance between the poles of an electromagnet, and analyser, was dispersed and formed into a spectrum in which was placed the strip of a bolometer. Any rotation of the plane of polarization such as would be caused by exciting the magnet, would then cause an increase or diminution of intensity of the beam transmitted by the analyser, which effect could be measured for any desired wave-length by the bolometer and the rotation calculated accordingly.

If the principal planes of the polarizer and analyser make an angle α, the intensity at any chosen wave-length as measured by the bolometer may be represented by

$$I = kI' \cos^2\alpha,$$

where I' is the intensity of this radiation incident on the analyser, and k takes account of absorption. Then any small rotation δ of the plane of polarization, produced either by turning the polarizer, or by the action of the magnetic field, will produce a change in I, disregarding signs,

$$dI = 2\delta kI' \sin\alpha \cos\alpha,$$

or, dividing by I and solving for δ,

$$\delta = \tfrac{1}{2}\frac{dI}{I}\cot\alpha \text{ radians, or } \frac{90}{\pi}\frac{dI}{I}\cot\alpha \text{ degrees.}$$

As pointed out by De la Provostaye and Desains, dI will evidently be greatest for $\alpha = 45°$, which is accordingly the angle chosen. For any rotation the value of δ may be obtained, if $\alpha = 45°$, from

$$dI = kI'\{\cos^2(45° \pm \delta) - \cos^2 45°\} = \mp\tfrac{1}{2}\sin 2\delta \,.\, kI',$$

or

$$\delta = \tfrac{1}{2}\sin^{-1}\frac{dI}{I}.$$

In this way the magnetic rotation for any chosen wave-length may be determined by a measurement of the intensity I of that wave-length in the spectrum, and of the change in intensity dI when the magnet is excited, or better, when it is reversed, since this gives a doubled effect. Both I and dI are determined as galvanometer-deflexions, and from their

* *Loc. cit.*

ratio the rotation is at once calculated. This method as so far described has about the same possibilities of sensibility as that of Moreau,—with the advantage, however, that the rotation is obtained almost directly, instead of as the difference between two large angles ; but its ability to measure small rotations may be greatly increased by the following simple scheme. Suppose it is possible, after measuring the intensity I, to increase the bolometer sensibility in a known ratio, say twenty times, and at the same time to shift the zero of the galvanometer so that the reading, which the radiation I in connexion with the greatly increased sensibility would otherwise throw far off the scale, may still be brought within the field of vision. In this way, the change of intensity dI, which is in general only a small fraction of I, may be effectively magnified and made to produce a galvanometer deflexion of the same order as the latter, with a corresponding increase in the possible accuracy of measurement, for the change of sensibility may be readily effected by cutting out resistance in series with the galvanometer, and the shift of the zero by simply adjusting the balance of the bolometer.

Description of Apparatus.

Before describing the apparatus in detail, certain special requirements of the problem should be noticed. Since rotation appears as a change of perhaps only a few per cent. in the energy at any wave-length, it is evident that for accurate measurement a very intense spectrum is demanded. This must be formed moreover of light which has been polarized, transmitted down the axis of the magnet, analysed, and which has suffered a number of other losses by reflexion and absorption ; hence the necessity for a very brilliant source. It is clear, too, that this source must be exceptionally constant, for changes of intensity might be falsely interpreted as rotations.

These requirements have been met by the use of a Nernst glower, with special protective covering, as a source. The formation of a parallel beam, as is generally used in polarimetry, which should be of sufficient intensity, and of small enough cross-section to suit the necessarily small apertures in the poles of the magnet, was found impracticable : a conical beam was therefore used, which converged to a focus at the only points where small aperture was demanded,—between the poles of the magnet and on the slit of the spectrometer. The latter instrument had short-focus mirrors, and was designed to give as intense a spectrum as possible without

the sacrifice of purity to the extent of causing a dispropor-
tionately large error in the measurement of wave-length.
As to the actual intensity of the spectrum, it may be stated
that with the arrangement used, galvanometer-deflexions were
not uncommon, measured with reduced sensibility, which
would correspond, on the basis of full sensibility, to 1500
centimetres on the scale.

The general arrangement of apparatus may be seen from fig. 1
(p. 48). Light from the Nernst glower G_1 was polarized by
reflexion at the pile of thin glass plates P, and after reflexion
at M_1 formed a real image of the glower between the poles
of the magnet, and again on the slit, after reflexion at M_2
and from the pile of analyser-plates A. This was drawn out
into a spectrum at the bolometer-strip B by a 60° rock-salt
prism. The latter had the customary Wadsworth minimum
deviation mounting, the plane mirror being a Brashear flat
4×10 cm. The prism itself had faces 4×5 cm., polished by
Brashear, and was protected by an enclosure, not shown in
the diagram, mounted on the spectrometer-table and con-
taining sulphuric acid. The table was rotated by a micro-
meter-screw connected with a steel strap pulling tangentially
over a lever arm. One division of the micrometer-head
corresponded to a rotation of 0·06 minute of arc. The
mirrors M_1, M_2, M_3 were each 40 cms. in local length, while
M_4 was 30 cms. They were mounted, together with the
other parts of the spectrometer, on heavy slabs of slate.

The broken lines represent various adjustments which, by
means of cords, could be conveniently made by the observer
without moving from his position in front of the galvano-
meter scale. (1) Could be used to rotate the polarizer ;
(2) allowed the carbon-bisulphide tube to be thrown in or
out of the path of the beam ; (3) operated a shutter in front
of the slit ; (4) rotated the mirror M_2 and hence shifted the
image of the glower on the slit.

Neither bolometer nor galvanometer present any especial
novelty of construction, save that the latter was made as
compact as possible to allow effectual shielding. It was of
the type described by Mendenhall and Waidner [*], and had
four coils of twenty ohms each, measuring 16 mms. outside
and 2 mms. inside diameter. The system, which had of course
quartz-fibre suspension, weighed something less than 2 mg.
and had a mirror 1 mm. square. Deflexions of the image
of a fine wire in front of a Nernst glower G_2 were observed
on a scale a metre and a half distant. Fairly good magnetic
shielding was secured by surrounding it with six concentric

* Amer. Journ. Sci. xii. p. 249 (Oct. 1901).

Fig. 1.—Arrangement of Apparatus.
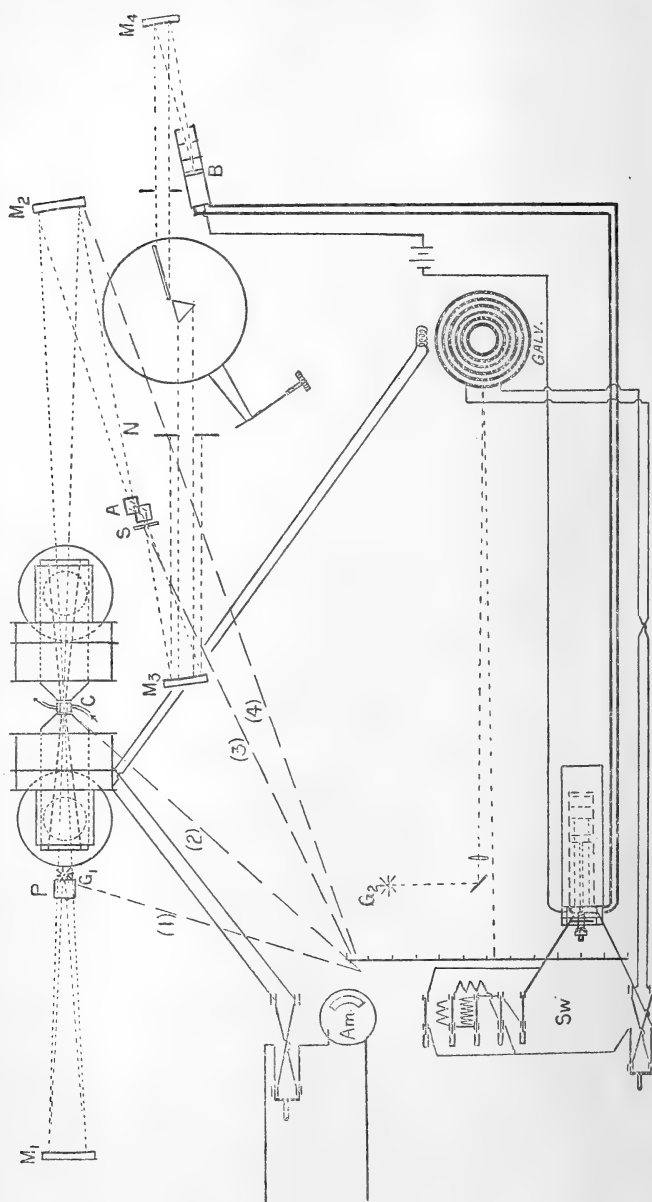
sections of annealed steel pipe. This arrangement could be made to give a sensibility of 4×10^{-11} for 5 ohms resistance, but considerations of steadiness and of proportionality of deflexions led to reducing this to about 2×10^{-10} amp. per mm. with 10 sec. period, for ordinary usage.

The bolometer used in most of the work had strips $\frac{1}{2}$ mm. wide and 8 mms. long. Heavy copper leads connected it with the balancing-bridge, and switches, Sw, which permitted reversal of galvanometer connexions and also changes of sensibility in the approximate ratios—3·5, 9, and 28 times respectively. At highest sensibility a candle at a metre's distance gave a deflexion corresponding to 200 centimetres on the galvanometer-scale.

The magnet was designed to give the greatest field-strength over ranges of several centimetres, with the smallest external field possible. The pole-pieces were of Swedish iron 9 cms. in diameter, and were bored with conical holes narrowing from 5 cms. diam. at the outside ends, to a slit 5 mms. wide and 16 mms. high at the conical ends. The advantages of a rectangular opening over a circular one are obvious, considering the shape of the glower used as a source. When excited with about 100,000 ampere-turns, it gave fields as high as 26,000 lines per cm. over an area of a few square millimetres, using pointed pole-pieces. It will be noted that this is not as great as the fields which may be obtained with the du Bois ring type of magnet ; but it may be remarked that in the present work the air-gap was never in practice much less than 3 cms., with, of course, a necessarily great loss of field-strength : therefore, since the ring type depends for its powerful fields, not on an excessive number of ampere-turns, but on the low reluctance of its magnetic circuit, it is doubtful whether its performance under the same circumstances would excel that of the present type. Moreover, the varied adjustments of which it must admit would be impossible in a magnet of the ring type,—at least without very great modification.

The external field was small and its effect at the galvanometer was compensated for by a secondary coil of high resistance, placed near the latter instrument and connected in shunt with one of the main magnet-coils. When run at full power there was necessarily considerable heating about the magnet, and to protect the neighbouring apparatus from this, the whole magnet was enclosed in a large wooden box (not shown in diagram) whose sides were water-jacketed.

Analyser and polarizer were each made up of a number of plates of microscopic cover-glass 3·5 × 5 cms. and 0·2 mm.

thick. Reflexion from thin glass plates, while by no means free from objection, seemed to be the only practical scheme of obtaining polarized radiations for this work ; for the use of a polarizing grating would involve too much loss of energy, and the absorption of any doubly-refracting substance makes its use prohibitive beyond about $\lambda = 2\cdot5\ \mu$. The polarizer carried a divided circle and was so mounted that it could be rotated about the axis of the beam. The glower was arranged to rotate with the polarizer, and at the same time to turn about an axis of its own, so as to keep its image on the slit always vertical. The analyser also had a number of possible adjustments, and carried with and parallel to it a plane silver surface, by reflexion at which, after passing the polarizer, the beam would be brought out parallel to its original direction.

Because of its rather exceptional performance as regards brilliancy and constancy, the glower and its mounting are worthy of special mention. When unprotected and in the open air, a Nernst glower may suffer variations of intensity of upwards of one or two per cent., even when supplied with a perfectly constant E.M.F. This is doubtless due to irregularities of convection ; and at the suggestion of Prof. Mendenhall it was tried inside an enclosure made of plaster-of-Paris or cement. This gave a much greater constancy, and as the objections to the plaster enclosure, which was soon destroyed by the heat, have since been removed by making it of firebrick, it leaves little to be desired as a source for infra-red work. Specimen curves have been obtained, using 96-volt direct-current glowers made by the Nernst Lamp Company of Pittsburg, and furnishing the current with a storage battery, which show a maximum variation of intensity of less than $\frac{1}{10}$ per cent. for several minutes at a time.

Some of the special adjustments may be mentioned. The analyser and polarizer piles of plates were set with the aid of a nicol at the angle of incidence which gave most completely polarized reflected light. It might be supposed that this setting would have to be changed to suit the different wave-lengths used; but as a matter of fact, as will be pointed out in a discussion of the sources of error, it was not a sensitive adjustment, and might be made once for all with sufficient accuracy for all wave-lengths within the range over which it was possible to work. The angle between the principal planes of the polarizer and analyser was determined by placing a large nicol with divided circle at N, and setting so as to extinguish, first, by crossing with the principal plane of the polarizer, and, second, with that of the analyser. In

general this angle was made 45°. When ready to take a set of readings, the small compensating-coil was adjusted in position till the galvanometer-zero was unchanged by the action of the magnet.

Because of the greater freedom from magnetic as well as mechanical vibrations, the observations were all made at night. With the spectrometer set at the desired wave-length, the total energy I was first measured with reduced sensibility. Then, with the slit still open, the sensibility was increased step by step, the bolometer at the same time being balanced to keep the reading on the scale. When the magnet was excited the galvanometer showed a deflexion, indicating either an increase or decrease of intensity dI. The magnet-current was then reversed, the new deflexion observed, and the operation repeated so as to give usually five readings in either direction. The galvanometer connexions were then reversed, and the process repeated; for in this way, it will be observed, any small outstanding effect due to the direct action of the magnet on the galvanometer would be eliminated, while at the same time a second set of readings was obtained which served to form a very good check on the first. An occasional set of readings with the carbon-bisulphide cell removed served to show if any other cause was also acting to produce rotation.

The mean difference of the galvanometer readings when the magnet-field was direct, and when it was reversed, gave $2\,d$I, and the first measurement of the total intensity gave I, and their ratio, on the basis of the same sensibility, determined the angle of rotation. The method of computing the mean dI was mainly graphical and need not be described in detail, but it will serve to remark that it was designed to render as ineffective as possible errors due to bolometer drift and variations of the source, as well as to eliminate completely any chance of personal bias as to any one or more observations.

Sources of Error.

It is to be particularly noted at the outset, that only those errors are of first importance which are selective in action, and hence affect the determinations for different wave-lengths unequally. For, while it is of course desirable to know the absolute magnitudes of the rotations, for convenience in checking the results by other methods, if the error can be shown to be a constant one which applies equally to all wave-lengths it can have no effect on the correctness of the results aimed at, *i. e.* the form of the rotatory dispersion

curve, and hence need be only approximately determined. This point will be mentioned again in connexion with certain of the errors to be considered.

Errors may otherwise be conveniently grouped into two general classes : those incidental to the measurement of the ratio dI/I, and those occurring in the interpretation of this result as rotation for radiations of a definite wave-length. Those of the first class will in the main be somewhat compensating, and their effect may be best observed by plotting from the above table dI/I with wave-length, and noting how closely the points follow a regular curve. Those of the second class may be systematic, and their effect is more difficult to estimate directly.

As sources of errors of the first kind may be mentioned :—

> Variations in brilliancy of the Nernst glower.
> Disturbing effect of magnet on galvanometer.
> Bolometer unsteadiness and drift.
> Changes of temperature due to heating of magnet.
> Inaccurate determination of the sensibility reduction ratios.

Considering them briefly in the order given,—the variations of the glower, as explained above, were small, and their effect was eliminated by taking the average of several deflexions in each direction, in rapid succession. The graphical method of computing this average was especially useful in enabling one to detect and allow for such accidental variations.

It was impossible to eliminate entirely the effect of the magnet on the galvanometer even with the most careful shielding and with the aid of the small compensating solenoid, because of the unequal heating effects in the different coils. However, by frequently adjusting this solenoid, by altering its distance from the galvanometer, the effect might be made small ; and by reversing the galvanometer connexions and averaging the two sets of readings, it was practically eliminated.

The matter of bolometer drift was rather troublesome, and would seem to be due partly to the high sensibility which was demanded of the instrument, and partly to the fact that the method of observation necessitated frequent changes of balance. However, the effect of drift was far less disturbing in this than in most spectrobolometric problems, and, like errors due to the variations of the glower, it was largely eliminated from a series of readings by taking the graphical mean.

Heating of the magnet doubtless gave rise to slight changes

of field-strength, although determinations with a small test-coil showed the effect to be small. But any possibility of this error being selective was removed by so choosing the succession of wave-lengths to be tested, that the rotation for the same or neighbouring parts of the spectrum might be measured at different times under very different heating conditions. The temperature of the carbon bisulphide was kept constant to within one or two degrees by a water-jacket around the containing tube, hence the error from this source would be less than one per cent.

Sensibility ratios were determined by measurements of the same intensity of radiation with the two sensibilities, the ratio of which was desired. Although the discrepancy between successive determinations made in this manner was sometimes as much as three or four per cent., so many tests were made that the probable error of the accepted values was very much smaller.

Judging from the agreement in different results, for the same or neighbouring wave-lengths, the cumulative effect of the errors of this class is to produce an uncertainty of probably less than two or three per cent. over the greater range of spectrum studied, although at the ends it is considerably increased. To be certain that this rotation was not in part due to some other cause than the Faraday effect in the carbon-bisulphide, the cell containing the latter was removed and a number of tests made, which, as may be seen from the table of results, showed that such spurious rotation was too small to be definitely detected. The rotation due to the thin glass ends of the cell was entirely negligible, being only about one-fifth of one per cent. of the other.

As causes of the second sort of errors may be mentioned:—

Inability of the pile of plates to polarize completely.
The use of a non-parallel beam of light.
Impurity of spectrum due to stray light and to the use of a broad bolometer strip and slit.

Regarding the first source of error, it is evident that the light would be incident on the polarizer-plates over quite a range of angle, because of the appreciable size, as well as nearness of the glower, which was only about 10 cms. distant; hence complete polarization could not be expected in the reflected light. Then, since the formula used to reduce the quantity dI/I to a rotation through a certain angle was deduced on the supposition of completely polarized light, the results obtained with it must be subject to a correction which will now be calculated.

Let I' represent the intensity, for any chosen wave-length, of the beam of natural light incident on the polarizer. Then the reflected beam will be made up of two parts,—

$\frac{1}{2} K_1 I'$ (polarized in the plane of incidence);

and

$\frac{1}{2} a K_1 I'$ (polarized perpendicular to the plane of incidence),

where K_1 is a factor which takes account of the percentage of light reflected by the surfaces, and a is a constant to be determined later.

On emerging from the analyser, whose plane of incidence makes an angle α with that of the polarizer, the intensity is

$$I = \tfrac{1}{2} K_1 K_2 I' \cos^2 \alpha + \tfrac{1}{2} K_1 K_2 I' b \sin^2 \alpha$$
$$+ \tfrac{1}{2} K_1 K_2 I' a \sin^2 \alpha + \tfrac{1}{2} K_1 K_2 I' a b \cos^2 \alpha,$$

where b and K_2 are constants for the analyser corresponding to a and K_1 for the polarizer. In general a and b will be small and their product may be neglected; hence, dropping the last term and collecting,

$$I = \tfrac{1}{2} K_1 K_2 I' \{ \cos^2 \alpha + (a+b) \sin^2 \alpha \}.$$

A small rotation δ will cause a change,

$$dI = -\delta K_1 K_2 I' (1 - a - b) \sin \alpha \cos \alpha.$$

Solving,

$$\delta = -\frac{dI}{I} \frac{\cos^2 \alpha + (a+b) \sin^2 \alpha}{(1-a-b) \sin 2\alpha},$$

or, if $\alpha = 45°$,

$$\delta = -\tfrac{1}{2} \frac{dI}{I} \frac{1+a+b}{1-a-b}.$$

The constants a and b are seen to represent the ratios of intensities of the light polarized perpendicular to and in the plane of incidence, for both polarizer and analyser. They may be determined with the aid of a nicol or other completely polarizing agent placed at N, fig. 1. If its principal plane is turned at right angles to that of the polarizer, the intensity of the light on emerging from the analyser will be

$$I_1 = \tfrac{1}{2} a K_1 K_2 K_3 I' \sin^2 \alpha + \tfrac{1}{2} ab K_1 K_2 K_3 I' \cos^2 \alpha,$$

where K_3 allows for absorption in the nicol.

If crossed with the analyser instead it will be

$$I_2 = \tfrac{1}{2} b K_1 K_2 K_3 I' \sin^2 \alpha + \tfrac{1}{2} ab K_1 K_2 K_3 I' \cos^2 \alpha.$$

When set with principal plane parallel to that of polarizer or analyser, respectively, the intensity will be given by

$$I_3 = \tfrac{1}{2} K_1 K_2 K_3 I' (\cos^2 \alpha + b \sin^2 \alpha),$$

or

$$I_4 = \tfrac{1}{2} K_1 K_2 K_3 I' (\cos^2 \alpha + a \sin^2 \alpha).$$

Making $\alpha = 45°$, and dropping the second terms of the four equations, as they may be neglected with respect to the first, a and b become on solving,

$$a = I_1/I_3 = I_1/I_4, \quad b = I_2/I_3 = I_2/I_4,$$

and hence may be readily measured as the ratio of two intensities.

It might reasonably be expected, since the plates were set at the same angle for all wave-lengths, that a and b would depend largely on the wave-length tested. Such, however, has not been found to be the case, at least to any appreciable extent, over the range of spectrum for which a nicol was transparent enough to be used in measuring them, say as far as $\lambda = 2 \cdot 5 \ \mu$. Then, since from there to $\lambda = 4 \cdot 3 \ \mu$ the variation in the refractive index of glass, and hence of the polarizing angle, is small, a and b could be regarded as constant, within the limits of accuracy required in the work, over the whole range of spectrum for which it was possible to measure rotations. This was undoubtedly because the variation of the angle of complete polarization with wave-length was much smaller than the unavoidable variation of the angle of incidence over the surface of the plates. The tests gave the value of a as $\cdot 025$ and b as $\cdot 035$, making a correction of $12 \cdot 8$ per cent. to be applied to the rotations as determined by the expressions $\tfrac{1}{2} dI/I$ or $\tfrac{1}{2} \sin^{-1} dI/I$; for while the correction has been calculated only for small angles of rotation, it may be applied to any rotation actually measured, with sufficient accuracy. With regard to the magnitude of the correction, it may be noted too that, so long as it applies equally to all wave-lengths, its actual value is a matter of secondary importance.

The use of non-parallel light resulted in the edges of the beam being polarized in slightly different planes from that of the centre. This would cause an effective variation of α for different parts of the beam, but such small uncertainty would be less than that of the determination of α itself. Fortunately such errors, as well as those incidental to the setting of the polarizer and analyser, were also of secondary importance, being non-selective in character. To be free from elliptical polarization which might result from reflexion at the mirrors

M_1 and M_2, the polarizer was in general set so that its plane of incidence was parallel and perpendicular respectively to the planes of incidence on these two mirrors. It should be noted that the elliptical polarization which would inevitably be introduced by the oblique reflexion at the silver surface, immediately after passing the analyser, as well as the polarization which would occur at the faces of the prism, could have no disturbing effect, since after the radiation had once passed the analyser its state of polarization was of no moment.

Purity of the spectrum was tested by double dispersion, and although, as might be expected, considerable impurity due to overlapping slit-images was found, only a trace of stray radiation could be discovered. To take account of the error due to impurity, correction was made for the width of slit and bolometer-strip. The ·5 mm. strip subtended an angle of 5′, corresponding to from ·02 μ to ·4 μ in the rock-salt spectrum. The slit was even wider as a rule, and it might be thought that such an arrangement, while necessary to secure sufficient energy for the measurements, would involve such a large correction as would seriously prejudice the accuracy of the work. But, while by no means negligible, the corrections in the present case prove to be much smaller than in most other spectrobolometric problems involving a like amount of impurity, because they are of a differential nature; for the rotations were determined, not by a single measurement of energy at any point of the spectrum, but by the ratio of the two quantities $d\mathrm{I}/\mathrm{I}$, and hence only the difference of the separate corrections to $d\mathrm{I}$ and I need be applied. These separate corrections were determined for each of the curves of $d\mathrm{I}$ and I plotted with wave-length (the curve of $d\mathrm{I}$, while not an energy-curve in the ordinary sense, is exactly analagous to one and may be treated in a similar manner) by the simple method of correction for slit-width due to Lord Rayleigh, and outlined in Preston's 'Heat,' p. 606 (last edition). The difference, or resultant correction, varied from 2 per cent. at $\lambda = 1\,\mu$ to -4 per cent. at $3\,\mu$. That the correction should be small was indicated by measurements of rotation for the same wave-length, with different slit-widths, which gave results in good agreement.

Results.

Table 1. is a synopsis of the principal results. Rotations are given for forty-two points of the spectrum (including duplicates), as produced by carbon bisulphide in a tube

TABLE I.

Series	Wave-length	I.	$2\,d\mathrm{I}_1$.	$2\,d\mathrm{I}_2$.	Mean $2\,d\mathrm{I}$.	$\dfrac{2\,d\mathrm{I}}{\mathrm{I}}$.	$\delta = \frac{1}{2}\sin^{-1}\dfrac{d\mathrm{I}}{\mathrm{I}}$.	Corrected Rotation.
	μ	cm.	cm.	cm.	cm.		°	c
a	0·58	9·1	8·7	9·0	8·85	·972	14·51	* 16·50
,, ...	0·69	47·4	33·6	33·6	33·6	·710	10·40	* 11·82
,, ...	0·78	34·8	16·5	17·3	16·9	·486	7·03	* 7·98
,, ...	0·81	180	80·0	84·2	82·1	·456	6·58	* 7·49
,, ...	1·02	172	53·6	53·8	53 7	·312	4·48	5·12
,, ...	1·24	669	152	146	149	·223	3·12	3·58
,, ...	1·62	1147	155	154	154·5	·135	1·93	2·20
,, ...	,,	635	88·0	87·5	87·75	·136	1·95	2·22
,, ...	2·20	1065	77·7	79·8	78·75	·074	1·07	1·19
,, ...	2·37	543	35·1	34·6	34·85	·064	·92	1·02
,, ...	2·50	744	43·5	42·5	43·0	·058	·83	·91
,, ...	2·80	332	17·5	16·3	16·9	·051	·72	·79
,, ...	3·00	109·5	5·1	5·3	5·2	·048	·69	·75
,, ...	3·15	128·8	6·0	5·2	5 6	·044	·63	·69
.. ..	3·30	103·1	5·1	4·5	4·8	·046	·67	·72
,, ...	3·50	58·5	2·2	2·9	2·55	·044	·63	·68
,, ...	3·64	47·0	1·5	1·8	1 65	·034	·48	·53
,, ...	3·90	52·7	1·0	2·2	1·6	·030	·43	·47
b	3·00	341	15·4	16·1	15·75	·046	·66	·72
,, ...	3·31	144	5·9	6·1	6·0	·042	·60	·65
,, ...	3·50	99	4·7	3·6	4·15	·042	·60	·65
,, ...	3·64	77·8	3·2	3·3	3·25	·041	·58	·63
,, ...	3·77	82·7	2·8	3·0	2·9	·035	·50	·54
,, ...	3·90	86·0	2·9	3·0	2·95	·034	·49	·53
,, ...	4·03	76·5	3·4	3·2	3·3	·043	·61	·67
,, ...	4·16	45·1	1·6	2·0	1·8	·040	·57	·63
,, ...	4·31	32·1	1·3	1·7	1·5	·047	·68	·75
c	0·63	11·8	10·8	11·5	11·15	·949	14·16	16·10
,, ...	0·70	32·9	24·4	24·2	24 3	·738	10·83	12·33
,, ...	0·84	158	79·2	79·7	79·45	·503	7·27	8·25
,, ...	1·10	618	175	176	175·5	·284	4·08	4·65
.. ..	1·40	1140	213	208	210·5	·184	2·64	3·00
,, ...	1·64	1465	193	190	191·5	·130	1·87	2·12
.. ..	1·80	1490	157	155	156	·105	1·51	1·79
.. ..	2·10	1370	112·5	112·5	112·5	·082	1·17	1·30
,, ...	2·40	1180	77·4	74·7	76 05	·064	·92	1·01
.. ..	2·68	618	34·0	33·8	33·9	·055	·78	·85
,, ...	3·00	228	10·6	10·3	10·45	·046	·66	·72
.. ..	3·30	98·3	4·7	4·9	4·8	·049	·70	·76
,, ...	,,	98·3	5·2	5·0	5·1	·052	·74	·81
,, ...	3·65	72·8	3·2	3·2	3·2	·044	·63	·69
.. ..	4·03	54·4	2·0	2·4	2·2	·040	·58	·64
Carbon bisul-phide cell removed.	0·78	37	−·5	...	−·5	−·013		
	1·02	270	·6	·9	·75	·003		
	2·37	660	·7	...	·7	·001		
	3·00	165	−·1	00	−·05	−·0003		
	3·30	200	·1	00	·05	·0001		
	4·16	75	−·2	·2	00	00		

* Probably several per cent. too low, for reasons which will be mentioned.

4·125 cms. long and at a temperature of 29° C. The field-strength was approximately 6000 C.G.S. units, and was given by a magnet-current of 18·7 amp. The columns marked $2 d I_1$ and $2 d I_2$ give the means of the several galvanometer-deflexions as explained above, the second column being made with the galvanometer-connexions reversed. All deflexions are stated in terms of the highest sensibility, and have galvanometer-proportionality corrections applied in all cases; but because of a certain amount of unsteadiness which is unavoidable—due mainly to small variations of the source—readings are given only to the nearest millimetre, and in some cases—where a large reading is obtained from one which is actually very much smaller, by reducing to a constant sensibility as a basis—only to the nearest centimetre. The considerable differences which are seen to occur, in the values of I in separate determinations on the same wave-length, are due to different slit-widths, or to different brilliancies of the glower on the two occasions.

The last column has been obtained from the previous one by the application of the correction factors previously deduced, necessitated by the existence of partially polarized light and by impurity of the spectrum. These results have been plotted in fig. 2, the circles representing the points of series a; the crosses, those of b; and the triangles, those of c. The small circles connected by dotted lines represent the determinations of Moreau, plotted, for comparison's sake, to the same scale, so that the rotation at $\lambda = ·79 \mu$ shall be the same in each case.

From the nature of the problem it is impossible to do more than estimate the probable error of the results, but, as previously mentioned, it is believed that the best idea of it may be gained by noting the variation of the points from the mean curve. For, since the points were all determined independently and with many different adjustments and changes of slit-width, &c., it is difficult to see how any systematic error could remain undetected, save perhaps one which applied equally to all wave-lengths, and this would only alter the scale of the diagram. The probable error of some points is much less than of others. Thus, in making measurements on the shorter wave-lengths of series c, a water-cell was interposed in the path of the beam, to absorb the slight amount of stray infra-red radiation, which was thought to have caused too low values in the corresponding points of series a, so these points in series c are accordingly given greater weight in drawing the curve. Other things being equal, those points, for the determination of which the

greatest amount of energy was available as shown in the
I column in Table I., are the more accurate; hence the
probable error increases considerably at the ends, especially
at the infra-red end, where both rotating-power and energy

Fig. 2.

Variation of the Faraday Effect with wave-length; constant field.

Present work.
---------------- Moreau's results.
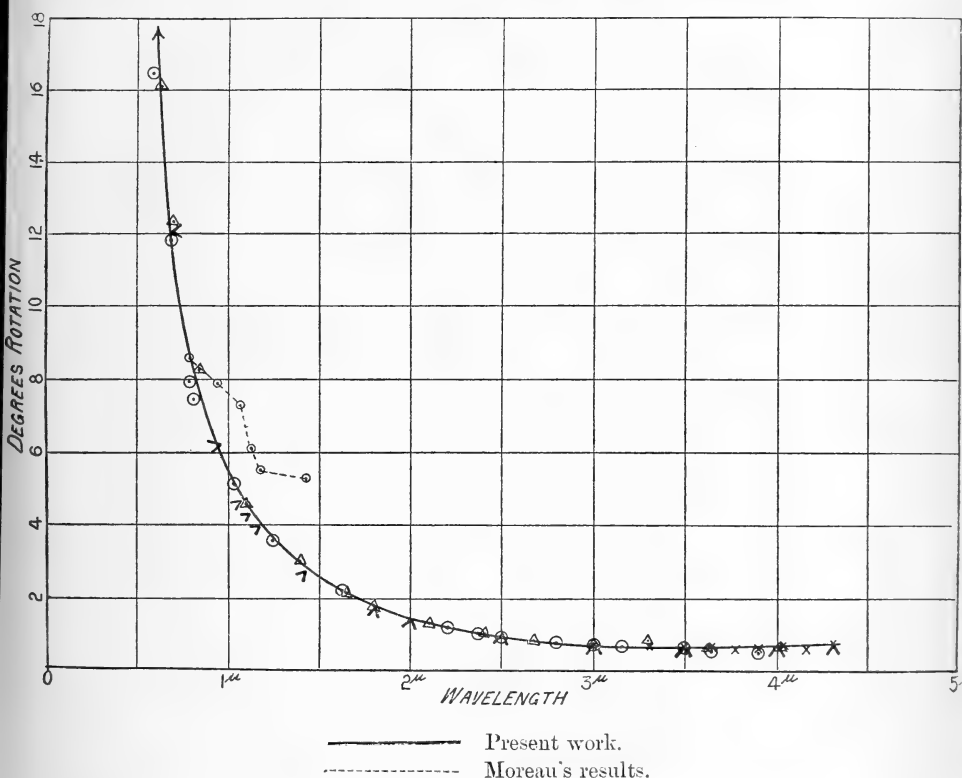
are rapidly decreasing. A check on the general accuracy of
the work has been furnished by visual measurements with a
half-shade polariscope using sodium light. These gave a
rotation of $17°·9$ as against perhaps $17°·7$ taken from the
curve, which is a considerably closer agreement than might
be expected, the discrepancy, indeed, being no greater than
the uncertainty in reading a point from the curve.

Table II. is a comparison of the results obtained by experi-
ment, including those of Moreau, with rotations calculated by
means of the four principal formulæ mentioned above, all

results, for the sake of comparison, being brought to the same scale by letting the rotation for the D line equal unity. The figures for the formulæ of Joubin and Maxwell are taken from a table given by Moreau *. The formula is, as given by this writer,

$$\delta = C \frac{n}{\lambda} + C' \frac{m}{\lambda}, \text{ where } m = -\lambda \frac{dn}{d\lambda}.$$

Log $C = 1 \cdot 29814$ and log $C' = 2 \cdot 43254$, if λ is expressed in terms of hundredth microns.

In the formula of Maxwell,

$$\delta = \frac{n^2}{\lambda^2} \left\{ n - \lambda \frac{d}{d\lambda} \right\} \left\{ k + k' \frac{n^2}{\lambda^2} \right\},$$

log $k = 2 \cdot 87687$, and log $k' = 4 \cdot 11114$. The figures given by Moreau have been extended to $\lambda = 2 \cdot 0\,\mu$, which, if the writer's knowledge is correct, is as far as the refractive index of carbon bisulphide is accurately known. The formula of van Schaik gives almost identical results with this of Maxwell in the infra-red, and hence need not be further mentioned.

The formulæ of Drude without modification are inadequate, for the presence of the carbon-bisulphide absorption-band at $\lambda = \cdot 212\,\mu$ will be of little effect in the infra-red, and will be entirely insufficient to cause the singular flatness of the curve in the neighbourhood of $3 \cdot 5\,\mu$. However, carbon bisulphide has also a well-marked absorption-band at $8 \cdot 05\,\mu$ †, and by adding a term which shall take account of this in the same way that the second term does the ultra-violet band, it is possible to fit the formulæ more closely to the facts. Accordingly they are written

$$\delta = n \left\{ \frac{a}{\lambda^2} + \frac{b}{\lambda^2 - \lambda_1^2} + \frac{c}{\lambda^2 - \lambda_2^2} \right\},$$

where

$$a = -0 \cdot 0136\,; \quad b = +0 \cdot 1530\,; \quad c = -0 \cdot 2370\,;$$
$$\lambda_1^2 = \cdot 045\,; \quad \lambda_2^2 = 64 \cdot 8\,;$$

and

$$\delta = \frac{1}{n} \left\{ \frac{a'}{\lambda^2} + \frac{b'\lambda^2}{(\lambda^2 - \lambda_1^2)^2} + \frac{c'\lambda^2}{(\lambda^2 - \lambda_2^2)^2} \right\},$$

where

$$a' = +0 \cdot 1167\,; \quad b' = +0 \cdot 2379\,; \quad c' = +3 \cdot 482.$$

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* *Loc. cit.*

† Ångström places this band at $\lambda = 8 \cdot 4\,\mu$ (Kayser's 'Spectroscopie,' Bd. iii. p. 371), which value would serve the purpose almost as well as the above with a slightly different choice of constants.

The constants a and b, a' and b', are those given by Drude for the formulæ as applied to rotatory dispersion in the visible spectrum. c and c' have been determined for wave-length 3μ. Their presence has practically no effect on the formulæ as applied to the visible spectrum, so the validity in that region has not been sacrificed. The values of n beyond $\lambda = 2\,\mu$ used in calculations with these two formulæ have been obtained by prolonging the dispersion-curve already known. Unless the ordinary dispersion of carbon bisulphide between $2\,\mu$ and $4\,\mu$ is very anomalous, which is unlikely, the values so obtained are more than sufficiently accurate for the purpose.

As may be seen, both the formulæ of Drude, as modified, fit the case much better than the others. As to distinguishing between them, throughout the greater range of spectrum tested neither has a decided advantage, although the first may show a slightly better agreement. But beyond $\lambda = 3\cdot50\,\mu$, where the curve is very flat, and indeed appears to take a slight upward turn, the state of affairs is much better represented by the second formula, which is the one deduced on the supposition of the Hall effect. The points calculated by this formula have been plotted in fig. 2, each being denoted by the apex of an inverted letter V.

Table II.

Wave-length.	Ob-served.	Moreau.	Joubin.	Maxwell.	Drude (1).	Drude (2).
$\mu.$						
·589	1·00	1·00	1·00	1·00	1·00	1·00
·700	·660	......	......	......	·675	·670
·945	·332	·48	·44	·35	·335	·348
1 076	·272	·44	·38	·28	·271	·265
1·126	·248	·37	·35	·25	·245	·242
1·170	·231	·33	·32	·22	·229	·225
1·419	·158	·32	·28	·13	·157	·152
1·80	·099	......	......	......	·098	·095
2·00	·079	......	......	·06	·081	·078
2·50	·052	......	......	.....	·055	·053
3·00	·041	......	......	.....	·041	·041
3·50	·036	......	......	......	·032	·037
4·00	·037	......	......	......	·028	·038
4·30	·042	......	......	......	·026	·041

Part II.

The Kerr Effect [*].

Du Bois [†] has shown that, for the visible spectrum, the rotation in the Kerr effect in general increases with longer wave-length in the light used. This is notably true in the case of iron or steel, while cobalt and nickel exhibit weakly defined minima in the green and yellow respectively, and magnetite appears to reach a maximum in the yellow. Inasmuch as this is entirely different from the magnetic dispersion shown by most substances—*e. g.* carbon bisulphide —and is not satisfactorily accounted for by theory, it is a matter of considerable interest to determine whether this anomalous nature persists throughout the whole spectrum, or is merely characteristic of the limited region studied, which includes less than $\cdot 3 \mu$.

This necessitates infra-red measurements, and accordingly, when it was found in connexion with the work on the Faraday effect that rotations of the order of a degree could be measured, the apparatus was modified to suit the case of reflexion, and the effect found to be measurable. The results so far obtained show fairly good agreement among themselves, indicating as great an accuracy as might reasonably be expected, in view of the difficulties of the problem. Since, however, they are of quite an unexpected character and point to certain rather important conclusions, it would seem most desirable to verify them, with apparatus modified to secure greater sensibility, and with more perfect surfaces, and thus to explain or remove, if possible, inconsistencies which must now detract from their certainty. To re-model the apparatus, however, and to make a more complete study of the problem, including the important case of rotation by transmission through thin films of magnetic metals, may require considerable time, so the first stage of the work is here presented, and is not intended to be regarded as final and complete, but rather in the light of a preliminary report.

Method.

The rotation was produced by reflexion at the surfaces of two small mirrors of the metal to be tested, mounted near the ends of the pole-pieces of the magnet. To double the effect for greater facility of measurement, and at the same

time to bring the beam out parallel to its original direction,
and hence permit the use of the already existing apparatus
with the smallest possible change, double reflexion was
resorted to, the two magnetic mirrors being held in a small
brass frame mounted between the pole-pieces of the magnet.
These had an air-gap of about 3 cms., and were slightly
shifted out of line to allow the passage of the beam through
them as before. To avoid as much as possible the many
complications of the problem of both magnetic and optical
character, such as elliptical polarization, the incidence was
made as nearly normal as possible, or about 8° at each
mirror. While this is somewhat larger than that used by
du Bois in his work in the visible spectrum, it is well within
the limit of 15° for which the effect, as shown by Righi *, is
practically the same as for normal incidence.

The method of making observations and their reduction
was essentially the same as that already described, and the
sources of error were also largely identical, although their
relative importance was very different. Thus, errors of
adjustment, of impurity of spectrum, and the like, the effect
of which was previously shown to be a matter of only a few
per cent., were not worth considering here in view of the
necessarily large errors of observation, while the question of
deformation of apparatus, due to magnetic traction, which
was entirely negligible before, became of capital importance,
for it was almost impossible to mount the mirrors so that they
would not be warped or shifted by the magnet-field. How-
ever, since the mirrors possessed no magnetic polarity, this
effect, which in general gave rise to an increase or decrease
of intensity of the radiation measured, did not change sign
on reversing the magnet, and hence readily separated itself
from the true Kerr effect. There is a certain amount of
elliptical polarization connected with the rotation in the Kerr
phenomenon, even for small angles of incidence, and it might
be thought that this would give rise to an error, but it can
readily be shown that such error would be only of the second
order, and since direct visual tests showed the amount of
elliptical polarization to be actually very small, its effect was
considered entirely negligible.

The apparatus was that used for the Faraday tests, with
the modifications already mentioned and some minor changes,
such as the increase of the number of polarizer and analyser
plates from six to twelve. The substances tested were steel
(hardened), cobalt, nickel, magnetite, Heusler's metal, and
silver, in the form of little polished plates each 6 by 16 mms.

* *Ann. Chim. Phys.* [6] ix. p. 132 (1886).

and about 3 mms. thick. Silver was included in the tests to indicate if any external cause was acting to produce rotation, and hence give rise to a spurious effect. The matter of polish * was rather troublesome, save in the case of steel ; so, to avoid as much as possible any error due to poor surfaces, several different specimens of each metal were generally tested, and those re-polished several times.

Results.

The results are shown in the curves of figs. 3–6. It has not been thought necessary to give detailed tables of galvanometer-deflexions, but it may be remarked that, by increasing the slit-width and brilliancy of the glower, these

Fig. 3.—Variation of Kerr Effect with wave-length. Constant field.
Glass-plate polarizing apparatus.
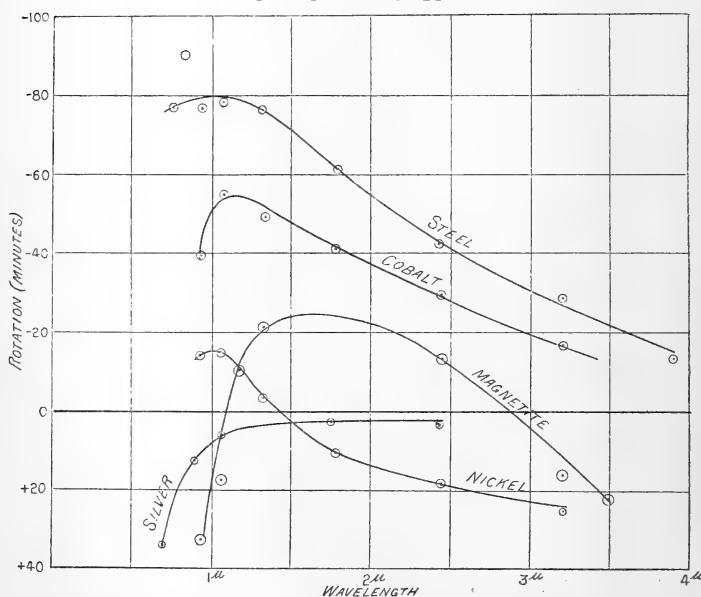
were in most cases made large enough to be perfectly definite, amounting for some of the points in the steel curves to as much as 30 cms. for values of $2 dI$, although in the case of nickel they were rarely more than 4 or 5 cms. It

* Just whether surface impurities have any appreciable effect on the dispersion in the Kerr rotation has not been determined. Micheli (Drude's *Ann.* i. p. 565, 1900) found that surface impurities tend to lessen the "critical angle" as observed for equatorial magnetization, but no conclusions can be drawn from that to apply to the present case.

should be stated, however, that, because of unavoidable un-
steadiness, points determined by deflexions of less than
10 mms. must be regarded as uncertain, and this applies
particularly to those points near the limits of the spectral
region covered. In the curves, the rotations given, measured
in minutes of arc, are as determined for the two surfaces—
unless otherwise specified; and for reversal of the magnet,
and hence to be put on the ordinary basis for the measure-
ment of the Kerr effect, should be halved so as to obtain the

Fig. 4.—Variation of Kerr Effect with wave-length. Constant field.
Double-refracting polarizing apparatus.
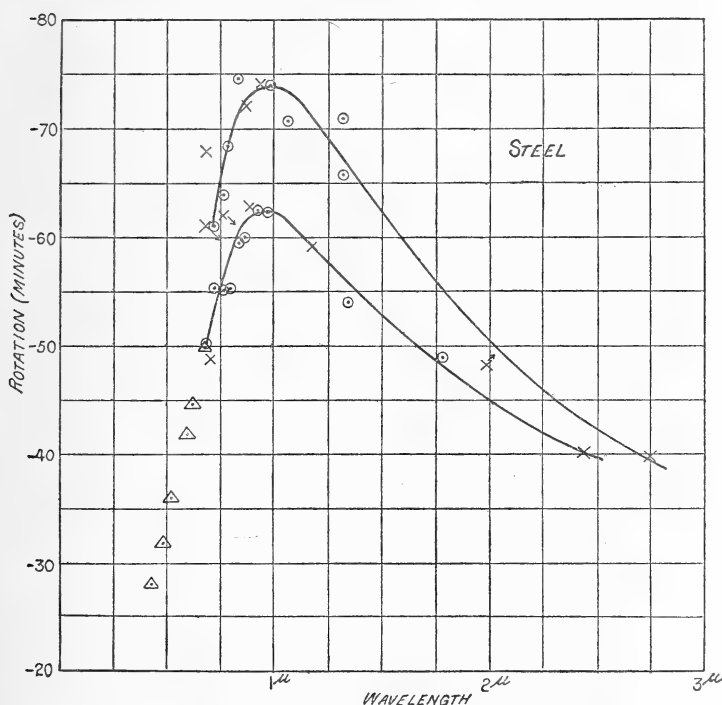
rotation for a single surface. On the same basis as the
results on the Faraday rotation, they are really fourfold,
since the rotations in the former case were stated for only a
single throw of the magnet. As is customary, the sense of
the rotation is taken as negative, being opposite in direction
to current in the magnet-coils.

To check the results with as large a change as possible in
the apparatus, the curves were all repeated with the glass-
plate polarizer and analyser replaced by a double-image prism

and nicol. They allowed measurement as far as $\lambda = 2\cdot4\,\mu$ in the infra-red, and this was almost as far as dependable results could be obtained with the use of the plates. While certain characteristic differences appear between the curves obtained with the two different polarizing arrangements,—both in the

Fig. 5.—Variation of Kerr Effect with wave-length. Constant field.
Glass-plate polarizing apparatus.
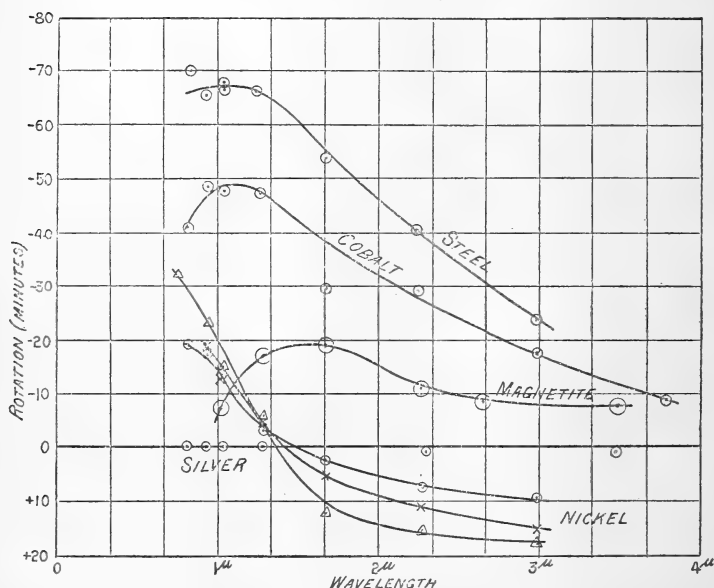
matter of form, which may be due to errors arising from oblique incidence at the surface of the nicol, and as regards absolute magnitude, which appears somewhat greater when measured with the pile of plates,—the results are on the whole consistent. The curves of fig. 4 and 6 were made with the doubly refracting arrangement.

The first striking feature which appeared in the results was the marked diminution of rotation with increase of wavelength, as is shown in the curves, instead of the reverse effect which was to be expected from the results of observations on the visible part of the spectrum. This indicated the existence of maxima which, in the cases of the three magnetic metals, appeared to lie somewhat to the left of $1\,\mu$, but did not prove easy to obtain because dependent on points just beyond the range of the visible and not far enough into the infra-red for accurate measurement, because of lack of energy. However, in several cases the maxima were clearly indicated, while

they are especially well brought out in fig. 4, which shows the results of a more detailed study of steel. Although the magnetic and other conditions were considerably different for the two curves, they still agree very well. Those points for

Fig. 6.—Variation of Kerr Effect with wave-length. Constant field. Double-refracting polarizing apparatus. (Magnetite curve drawn by comparison with fig. 3.)
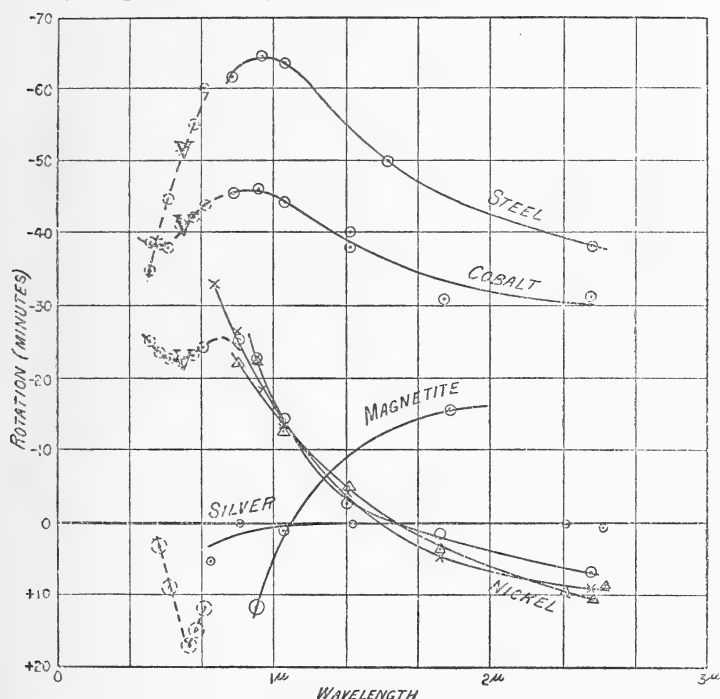
which the two sets of readings—which, as previously explained, were always made for each point—agreed most closely, were regarded as having the least probable error and have been weighted accordingly in drawing the curves. They are plotted as circles, and the others as crosses, while the curves to which certain ambiguous points belong are indicated by small arrows. To show that the slope on the short wave-length side agrees with that of du Bois' dispersion-curve of iron, made for the Kerr effect in the visible spectrum, the latter curve has been plotted to such scale that it joins on to the lower of the given curves at ·67 μ, the points being represented by triangles.

Cobalt gave results similar to steel and possesses no especial

interest; but nickel, curiously enough, appeared to reverse the direction of rotation at about 1·4 μ. This, being such an unexpected result, was for a time regarded as spurious, but as repeated trials on three different samples verified it in all cases where the sensibility was high enough, it is now believed to be genuine. The samples tested were (1) nickel plated on polished brass (points plotted as crosses) ; (2) cast-nickel of supposedly exceptional purity from a magnetic standpoint, although still containing a trace of iron (plotted as circles) ; and (3) commercial rolled nickel (plotted as triangles). As may be seen, there is considerable discrepancy between the various nickel curves. Thus, in fig. 3, this curve, which was made with a rolled-nickel specimen, is unlike its counterparts in the other two sets, while the different curves of fig. 5 seem to show considerable separation after crossing the axis. Whether these differences are real and characteristic of the different specimens tested, or whether they are spurious, cannot yet be stated. Although the measurements on nickel were attended with more difficulty than in most of the other cases, and the probable error of any point is therefore large, it can hardly account for the consistent differences which appear between the curves. In view of this disagreement it is rather odd that all the curves should cross the axis at nearly the same point, as they seem to at about 1·4 μ.

Magnetite proved very difficult to test, and the different curves show poor agreement. Because of its low reflecting-power only a single surface could be advantageously used, the place of the other being taken by silver; so the results in the curves of figs. 5 and 6 are for this single surface only, although they have been doubled in fig. 3, partly to put them on the same basis as the others for comparison's sake, and partly to avoid confusion. The most noticeable feature of the curve for this substance is that, for a good share of the infra-red at any rate, the rotation is in the same sense as for steel, although the positive part of the curve, which would agree with the positive results found in the visible spectrum by du Bois, is well indicated. The second reversal, as found in fig. 3, could not be verified in later observations.

The points on the silver curves were determined with the use of two surfaces of silver plated on brass, and in this way the zero from which the other rotations are to be reckoned was obtained. The curve shows the characteristic Faraday dispersion form, and in the proper, that is positive, sense. It is doubtless due to the effect of the weak field in the neighbourhood of the analyser and polarizer, for its magnitude could be altered and even made to disappear, as is seen in

figs. 5 and 6, by a proper arrangement of the exciting coils of the magnet. In the latter two sets of curves the pole-coils only were excited, while in the set of fig. 3 one field-coil was added. It was also largely dependent on whether the pile of plates or doubly-refracting substance was used as polarizing agent. The points in the set of fig. 3 have all been corrected by the use of the silver curve, although this has not seemed necessary in the last two sets. Unfortunately no silver correction-curve was made for the steel curves of fig. 4, its importance not being realized at the time; hence there is some doubt as to whether they correctly represent the position and shape of the maximum, although it is probable that the correction in this case would be small, from the fact that a single determination at $\lambda = 1 \cdot 34 \mu$ with the steel covered with silvered glass surfaces gave no effect. The silver points have been plotted by somewhat smaller, and the magnetite by larger, circles than the others, to avoid confusion.

As a further check on the accuracy of the results, some direct visual observations were attempted. The method was rather crude, for the use of a half-shade apparatus and monochromatic light failed to give sufficient illumination with the existing arrangement of apparatus; but it was found that fairly good settings for extinction could be made on the polarized image of the glower itself, and the rotations so determined have been plotted in fig. 6 by the letter **V**. This use of white light would of course give the integral of the effect for the whole visible spectrum; but the range, save in the case of steel, is small, and in any case it is assumed to correspond to the rotation for some wave-length near the maximum of visibility, or, nearly enough, for sodium light. The agreement with more careful measurements by other observers is good, considering the necessary uncertainty of field-strength in the present case, for while measurements with a test flip-coil indicated about 10,000 C.G.S. units, it would vary considerably over the surface of the specimen, which had to have an appreciable area to meet the other requirements of the problem. These points in the visible spectrum have been conveniently used to connect the du Bois rotatory dispersion-curves with those of fig. 6, the former results being plotted to such a scale that the rotation for the sodium line shall be the same as in the present case. The curves join together fairly well, save in the case of magnetite, where, the visible spectrum measurements being indecisive, the scale is arbitrarily chosen.

Rather curiously, Heusler's metal gave entirely negative

results, both in visual and infra-red observations, and while the existing arrangement could not be considered as very suitable for the detection of minute rotations, still it may be stated with reasonable certainty that the effect, if it exists at all, is less than a tenth of the rotation for steel.

Interpretation of Results.

As to the physical interpretation of the curves, not as certain conclusions can be drawn now as, it is hoped, may be possible after the work has been extended as already mentioned, but certain points may be considered. The most evident feature is the striking similarity between the completed curves as given in fig. 6 and a typical dispersion-curve running through an absorption-band. Now the general theory of magnetic rotation calls for an anomalous character of the rotatory dispersion-curve in the neighbourhood of an absorption-band, very similar in fact to the case of ordinary dispersion, and indeed intimately connected with it in theory. Whether the curve should have oppositely directed branches on the two sides of the band, or not, depends on whether the hypothesis of molecular currents, or of the Hall effect, be admitted in forming the equations. (See Drude's 'Optics.') The latter condition is now well known to occur in the case of sodium vapour, while curves of the former type have been obtained by Schmauss [*], and quite recently by Wood [†], who worked with a saturated solution of praseodymium chloride. In view of this, the most natural conclusion to be drawn from the curves above given is that the magnetic metals present a case of anomalous rotatory dispersion similar to that found in certain absorbing solutions, with the difference that the region of resonance-absorption, if such it be, instead of being very limited in extent, covers nearly the whole visible spectrum as well as a small part of the infra-red ; in the case of steel it may even extend into the ultra-violet. This might be construed as evidence pointing to the existence of free resonance periods in metals, although the definite vibration time characteristic of common cases of resonance would seem to be lacking. This general view is somewhat strengthened by the fact that the magnetic metals show an increase of refractive index, as well as of magnetic rotation, towards the red, in the visible spectrum; and while the dispersion-curves as given by du Bois and Rubens [‡] show a regular rise without maxima

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[*] *Ann. der Phys.* ii. p. 281 (1900), and also several later papers in this publication.

[†] Phil. Mag. May 1905.

[‡] Wied. *Ann.* xli. p. 522 (1890).

or minima as in the present case, it is not unlikely that, could they be extended on each side of the visible spectrum, more of a resemblance to the rotatory dispersion-curves would appear.

The preceding explanation seeks to interpret the magnetic rotation of the Kerr effect as a characteristic property of the metal itself, having the same physical significance as the simpler case of magnetic rotation in transparent substances. Another point of view may be obtained following the suggestion made by Voigt [*] some years ago, and later developed by du Bois [†]. The observations of Kundt [‡], and later of Lobach [§], on rotation by transmission through thin films of magnetic metal in a magnetic field, have suggested the explanation of the Kerr effect as a special case of rotation by transmission, with reflexion below the surface of the metal. This is the now commonly accepted explanation, but it does not account for the fact that the rotations in the two cases are opposite in sign; and to explain this Voigt and du Bois suggest that one of the circularly-polarized components may in general penetrate deeper into the metal than the other, and hence introduce a difference of phase which may give any sort of a rotation positive or negative. A point in favour of this argument is the peculiar reversal effect which was found in nickel, and which may most easily be explained on this basis, although as regards the negative rotation, it is of interest to note that the Hall effect in nickel is opposite in sense to what it is in iron and cobalt; and on the basis of the Hall effect as an explanation of the Kerr phenomenon, nickel should show a negative rotation.

But even if this view of Voigt and du Bois on the nature of the Kerr effect be accepted, it does not necessarily mean that the rotation dispersion-curves, as found, are without physical significance, but merely that certain effects which have been attributed to the property of magnetic rotation would then be considered as caused by this, which might be called the differential reflecting-power of the metal, which causes the reflexion of the two circularly-polarized components at different mean depths. For it is not unreasonable to suppose that this, if it exists at all, is just as characteristic a property of the metal, and as intimately related to other optical properties, as the power of magnetic rotation by transmission itself.

The negative results given by Heusler's metal are quite remarkable, for, to the best of the writer's knowledge, this is the only case in which this peculiar metal has failed to

* Wied. *Ann.* xxiii. p. 493 (1884).   † *Ibid.* xxxix. p. 25 (1890).
‡ *Ibid.* xxiii. p. 236 (1884).   § *Ibid.* xxxix. p. 347 (1890).

exhibit its magnetic properties. If more careful tests on it, as well as on the non-magnetic alloys of nickel, can show that the Kerr effect is not entirely dependent on magnetic properties, a considerable advance will be made towards the explanation of the phenomena.

Summary.

1. The electromagnetic rotatory dispersion of carbon bisulphide has been measured by infra-red methods over a range of spectrum extending from the sodium lines to $\lambda = 4\cdot3\,\mu$, and found to be correctly represented by a formula which takes account of the absorption-band beyond $8\,\mu$, showing that an infra-red absorption-band may affect the rotatory dispersion, much as it does ordinary dispersion, over a considerable range of spectrum.

2. The magnetic metals and magnetite show, after wave-length $1\,\mu$ in the infra-red, a decrease of the Kerr rotation, with increase of wave-length. The complete rotatory dispersion-curves, made by supplementing the results for the infra-red by existing observations for the visible spectrum, show a marked resemblance to a typical dispersion-curve in the region of an absorption-band, indicating the existence in metals of something analogous to a region of resonance-absorption, extending over the visible spectrum.

3. The particular cases of nickel and magnetite are notable, for the rotation appears to vanish for a particular wave-length in each case, and then change in sign. The Kerr rotation for the magnetic alloy, Heusler's metal, if it exists at all, is less than one tenth of that for iron or steel, although the magnetic properties of the metal are quite comparable.

4. Although the results do not allow of definite conclusions as to whether the hypothesis of molecular currents, or of the Hall effect, should be accepted in explaining magnetic rotation, the indications are that the latter theory holds for carbon bisulphide—*i. e.* it presents a case analogous to that of sodium vapour ; while the curves of the magnetic metals require the former explanation, although the reversal of nickel might perhaps be considered as evidence for the Hall effect.

Physical Laboratory, University of Wisconsin.
July, 1905.

IV. *A Modified Form of Apparatus for the Determination of the Dielectric Constants of Non-Conducting Liquids.   By* V. H. VELEY, *F.R.S.**

WITHIN recent years the determination of the dielectric constants of liquids has attracted considerable attention, partly on account of the simplification of the methods, and partly with a view of ascertaining the validity of the Maxwell equation

$$(\mu^2 - 1)M/(\mu^2 + 2)d = (K - 1)M/(K + 2)d, \quad . \quad . \quad (1)$$

or that of Clausius and Mosotti

$$K = 1 + 2d/1 - d. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As regards the latter point the determinations have led to no satisfactory conclusion ; but as regards the former, the application of the bridge method, the substitution of the telephone for the galvanometer or electrometer, of liquid resistances (Nernst[†]) for many yards of wire (Paluz[‡]) or

Fig. 1.
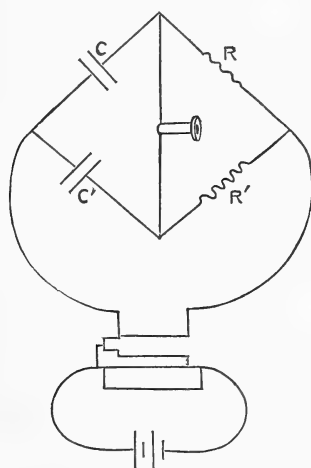
high-resistance coils, have all served to simplify and render more accurate determinations based upon the conditions of balance according to the principle represented by the diagram (fig. 1).

Herein, if C and C′ are two condensers, R and R′ two capacity free resistances, then a balance is attained when

$$R : R' = C' : C. \quad \ldots \quad \ldots \quad (3)$$

The general method of working has been to keep one resistance and one capacity constant, and to vary the other resistance concomitantly with the other capacity so as to restore the balance.

The method to be described differs only from the above in keeping both resistances constant, and varying one capacity concomitantly with the other, as also to measure the capacities in terms of a length according to the principle that *cæteris paribus* the capacity of a condenser varies inversely with the distance between the metallic plates, provided that such distance is small relatively to the area of the plates.

Before passing on to a description of the apparatus, it seems desirable to discuss the method worked out by Nernst (*cf. supra*) and certain of his pupils in Germany, and in this country by Sir James Dewar and Prof. Fleming[*]. The disadvantage of this method appears to consist in the fact that the dielectric constant is determined indirectly, since an air-condenser is not balanced simultaneously against a condenser containing the liquid under investigation; on the other hand, its advantages are (1) the substitution of a small induction-coil instead of a rapidly alternating commutator, and (2) the substitution of liquid resistances instead of bobbins.

As regards the latter improvement, I might mention that for some of my earlier experiments Herr Wolff of Berlin wound for me two 10,000 (international) ohm coils of manganin according to Chaperon's method; these coils were placed in the same box and provided with connecting screws so that either one coil could be placed in each of the two arms of the bridge, or both coils could be put into the same arm. But these coils were discarded as self-induction was not avoided, and consequently it was impossible to determine the point of minimum sound of the telephone owing to the consequent and irritating after-tone.

However, by using liquid resistances the above difficulty was overcome. My apparatus, as stated above, consisted of two condensers, two liquid resistances, telephone, and induction-coil with actuating battery; the only point of novelty consists of the first named, while of the remainder various forms were tried with a view of obtaining more satisfactory results.

[*] Proc. Roy. Soc. lxii. p. 250 (1898).

Description of Condensers.

These were constructed for me by Mr. Pye of Cambridge, according to the specifications of the Rev. F. Jervis-Smith, F.R.S., who kindly assisted me in working out the principle of measuring capacities and hence dielectric constants in terms of lengths.

Each condenser consisted essentially of a gilt nickel dish 46 mms. deep (A in sketch) and 70 mms. internal diameter, provided with a binding-screw S ; within each there was a moveable gilt disk B about 58 mms. diameter (actually 59·62 mms. for one, and 57·87 mms. for the other) and 5 mms. thick, provided with a vertical metal spindle 56 mms. long, C, at the end of which was a binding-screw S′ and a metallic T-piece, in the hollow of the horizontal arm of which a glass

Fig. 2.
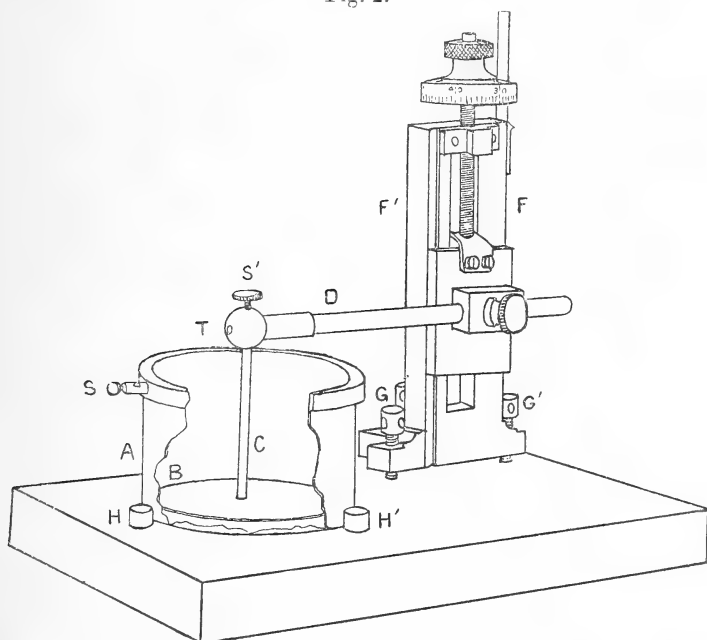
rod D was cemented in. This form of condenser is so far similar to that described by Cohn and Arons[*], except that the position of the disk is variable, and the capacity of the condenser thereby alterable.

The following device was adopted for moving the disk vertically up and down, as also for ensuring that its lower surface remained strictly parallel with the upper surface of

[*] Wied. *Ann.* xxviii. p. 461 (1886).

the bottom of the dish.   The above mentioned glass rod passed through a metal collar in which it could be rigidly fixed by a screw ; the collar was connected with a sliding piece working upon two upright struts, F, F', one smooth and the other provided with V-groove ; at the upper end of the sliding piece was fixed a micrometer-screw, the head of which, divided into 50, revolved once for each ·5 millimetre on a vertical scale, so that ·01 mm. could be read directly and less than half that amount by estimation ; the vertical play of the screw amounted to 31 mms.

The glass rod could be fixed rigidly in any desired position by a clamping screw.   The vertical struts were fixed upon a white marble slab by three levelling-screws (two G, G' shown) arranged in a triangle ; at the other end of the slab was placed the gilt nickel dish kept in position by three ebonite knobs (two H, H' shown) let into the slab.

The corrections for the zero point and the graduation errors of the vertical scale were determined by taking away the dish, placing the apparatus in a horizontal position, and measuring the displacement of the edge of the rim of the disk by a Hilger travelling microscope.   A table of corrections for each condenser was drawn out once for all.

One of these condensers was throughout kept as an air-condenser, while the other was used to contain the liquid dielectric ; during working, each condenser was closed with an ebonite plate cut into half and with a small semicircular piece cut out of each half for the spindle to pass through ; these plates served to exclude dust.

The two dishes were regarded as the outer, and the two disks with their spindles as the inner plates of a pair of diagrammatic condensers.

The major portion of the capacity* of such a condenser (or leyden) is made up of three terms : (a) that due to the area of the lower surface of the disk and the corresponding area of the dish, (b) that due to the area of rim of the disk and likewise, (c) that due to the area of spindle and likewise.   Of these, (a) varies with the distance and nature of dielectric, (b) is constant according to the method of working adopted, (c) varies according to the length of spindle immersed within the dish.   In actual working, the alteration of capacity due to (c) varied about 1 per cent. on an average ; as the maximum

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* The total capacity of such a condenser is a very complex matter, and could only be solved by the method of conjugate functions as discussed by Clerk Maxwell, 'Electricity and Magnetism,' Articles 195 *et seq.*, and J. J. Thomson, 'Recent Researches in Electricity and Magnetism,' pp. 208-250 (Clarendon Press, 1893).

variations in readings amounted to 1 per cent., the correction to be applied would be of the second order.

Liquid Resistances.

In earlier experiments these consisted of electrolytic cells of the H-bottle shape (Kohlrausch type) filled with redistilled water ; the electrodes were platinized in the usual way. One cell was put into each of the two arms of the bridge, the two cells being immersed side by side in a beaker of cold water to eliminate effects of temperature. The resistance introduced into each arm was approximately 9000 ohms, which was found sufficient for air or liquids of low dielectric constant. In later experiments, liquid resistance-tubes of the Nernst type were used, filled with Manganini's (mannite-boracic acid) solution ; these were found to be more suitable for liquids of relatively higher dielectric constant.

Inductoria.

Two forms of inductorium were used, namely, that of Ostwald-Luther and the wire form (Saiten-unterbrecher). My experience was to the effect that the latter gave the sharper note, and consequently a more distinct minimum in the telephone, but was more liable to break down in the course of work, but this might be due to some slight mechanical defect in the particular instrument.

My observations were in complete accordance with those of Nernst[*], that a more decided minimum is produced in the telephone when the inductorium gives a buzzing, rather than a humming or singing sound. The inductoria were actuated by a small chromic-acid battery or accumulator.

Telephone.

The form of telephone used was that of Mix and Genest (resistance about 130 ohms) with an insulating handle ; it was found preferable to hang up the telephone on a glass rod ; when the wire inductorium was used, the note of the telephone, except at or about the minimum point, could be heard some feet away, so that the initial adjustments of the micrometer-screws could be made with comfort and convenience. The whole apparatus (namely, liquid resistances, condensers, connecting wires) was disposed symmetrically as far as possible, and each of the two former pairs was placed together.

[*] Nernst (*loc. cit. supra*) writes : " Ich habe oft beobachtet dass das Minimum leidet, wenn die Saite summt oder singt, anstatt rasselt ; ich vermag dies nicht sicher zu erklären."

Method of Working.

Firstly, an observation was made with both condensers with air as dielectric, the disk of one (on the right hand, called R) being clamped in a fixed position, and the distance of the disk of the other (on the left hand, called L) from the upper surface of the dish being varied by the micrometer-screw, until the minimum sound in the telephone was attained.

Secondly, the liquid to be experimented with was then introduced into the R condenser by means of a pipette (the dish being slightly tilted to avoid the inclusion of air-bubbles) to the point at which the surface of liquid was level with the bottom of the rim of the disk, when the dish was replaced in a vertical position. Superfluous liquid accidentally introduced was removed by the pipette or blotting-paper. The distance of the disk from the dish in the L condenser was varied by the micrometer-screw until the minimum sound was again attained. The ratio of the distance of the disk in the second observation to that in the first is the ratio of the dielectric constant of the liquid to that of air ($=1$). Hence, therefore, as explained above, the dielectric constants can be determined in terms of a length, measured by one micrometer-screw.

Two sets of observations of the first operation are quoted to show the degree of accuracy:—

First Set.

L (Air) condenser.   R (Air) condenser fixed at
Scale Readings.   7·615 mm. on scale.
8·650
8·645
8·645
8·710
8·681

The mean value of the readings of the L condenser is 8·666 ± ·0086.

Second Set.

L (Air) condenser.   R (Air) condenser fixed at
Scale Readings.   12·595 mms.
14·05
14·12
14·13
14·11
14·12
14·15
14·12

Mean   14·12

It will be seen from the above that the results are approximately within an error of 1 per cent.

The dielectric constants of certain organic liquids were determined to test the accuracy of the method.

Benzene.

The sample (thiophene free) was purchased of Kahlbaum, and subsequently purified by freezing and distilling over sodium; the following results were obtained :—

Temperature 17°.

L (Air).	R (Air) condenser fixed at
7·680 mms.	7·615 mms.

L (readings).	Benzene introduced.
3·3	
3·4	
3·4	
3·42	
3·4	

Mean   3·384

$$\text{Dielectric constant } K = \frac{7\cdot68}{3\cdot384} = 2\cdot27.$$

Nernst[*] gives 2·258 for carefully dried benzene, Thwing[†] 2·310 as a mean of nine observers; the value given above is approximately the mean of these two values.

Another set of observations, probably less accurate, was made in the reverse manner, namely, by varying the benzene condenser and keeping the air-condenser constant; the mean value of nine concordant observations gave $K = 2\cdot267$, which differs from the above approximately in the ratio 1/600.

Carbon Tetrachloride.

Purchased of Kahlbaum, purified by fractional distillation, the portion coming over between 76°·59 and 76°·99 being collected. Mean boiling-point 76°·79 at Bar. 750·4 (corr.). Density 4/4 = 1·62611 (corr. to vacuum). Thorpe[‡] gives the values for B.P. = 76°·76 at 755·4, and D. 0/4 = 1·63195. Young[§] for B.P. = 76°·75 at 750·1. These several values are concordant.

[*] *Loc. cit. supra*, p. 658.
[†] *Zeits. f. physikal. Chem.* xiv. p. 300 (1894).
[‡] Phil. Trans. 1894 (A), p. 494, and Journ. Chem. Soc. 1880, Trans. p. 59.
[§] Journ. Chem. Soc. 1891, Trans. p. 912.

The following values for K were obtained in three sets of observations (temp. = $12°\cdot5$) : —

Set (1) of 5 observations ...... Mean = $2\cdot071$
Set (2) of 4   „   ...... „ = $2\cdot027$
Set (3) of 5   „   ...... „ = $2\cdot049$

Total Mean = $2\cdot049$

Drude[*] gives the value K = $2\cdot18$ at 17°; Turner[†], K = $2\cdot20$ at 14°; but neither of these observers purified their samples or give physical data as criteria of purity.

Ethylene Chloride.

Purchased of Kahlbaum, portion on distillation at boiling-point $83°\cdot71 \pm \cdot15$ at Bar. $757\cdot1$ (corr.) collected; D. 4/4 = $1\cdot27536$ (corr.).

Thorpe[‡] gives the values B.P. = $83°\cdot5$ at Bar. $753\cdot9$ and D. 4/4 = $1\cdot27432$ (corr.) One set of five observations gave the value of K = $11\cdot29$ at 17°.

Turner[†] gives K = $10\cdot98$, but no details of purification or physical data.

Monochlorobenzene.

Purchased of Kahlbaum, portion on distillation at boiling-point $131°\cdot55 \pm \cdot2$ at Bar. $736\cdot97$ collected; D. 4/4 = $1\cdot12355$.

Ramsay and Young[§] give the value B.P. = $132°$ at Bar. $758\cdot8$, Young[||] D. 0/0 = $1\cdot12786$.

One set of seven observations gave the value of K = $10\cdot95$ at $10°\cdot8$; a previous observation for this substance does not appear to have been recorded.

The necessity for purifying halogen organic compounds before determining their dielectric constants appears to have been overlooked; most of such compounds decompose slowly, though slightly, in the presence of sunlight with liberation of traces of the halogen acid, which would impart a conductivity to the liquid, and thereby render useless determinations of the dielectric constant, except possibly by the method of Nernst. Thus, to select an example, the constant for chloroform has been determined by three observers, none of whom

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[*] *Zeits. f. physikal. Chem.* xxiii. p. 309 (1897).
[†] *Ibid.* xxxv. p. 428 (1900).
[‡] Journ. Chem. Soc. 1880, Trans. p. 182.
[§] *Ibid.* 1885, Trans. p. 654.
[||] *Ibid.* 1889, Trans. p. 488.

purified the sample used, and one definitely states that he used a "commercial preparation"; this would, without doubt, contain both moisture and hydrochloric acid, namely, a concentrated solution of the latter, the amount of which would increase with time and other conditions.

Personally I experienced such difficulty in purifying a sample of ethylene dibromide (though of correct melting- and boiling-point) so as to be fit for a determination of the dielectric constant, that the attempt was for a time abandoned. Turner[*] gives a value 4·865 for this substance, but makes no allusion to such difficulty.

It is hoped to proceed with further determinations with this form of apparatus and method of working; meanwhile it is thought that they possess sufficient novelty and give sufficiently accurate results to merit description.

V. *The Theory of Phasemeters.*
By W. E. SUMPNER, *D.Sc.*[†]

PHASEMETERS are instruments of the dynamometer type for indicating the phase relations of the currents and potentials in alternating-current circuits. With few exceptions they are made for use on multiphase circuits. Single phase instruments have been constructed, but they are not very satisfactory, as their calibration alters with both frequency and wave-form.

Such instruments consist essentially of two sets of coils, of which one set is fixed and the other forms a single moving system which is not provided with any form of control. The currents in one set of coils are determined by the voltages of the main circuits; while those in the other set are produced by the circuit currents. The total number of coils used in the two systems must be at least three, and as a rule actual instruments only contain this minimum number of coils: the fixed system usually consists of a single coil conveying one of the line currents, while the moving system consists of two independent but relatively fixed coils, traversed by currents produced, through suitable non-inductive resistances, by two of the voltages of the multiphase circuit. There is one instrument made commercially for three-phase circuits which is of a more elaborate construction, and contains three fixed coils for the currents, and a moving system—also of three coils—for the voltages. Such an

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[*] *Loc. cit. supra.*
[†] Communicated by the Physical Society: read October 27, 1905.

instrument is more complicated to make and to connect to the circuit, but it possesses certain advantages as will be shown later. With rare exceptions the magnetic circuits of phasemeters are air circuits, containing no iron, so that the magnetic fields associated with them are weak, and the instruments in consequence are of somewhat delicate construction. There, however, appears to be no reason why the use of iron should be avoided in these instruments, and in what follows the writer proposes to show, that the theory of these instruments is the same whether they contain iron or not, and however the coils may be arranged ; that they can be calibrated by direct current methods although for use on alternating current circuits ; and that a new type of instrument, containing iron, conforms to the theory given. An investigation will also be made of their action when the circuit currents are of unequal magnitude.

The study of these instruments really resolves itself into an examination of the behaviour of multiple magnetic circuits when actuated by independent currents.

Theory of Four-Circuit Phasemeters.

We shall first consider the case of a three-phase instrument having three fixed coils for the circuit currents, and one movable coil, in series with a non-inductive resistance, traversed by a current produced by one of the circuit voltages.

In all cases, it will be assumed (i.) that the induction density at any point due to the current $\mathbf{A}$ in a fixed coil is represented by $\mathbf{A}F$, where $\mathbf{A}$ is the instantaneous value of the current A, and F is a quantity dependent merely on the coil and the position of the point considered ; and (ii.) that the principle of superposition holds, viz. :

$$\mathbf{B} = \mathbf{A}_1 F_1 + \mathbf{A}_2 F_2 + \mathbf{A}_3 F_3,$$

where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are the currents in the three fixed coils and the quantities F are functions of the position of the point corresponding to these fixed coils. The addition must be made vectorially if the subsidiary fluxes are differently directed.

There is of course no doubt about either of these assumptions if the magnetic circuits associated with the fixed coils pass wholly through non-magnetic and non-conducting media such as air. When the path of the lines of force lies partly through (suitably laminated) iron it is necessary, however, to consider the effects of varying permeability and hysteresis. But any magnetic circuit with which the moving coil is

concerned must necessarily contain an air-gap to permit the movement of this coil, and for all dimensions suitable for instruments this air-gap must be such that the reluctance of the iron path is negligibly small compared with that of the gap. The flux densities in the iron under these circumstances can only reach moderate values for which the permeability is fairly constant, while even a considerable change in the permeability will not appreciably affect the magnetic reluctance of the circuit. Similar considerations show that hysteresis cannot produce any sensible effect in magnetic circuits the reluctance of which is almost entirely that of an air path.

We may therefore assume that, however unsymmetrical the windings of the three fixed coils may be, the induction density $\mathbf{B}$ at any point on the conductors * of the moving coil for any deflexion x follows the law

$$\mathbf{B} = \mathbf{A}_1 \mathbf{F}_1 + \mathbf{A}_2 \mathbf{F}_2 + \mathbf{A}_3 \mathbf{F}_3, \quad \ldots \quad (1)$$

where $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ are three quantities dependent only on x and quite independent of time, while $\mathbf{A}_1, \mathbf{A}_2$, and $\mathbf{A}_3$ are the instantaneous values of the currents in the three coils.

If the current in the moving coil is denoted by $\mathbf{V}$, the product $\mathbf{BV}$ represents the momentary value of the force per unit length acting on the portion of the coil considered. It will also be seen that we can regard the quantities $\mathbf{F}$ as line averages for the whole moving coil (for the deflexion x) and that the turning moment acting on the latter may be written

$$\overline{\mathbf{BV}} = \mathbf{F}_1 \overline{\mathbf{A}_1 \mathbf{V}} + \mathbf{F}_2 \overline{\mathbf{A}_2 \mathbf{V}} + \mathbf{F}_3 \overline{\mathbf{A}_3 \mathbf{V}}, \quad \ldots \quad (2)$$

where, if the currents are constant $\overline{\mathbf{AV}}$ denotes the product of the corresponding quantities after taking account of their sign, while if the currents are periodic the expression represents the average value of such product.

Now as the moving coil is not provided with control, if the above quantity is not equal to zero the coil will turn until this is the case. Such a position must exist, since if the coil be turned through 180 degrees in either direction, the torque acting on the coil will be reversed in sign, and thus, as the coil turns, must pass through a zero value in either case. One of these corresponds with a stable, and the other with an unstable, position of equilibrium.

If the second member of (2) be equated to zero, we get an equation for the deflexion x, the solution of which must be independent of the absolute, though not of the relative,

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* Strictly speaking $\mathbf{B}$ is the component of the flux density perpendicular to both the conductor and the direction of motion.

magnitudes of the currents **A**, and which must also be unaffected by the magnitude of **V**. For alternating currents the position of balance can only depend on the phase differences of the currents **A** with regard to the voltage **V**, and on the relative magnitudes of these load currents. For direct currents the equilibrium value of x can only depend on the ratios between A_1, A_2, and A_3.

Now in the theory of phasemeters as hitherto given, the following assumptions are made :—

(i) That the currents and voltages vary with time according to the sine law.

(ii) That the functions F are of equal magnitude and vary with the deflexion x according to the sine law.

(iii) That the load currents are balanced between the phases, that is to say, the magnitudes of the currents A_1, A_2, and A_3 are equal.

It is well known that assumptions (i) and (iii) are only true in practice under most exceptional circumstances, while assumption (ii) is only true when the moving coil is of very small dimensions compared with those of the fixed coils and is placed at their common centre. This cannot be the case in an actual instrument. In what follows, we shall not find it necessary to make any of these assumptions, but we shall confine ourselves at present to the case of balanced loads for which the magnitudes of A_1, A_2, and A_3 are equal.

By the *magnitude* A, of any cyclic function **A**, of a variable t and of period P, we understand that quantity whose square is equal to

$$A^2 = \frac{1}{P} \int_0^P \mathbf{A}^2 \, dt.$$

It follows from the foregoing that with balanced loads the condition of equilibrium at the deflexion x is

$$F_1 \overline{\mathbf{A}_1 \mathbf{V}} + F_2 \overline{\mathbf{A}_2 \mathbf{V}} + F_3 \overline{\mathbf{A}_3 \mathbf{V}} = 0, \quad . \quad . \quad . \quad (3)$$

and is such that the coefficients of F_1, F_2, F_3 are proportional to quantities which merely depend on the phase differences between the currents and the voltage **V**, and also that if we pass steady currents through all the coils and adjust those through the fixed coils till they are proportional respectively to $\overline{\mathbf{A}_1 \mathbf{V}}$, $\overline{\mathbf{A}_2 \mathbf{V}}$, $\overline{\mathbf{A}_3 \mathbf{V}}$, we shall get the same deflexion x. A given deflexion thus always corresponds with a particular power factor of the load ; and in order to calibrate the instrument by means of direct-current tests, it remains to show how to calculate the power factor from the ratios of the direct currents used in order to produce the corresponding deflexion of the instrument.

Now however the instrument may be calibrated, its readings will only be correct when its coils are all connected to the circuit in the particular manner corresponding with the calibration. We must thus adopt for each fixed coil one direction as positive, and, if we distinguish the ends of the coil as positive and negative respectively, we shall understand by a positive current through the coil a current flowing from its positive to its negative end. Moreover, if we consider a current in the mains as positive when flowing in a particular direction along the mains, we have the following equation true at every instant, whatever the law of variation of the alternating currents may be :

$$\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = 0. \quad \ldots \ldots \quad (4)$$

From this it follows that the mean products forming the coefficients in equation (3) must be connected by the relation

$$\overline{\mathbf{A}_1 \mathbf{V}} + \overline{\mathbf{A}_2 \mathbf{V}} + \overline{\mathbf{A}_3 \mathbf{V}} = 0. \quad \ldots \ldots \quad (5)$$

That is to say, in the calibration by means of direct currents we must use three steady currents through the fixed coils such that their algebraic sum is always zero. This can easily be arranged by connecting together all the negative ends of the coils, and afterwards putting two of the coils in parallel through variable resistances, and in series with the third. The two selected for parallel connexion may of course be varied if necessary to calibrate the instrument throughout the scale.

But there is another consequence of (4) the truth of which is also quite independent of any assumption in regard to the variation of the currents with time. Equation (4) can be regarded as a vector equation such that if three vectors are drawn forming a triangle the sides of which are respectively proportional to the magnitudes of the three currents, the angles of this triangle will perfectly represent the phase relations of these three currents. It is also possible to find another vector representing the voltage $\mathbf{V}$ in both magnitude and phase so that the mean product of any two of the quantities considered is accurately equal to the scalar product of the corresponding vectors [*]. In all ordinary cases the vector $\mathbf{V}$ can be regarded as in the same plane as the current vectors, even if the currents vary in a manner widely departing from the sine law ; but, in any case, if $\mathbf{V}_p$ is the perpendicular projection of the vector $\mathbf{V}$ on the plane of

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[*] See "The Vector Properties of Alternating Currents and other Periodic Quantities," Proceedings Royal Society, 1897, vol. lxi. p. 455.

the current vectors, we have from elementary vector considerations the following relations between scalar products:

$$\overline{VA}_1 = \overline{V_p A_1}, \quad \overline{VA}_2 = \overline{V_p A_2}, \quad \overline{VA}_3 = \overline{V_p A_3}.$$

We also have

$$\cos \phi = \cos \alpha \cos \phi_p,$$

where ϕ is the phase-angle between **V** and any current **A**,

ϕ_p is the phase-angle between $\mathbf{V}_p$ and the same current **A**,

and α is the angle between the vectors **V** and $\mathbf{V}_p$.

Hence, it will be apparent that in any such equation as (3) or (5) we can regard the vectors **V** and $\mathbf{V}_p$ as interchangeable, and also that if we may neglect the square of the small angle α, we can regard $\cos \phi$ and $\cos \phi_p$ as identical.

Now the vector figure will be as represented in fig. 1.

In the case considered the load currents are equal, hence the vectors **A** will be of equal magnitude, and it follows that any two will differ in phase by 120°. If therefore ϕ is the angle between the vectors **V** and $\mathbf{A}_1$, and if we denote by

Fig. 1.
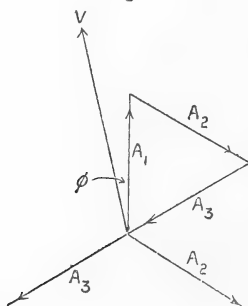
V and A the magnitudes of the voltage and current vectors, we have

$$\overline{V_1 A_1} = VA \cos \phi,$$

$$\overline{VA}_2 = VA \cos (120 + \phi) = -VA \cos (60 - \phi),$$

$$\overline{VA}_3 = VA \cos (240 + \phi) = -VA \cos (60 + \phi).$$

Substituting in (3) we have for balance the condition :

$$F_1 \cos \phi = F_2 \cos (60 - \phi) + F_3 \cos (60 + \phi). \quad . \quad . \quad (6)$$

If now in the direct-current test we make the steady currents such that

$$-A_1 = A_2 + A_3,$$

and

$$\frac{A_2}{A_3} = \frac{\cos (60 - \phi)}{\cos (60 + \phi)} = \frac{1 + 3^{\frac{1}{2}} \tan \phi}{1 - 3^{\frac{1}{2}} \tan \phi}, \quad . \quad . \quad (7)$$

the position of balance will be the same as in the corresponding alternating current test, when the phase of the current through the moving coil differs from that of the current A_1 by ϕ.

From (7) we have

$$\tan \phi = \frac{1}{3^{\frac{1}{2}}} \frac{A_2 - A_3}{A_2 + A_3}, \quad \cdots \cdots \quad (8)$$

from which ϕ and $\cos \phi$ can be found. If $\cos \phi$ so found is to represent the power-factor of the alternate current load, all that is necessary is to select for the voltage **V** that between the main conductor carrying the current A_1 and the neutral point of the system, and to apply it through a non-inductive resistance to the moving coil. If, however, instead of this resistance a condenser be used, the voltage selected should be that between the lines carrying A_2 and A_3, since the effect of the condenser will be to cause the moving coil current to be in quadrature with the voltage producing it. The shape of the curve connecting the deflexion x with ϕ or $\cos \phi$ will depend on the nature of the functions F in (6), and these are determined by the structure of the instrument. With actual phasemeters the coils are arranged with the object of making the quantities F of equal magnitude, and as nearly as possible sinuous functions of the deflexion x; and the angle between the planes of any two coils is made equal to the angle representing the phase difference of the currents passing through the coils. Such assumptions are equivalent to putting :

$$F_1 = f \sin x ; \quad F_2 = f \sin (x - 120) ; \quad F_3 = f \sin (x + 120);$$

and if we substitute these expressions in (6) and reduce, we easily find

$$\sin (\phi + x) = 0,$$

or the deflexion x in degrees is equal to ϕ when measured from an appropriate zero.

But the accuracy of the instrument as a phasemeter in no way depends upon the fulfilment of these conditions of construction. The introduction of iron into the magnetic circuits, and the use of unsymmetrical winding in the coils, will affect the shape of the curve connecting x and ϕ, but will not prevent x from being an accurate indication of ϕ. Nor is it even desirable that x should be proportional to ϕ. It would be much more useful to arrange to make x fairly proportional to $\cos \phi$. Phasemeters are wanted to measure not ϕ, but $\cos \phi$, or the power-factor, and this for load currents which in practice are always lagging. By making x proportional to ϕ the scale-readings for different power-factors are widely separated where least wanted, that is

between power-factors 0·9 and 1·0, and are close together for the values most wanted, for power-factors between 0·6 and 0.8, the values of ϕ for which only differ by 16 degrees. The result is to produce an instrument having a scale the greater portion of which is hardly ever used, while the portion which is most used is small, and the readings undesirably crowded. Assuming bilateral symmetry in the moving coils, a change in x of 180 degrees must necessarily correspond with a reversal of the currents, or with a change in ϕ of 180 degrees, but there is no need for the range of x for leading currents to be as great as that for lagging currents, and it is advantageous to get a large range of x for the values of $\cos \phi$ most needed.

Tests on Phasemeters having Iron Cores.

The foregoing theory has been fully tested by the writer in reference to three new instruments each provided with

Fig. 2.
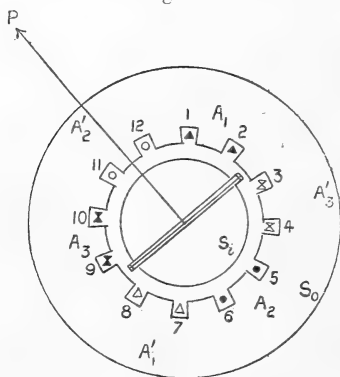
iron-cored magnetic circuits. The structure of one of these is sufficiently indicated in fig. 2. The iron parts are built up of thin circular stampings arranged in two fixed cylindrical coaxial blocks S_0 and S_i. The outer ones S_0 are provided with twelve slots on their inner periphery arranged symmetrically at angular intervals of 30°. In each pair of diametrically opposite slots a coil of 8 turns is wound, and adjoining coils are series connected in pairs to form the three fixed coils A_1, A_2, A_3. When positive currents are sent through the coils the currents flow downwards through the slots 1, 2, 5, 6, 9, and 10, and upwards through the remaining slots. This is indicated in the figure by the use of full signs and corresponding outline signs. The moving coil is rectangular in shape and turns with its sides in the circular air-gap about the common axis of the system. To this

moving coil a pointer P is attached, which reads on a fixed scale of degrees. For any given values of the currents in the fixed coils, the space distribution of induction-density in the gap tends to vary rapidly at the slots and to be nearly uniform in the space between one slot and the next. To overcome this effect, which was foreseen, successive stampings had been slightly sheared circumferentially, so that the line of centres for any slot, instead of being quite parallel to the axis, was slightly spiral in reference to it. The stepped nature of the curve connecting x and ϕ shows that the effect in question was only partially neutralized. But this tends

Fig. 3.
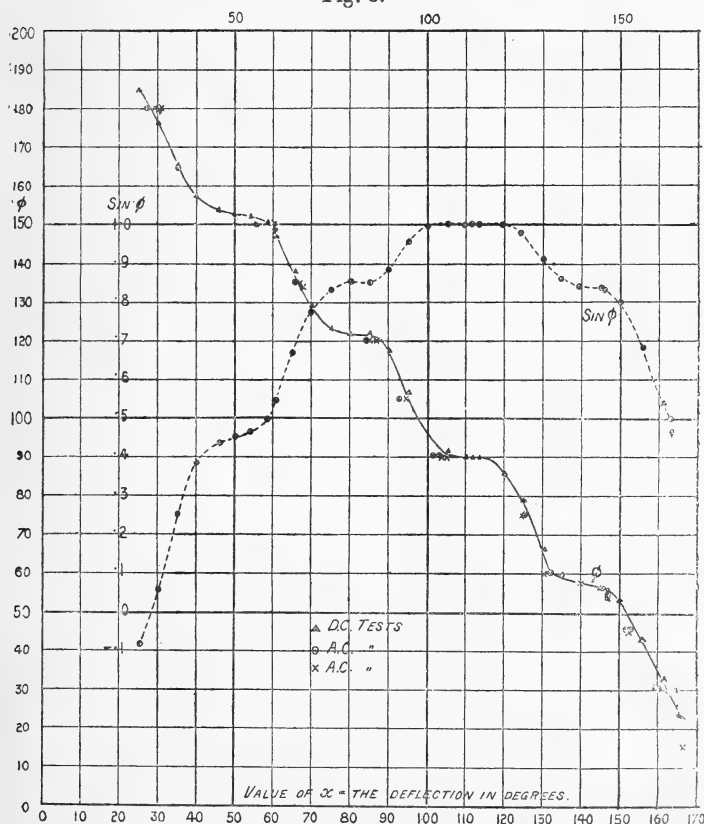
to make the correspondence of the direct-current and alternating-current tests the more convincing. It will be noted that the angular period of the steps in the curve is $30°$, the same as the angular interval between successive slots.

The results are all set forth in fig. 3. The direct current

tests were 33 in number, and are indicated on the curve by
delta signs.   About 40 tests were taken with alternating
currents, half of them with a non-inductive resistance in
series with a moving coil, and the rest with a condenser
substituted for this resistance.   These tests are separately
marked on the curve with rings and crosses.   The observations
are not all plotted, as several of the points were too close to
be distinguishable, but a fair selection is given.

A few typical observations with direct currents are given
in Table I. and will be sufficient to show the conditions of
the test.   The currents A_2, A_3 are in amperes, x is the
observed deflexion of the pointer in degrees, and ϕ is
calculated from formula (8).

<div align="center">TABLE I.</div>

A_2...	+2·2	+1·1	+·42	−·5	−2·8	−5·2	−6·2	−8·9	−9·3	−9·5
A_3...	2·7	7·0	7·2	7·45	7·6	10·9	7·2	8·2	4·3	0·5
x ...	30	40	50	60·5	70	80	100	119·7	139	161
ϕ ...	176·4	157·3	153·0	146·7	128·9	121·7	97·4	85·9	57·3	32·6

The value of ϕ so calculated corresponds in the alternating
tests with the phase difference between the current A_1
through the first fixed coil, and the moving coil current.
For both the direct and alternating current tests, the deflexion
x of the instrument could be read to, and appeared reliable
to, one-tenth of a degree.   The value of ϕ was determined
with almost equal accuracy for the direct current tests, the
conditions for which were simple and could easily be kept
constant, while the value of ϕ depended merely on the ratio
of the readings of two excellent instruments of the permanent
magnet type.   But ϕ could not be determined with such
accuracy in the alternating-current tests, for several reasons.
The testing   conditions with running machinery involved
cannot easily be kept very constant ; alternating-current
instruments are not so reliable as those for direct currents ;
while ϕ has to be determined from the readings of three
instruments instead of from those of two.   The alternator
used was provided with six terminals arranged for the supply
of two-phase and three-phase current.   The three-phase
terminals were connected to the positive ends of the fixed
coils through ammeters and banks of lamps, the negative
ends being connected together to form the neutral point.
The currents were adjusted to approximate equality, but only

one good hot-wire ammeter of the range required was available, and this was put in the A_1 circuit. At first the wattmeter method of testing ϕ was used, a Mather-Duddell instrument being utilized, with its current-coil placed in the A_1 circuit. The pressure circuit of this instrument was put in parallel with the moving-coil circuit of the phasemeter, and with a Weston dynamometer voltmeter. The current circuits were not altered during the tests. The six terminals of the alternator, together with the neutral point, formed seven points, any two of which could be selected for application to the pressure circuits. In this way many different values of ϕ could be obtained, and a new set of tests could be made with a condenser substituted for the non-inductive resistance in series with the moving coil of the phasemeter. Of course, when interpreting the results in the latter case, the value of ϕ as deduced from the wattmeter readings had to be changed by 90 degrees to get the corresponding phase-difference between the moving coil current and the current A_1. The wattmeter tests were made on the instrument shown in fig. 3, but not with the particular arrangement of coils there indicated; and the results found were in close agreement with the curve between x and ϕ as determined by the direct-current method. The wattmeter method proved laborious when a large number of tests were needed; and it was found sufficiently accurate, and much quicker and simpler, to assume that the six terminals of the alternator gave voltages in the precise phase relation indicated by the geometrical properties of fig. 4, in which the points A_1, A_2,

Fig. 4.
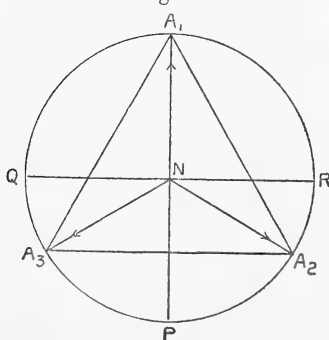
A_3 are 120° apart and correspond with the current circuits, while the points A_1P and QR are the extremities of two perpendicular diameters. The centre of the circle, N, denotes the neutral point. The current vectors are NA_1,

NA_2, and NA_3. The vector NA_1 represents zero phase, while the angle between NA_1 and the vector joining the two points selected for the voltage applied to the moving coil circuit is ϕ_1 the phase of this voltage. When a non-inductive resistance was used with the moving coil, ϕ_1 was the same as ϕ the phase of the moving coil current ; but when a condenser was used, it was found that ϕ_1 had to be diminished by 90°, or increased by 270°, in order to get ϕ the value to plot with x. A few of the observations taken are given in Table II. Particulars are given in the column V of the voltage points chosen for the pressure circuits. ϕ_1 was not a measured quantity, but one calculated on certain assumptions from fig. 4. Any error in these assumptions may serve to explain the slight divergence of some of the points obtained with the alternating-current tests from the curve deduced from the direct-current measurements, but the general agreement observed between the two sets of tests can only be made closer by putting any such error right.

TABLE II.

With Resistance.			With Condenser.			
V.	$\phi_1=\phi.$	$x.$	V.	$\phi_1.$	$\phi.$	$x.$
A_1A_3	150	60	A_3A_2	− 90	180	27
RP	135	67	NA_2	−120	150	55·5
NA_3	120	87	A_1A_2	−150	120	84·2
RA_3	105	94·5	RA_2	165	75	125
A_2Q	75	124	A_2P	120	30	158·7

Another complete set of tests was made on this instrument with one of the coils, A_2, reversed, so as to make the instrument unsymmetrical. The end of the coil A_2, formerly considered positive, was thus made negative. The direct-current test yielded a curve between x and ϕ of a most extraordinary character, yet a number of tests made by the alternating-current wattmeter method yielded points which plotted perfectly on the curve so found. The other two instruments referred to were also thoroughly tested, and with equally satisfactory results. Their structure differed from that indicated in fig. 2 merely in the arrangement and number of the fixed coils. There were three such coils in each case. In one instrument these were wound through six holes, stamped in the laminations of the outer stator S_0, and arranged symmetrically at angular intervals of 60°. In the

other instrument, the coils were wound in six slots stamped symmetrically in the periphery of the inner stator S_i. In either case, the curve found between x and ϕ was of the same general character as that shown in fig. 3, except that, as the windings were in these instruments 60 degrees apart, it was found that the flat parts of the curve covered a greater angular range, and that the angular period of the steps was $60°$ instead of $30°$. In each instrument the direct and alternating current tests were found to be in close agreement, and the direct-current calibration of one of them was found to coincide with a large number of alternating-current tests taken on the same instrument more than eight months previously.

It therefore appears clear, both from theory and experiment, that the accuracy of these instruments on balanced loads in no way depends upon the structure of the coils, or upon the presence or absence of iron, or upon the mode of variation of the currents. The theory given of the four-circuit instrument described will be seen to be equally applicable whichever of the two systems of coils is fixed, so that if the fixed system consists of a single coil and the movable system consists of three relatively fixed coils, the same theory applies.

Three-Circuit and Monophase Instruments.

If the three-coil system is reduced to a two-coil system, as in most actual phasemeters, only a slight modification of the theory is required, and the instrument can still be tested by direct-current methods. All that is needed is to put $F_1 = 0$ in equations 1, 2, 3, and 6. Equations 4, 5, 7, and 8 still hold good, and in the direct-current calibration there are only two currents to consider, and these may have any values. For a single-phase instrument, which merely differs from a multi-phase instrument in that its two-coil system is parallel connected, and the branch circuits made of different inductive properties, a similar theory applies. For a fixed frequency and wave form there will be a constant ratio between P and Q, the two alternating currents in the two-coil system, and a constant phase difference α between these currents. If, then, ϕ is the phase difference between the alternating current in the single-coil system and that through P, the corresponding direct currents through the coils P and Q which will produce the same deflexion will be

$$A_2 = P \cos \phi,$$
$$A_3 = Q \cos (\phi - \alpha).$$

If therefore α and the ratio P : Q are known, it is possible to determine ϕ for any ratio $A_2 : A_3$ of the currents producing the observed deflexion.

Power-factor for Unbalanced Loads.

It has been assumed throughout the foregoing theory that the load currents are balanced ; a most desirable condition in practice, but one rarely attained.   So far as the writer is aware, the behaviour of phasemeters on unbalanced loads has not previously been investigated.   There is a fairly general impression that they are inaccurate under these circumstances, and for this phasemeters have been blamed ; though the fault rather lies with those who assume that an instrument, which is only affected by a few of the conditions of a circuit, can be reasonably expected to measure the average of such conditions.

Now, in the first place, there is not even any generally accepted definition of the power-factor of a three-phase circuit when the load currents are unbalanced.   There are really three circuits, consisting of each line current and the voltage between this line and the neutral point.   There are thus three power-factors, one for each circuit.   Fortunately the voltages, being fixed by the generator, can always be regarded as equal and in symmetrical phase relation; otherwise the definition of the average power-factor of the load would be most complicated.   But we shall take the following definition of $\cos \phi$ for unbalanced loads :—

$$\cos \phi = \frac{A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3}{A_1 + A_2 + A_3} ; \quad . \quad (9)$$

where A_1, A_2, A_3 are the magnitudes of the three load currents, and ϕ_1, ϕ_2, ϕ_3 are the phase angles between these currents and the corresponding voltages to the neutral point. This formula for $\cos \phi$ merely amounts to defining the volt-amperes of an unbalanced three-phase circuit as being the product of the voltage to the neutral point and the sum of the three-line currents.   The relations of the different quantities are indicated in fig. 5, in which the voltage vectors are supposed to be of equal magnitude and symmetrically spaced. If this were not the case, we could still determine a neutral point such that the voltages between it and each line would be equal ; and the definition (9) would still hold good provided the phase angles ϕ_1,

Fig. 5.

ϕ_2, ϕ_3 were reckoned from the corresponding voltages to the neutral point so determined.
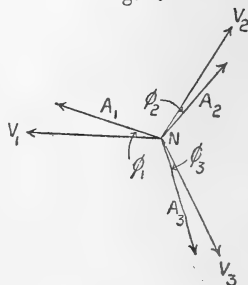
Symmetrical Six-Circuit Phasemeter.

Now the most likely form of phasemeter to be accurate on three-phase circuits with unbalanced loads, is naturally one in which each system of coils consists of three similar and symmetrically arranged coils, having a common line of symmetry, and with their planes fixed at 120 degrees from each other. One of these sets of coils must be actuated by the load currents, and the other set by the line voltages. With such an instrument, we must have three equations like (1) for the induction densities at the three moving coils, or we have :—

$$\left. \begin{aligned} B_1 &= A_1 F_{11} + A_2 F_{21} + A_3 F_{31} \\ B_2 &= A_1 F_{12} + A_2 F_{22} + A_3 F_{32} \\ B_3 &= A_1 F_{13} + A_2 F_{23} + A_3 F_{33} \end{aligned} \right\} ; \quad \ldots \ldots (10)$$

where in the quantities F the first suffix refers to the fixed coil, assumed to take one of the load currents, and the second suffix to the moving coil. Thus F_{23} means the line average effective induction density due to a unit current in the second fixed coil at the conductors of the third moving coil when the deflexion of the pointer is x. Now under the symmetrical conditions assumed we have :—

$$\left. \begin{aligned} F_{11} &= F_{22} = F_{33} = P \\ F_{23} &= F_{31} = F_{12} = Q \\ F_{32} &= F_{13} = F_{21} = R \end{aligned} \right\} ; \quad \ldots \ldots (11)$$

but Q and R are not necessarily the same. Hence we have :

$$\left. \begin{aligned} B_1 &= A_1 P + A_2 R + A_3 Q \\ B_2 &= A_1 Q + A_2 P + A_3 R \\ B_3 &= A_1 R + A_2 Q + A_3 P \end{aligned} \right\} ; \quad \ldots \ldots (12)$$

Now if the three moving coils are star connected, and their three free ends joined through three equal non-inductive resistances to the three mains, the currents through the three coils will be represented by V_1, V_2, V_3, the voltages between the neutral point and the corresponding mains. The condition for balance constituting an equation for x is :

$$\overline{V_1 B_1} + \overline{V_2 B_2} + \overline{V_3 B_3} = 0,$$

or

$$P\Sigma \overline{V_1 A_1} + Q\Sigma \overline{V_1 A_3} + R\Sigma \overline{V_1 A_2} = 0.$$

Now referring to fig. 5, and remembering that the voltage vectors are of equal magnitude, and are symmetrically spaced.

we have for the different angles required :—

$$\widehat{V_1A_1} = \phi_1, \quad \widehat{V_1A_2} = \phi_2 + 120, \quad \widehat{V_1A_3} = \phi_3 + 240, \quad \&c., \quad \&c.$$

If we substitute in the above equations, using such relations as

$$\overline{V_1A_3} = VA_3 \cos \widehat{V_1A_3},$$

and divide out by V we get,

$$\left.\begin{array}{l} P\Sigma A_1 \cos \phi_1 + Q\Sigma A_3 \cos (\phi_3 + 240) + R\Sigma A_2 \cos (\phi_2 + 120) = 0 \\ \text{or} \\ \quad P\Sigma A_1 \cos \phi_1 = Q\Sigma A_1 \cos (\phi_1 + 60) + R\Sigma A_1 \cos (60 - \phi_1) \end{array}\right\} \quad (13)$$

Let us define two quantities C and S such that

$$\left.\begin{array}{l} C = A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 \\ S = A_1 \sin \phi_1 + A_2 \sin \phi_2 + A_3 \sin \phi_3 \end{array}\right\} \quad \cdot \ \cdot \ (14)$$

We then find on expanding (13) that

$$PC = Q\frac{C - 3^{\frac{1}{2}}S}{2} + R\frac{C + 3^{\frac{1}{2}}S}{2}; \quad \cdot \ \cdot \ \cdot \ (15)$$

but from (14) we get

$$C^2 + S^2 = A_1^2 + A_2^2 + A_3^2 + 2\Sigma A_2A_3 \cos (\phi_2 - \phi_3) \quad . \quad (16)$$

Now we only propose to consider the case of a moderate want of balance for which the angles ϕ_1, ϕ_2, ϕ_3 differ from their mean value by only small amounts whose squares and products can be neglected compared with unity. Under these conditions, the cosine of the difference of any two of these angles can be considered unity, and (16) reduces to :—

$$C^2 + S^2 = (A_1 + A_2 + A_3)^2,$$

but from (9) and (14)

$$\left.\begin{array}{l} C = (A_1 + A_2 + A_3) \cos \phi \\ \therefore \qquad S = (A_1 + A_2 + A_3) \sin \phi \end{array}\right\} \cdot \ \cdot \ \cdot \ \cdot \ (17)$$

If now we substitute in (15) and simplify we obtain :

$$P \cos \phi = Q \cos (60 + \phi) + R \cos (60 - \phi), \quad . \quad (18)$$

and substituting from (11)

$$P = F_{11}, \quad Q = F_{31}, \quad R = F_{21},$$

we have

$$F_{11} \cos \phi = F_{21} \cos (60 - \phi) + F_{31} \cos (60 + \phi) ;$$

an equation exactly similar to (6) and showing that the deflexion is the same as on a balanced load of power-factor $\cos \phi$. Also it appears that for balanced loads only one of

the moving coils need be used if it is connected to its appropriate voltage. The deflexion will be the same whichever moving coil is connected to the circuit. For unbalanced loads it is, however, necessary to have all the coils so connected if the instrument is to read cos ϕ as defined in (9).

To calibrate such an instrument by the direct-current method, the process already described may be applied, the three coils of one system, but only one coil from the second, being used for the test.

To confirm this, tests were made on a commercial instrument of the symmetrical six-circuit type, manufactured by Messrs. Everett, Edgcumbe & Co. The moving system consisted of three star-connected coils designed for shunt connexion with the mains. A constant current was passed through one of the three fixed coils, various small measured currents were passed through the moving coils in accordance with the method above described, and the value of ϕ corresponding with each observed deflexion x was calculated by means of formula (8). The scale of the instrument was so constructed that with alternating currents the deflexion x was a direct measure of the phase angle. The observed value of x was found to agree with ϕ when a particular fixed coil was used for the current circuit. Two additional sets of tests were taken using the remaining fixed coils in succession. In one of these cases ϕ had to be increased by 120 degrees, and in the other diminished by the same amount, to make the converted value of ϕ agree with x. The tests made were comprehensive, each set consisting of about 35 determinations ranging over the whole scale. The three curves obtained by plotting the (converted) value of ϕ with x were nearly straight lines, and were nearly the same ; but they were by no means absolutely so, and the differences amounted in some cases to as much as ten degrees. The arrangement of the coils was thus not symmetrical in the mathematical sense assumed in the above proof, and it is difficult to estimate to what extent the indications of the instrument are dependent upon the balance, or lack of balance, of the load currents. To test such an instrument thoroughly with alternating currents on unbalanced loads would be so complicated and laborious that it is unlikely that such a test ever has, or will be, made. There would be needed no fewer than *nine* electrical instruments (three ammeters, three voltmeters, and three wattmeters) besides a number of troublesome adjustments and subsequent calculations. The labour of the nine or more observers would be of no avail unless all the readings for each test were taken accurately and simultaneously.

Symmetrical Five-Circuit Phasemeter.

Before leaving the consideration of symmetrical instruments, it may be well here to state the result of an examination of the case of one having five coils instead of six.

If the first coil of the moving system is omitted, making $F_{11} = F_{21} = F_{31} = 0$ in equations (10), (11), and (12), it will be found on making the changes which necessarily follow that equation (13) becomes

$$P[C - A_1 \cos \phi_1] = Q\left[\frac{C - 3^{\frac{1}{2}}S}{2} - A_3 \cos (\phi_3 + 60)\right]$$
$$+ R\left[\frac{C + 3^{\frac{1}{2}}S}{2} - A_2 \cos (\phi_2 - 60)\right];$$

now in the above equation the coefficient of P is equal to the sum of the coefficients of Q and R (see (21) and (22) below), and we can thus always find an angle χ such that

$$P \cos \chi = Q \cos (60 + \chi) + R \cos (\chi - 60),$$

and hence by (18) the reading of the phasemeter will be that due to a balanced load of power-factor $\cos \chi$.

With the help of (17) and relations proved below (21), (22), and (25), it will be found that

$$\frac{3A \cos \phi - A_0 \cos \theta}{\cos \chi} = \frac{3A \cos (\phi + 60) - A_0 \cos (\theta + 60)}{\cos (\chi + 60)}$$
$$= \frac{3A \cos (\phi - 60) - A_0 \cos (\theta - 60)}{\cos (\chi - 60)},$$

and assuming that θ and χ exceed ϕ by small amounts θ_0 and χ_0 respectively, it will be found that the above equations involve the relation

$$\chi_0 = \tfrac{1}{3}\theta_0, \quad \ldots \quad \ldots \quad (19)$$

or the phasemeter on an unbalanced load will read, instead of the true power-factor $\cos \phi$, the value

$$\cos (\phi + \tfrac{1}{3}\theta_0),$$

where θ_0 has the value given by (31) below.

Phase Error due to Unbalanced Loads.

If the instrument is unsymmetrical and of less complicated construction than those just considered, the next best form will be one having a three-circuit system for the currents, and a single-circuit system for the volts, like the one first described. Referring to equation (3) and assuming that the

load currents are no longer balanced, we see that instead of equation (6) we shall have as the condition for balance :

$$F_1 A_1 \cos \phi_1 = F_2 A_2 \cos (60 - \phi_2) + F_3 A_3 \cos (60 + \phi_3). \quad (20)$$

Now the vectors of fig. 5 are the same as those of fig. 6, in which latter the current vectors, and also the voltage vectors,

Fig. 6.
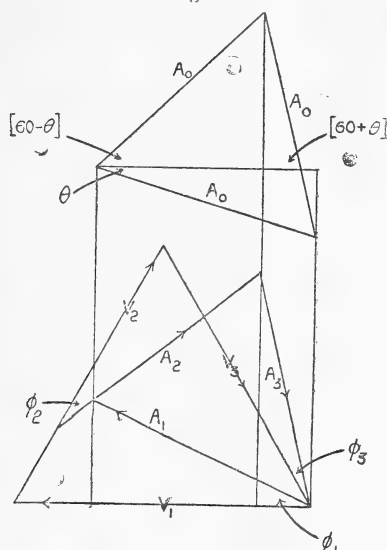
are drawn so as to form closed triangles. Bearing in mind that the voltage triangle is equilateral, it is readily seen that

$$A_1 \cos \phi_1 = A_2 \cos (60 - \phi_2) + A_3 \cos (60 + \phi_3) ; \quad (21)$$

and as we have, whatever θ may be,

$$\cos \theta = \cos (60 - \theta) + \cos (60 + \theta),$$

we see that we can always determine an angle θ from the consistent equations

$$A_0 = \frac{A_1 \cos \phi_1}{\cos \theta} = \frac{A_2 \cos (60 - \phi_2)}{\cos (60 - \theta)} = \frac{A_3 \cos (60 + \phi_3)}{\cos (60 + \theta)} \quad (22)$$

and that for such value of θ we have from (20)

$$F_1 \cos \theta = F_2 \cos (60 - \theta) + F_3 \cos (60 + \theta). \quad (23)$$

If we compare this equation with (6), we see that if the instrument has been correctly calibrated for power-factor on balanced loads, the reading of the instrument will, for the

H 2

unbalanced load assumed, be cos θ as determined from (22), and that this value is absolutely *independent of the quantities* F resulting from a particular structure of the instrument.   In other words, if a number of such phasemeters be connected to the mains in precisely the same manner, and if these instruments have all been calibrated for balanced loads, each of the instruments will indicate the same reading whatever the nature of the load, and however different the various instruments may be as regards internal structure.   But it does not follow that this common reading correctly gives the power-factor of the load.

There are certain relations between A_1, A_2, A_3, ϕ_1, ϕ_2, ϕ_3, resulting from the geometrical properties of fig. 6, but it cannot in general be true that the value of cos θ determined from (22) is the same as that of cos ϕ as defined in (9). This can readily be seen by putting $A_3 = 0$, in which case θ must be 30 degrees while ϕ may have any value.   It may all the same be the fact for small divergences of the load currents from A their mean value, that cos θ and cos ϕ differ but very slightly.   We thus want to find θ_0 where

$$\theta = \phi + \theta_0,$$

since if θ_0 is small, the error, considered as a fraction of the true power-factor, made in taking cos θ (the reading) to represent cos ϕ (the true power-factor) will be :

$$\frac{\theta_0}{\cos \phi} \frac{d \cos \phi}{d\phi} = -\theta_0 \tan \phi, \quad . \quad . \quad . \quad (24)$$

and θ_0 will correspond with the phase error of an ordinary wattmeter due to inductance in its pressure-coil.   For equal phase errors the two instruments will read erroneously by precisely the same percentage on loads of the same power-factor.   The question to determine is whether θ_0, for a moderately unbalanced load, is sufficiently great to render the phasemeter unsatisfactory.

While θ_0 is small the instrument remains a phasemeter, but when θ_0 becomes large the instrument tends to indicate the want of balance of the currents, rather than the average power-factor of the load.

If we put each of the ratios in (22) equal to A_0 and consider fig. 6, it will be noticed that A_0 is the length of each of the sides of an equilateral triangle drawn so that these sides make angles θ with the sides of the triangle representing the voltages, and so that the vertices lie on lines perpendicular to the voltage V_1, and passing through the angular points of the current triangle.   For balanced loads the A_0 triangle

must coincide with the current triangle, and for moderate variations from balance we may put :—

$$A_1 = A\,(1+\epsilon_1), \quad A_2 = A\,(1+\epsilon_2), \quad A_3 = A\,(1+\epsilon_3),$$
$$A_0 = A\,(1+\epsilon_0),$$

where A is the mean value of the load currents, so that

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0,$$

and where we may regard all the quantities ϵ as small fractions whose squares and products may be neglected compared with unity.

Since when the load currents are equal $\phi_1 = \phi_2 = \phi_3$, the above assumptions necessarily imply that if

$$\phi_1 = \chi + \theta_1, \quad \phi_2 = \chi + \theta_2, \quad \phi_3 = \chi + \theta_3,$$

where χ is the mean value of the quantities ϕ, so that

$$\theta_1 + \theta_2 + \theta_2 = 0,$$

we may also neglect the squares and products of the quantities θ.

Now, if we substitute in (9) and simplify, neglecting squares and products of small quantities, we easily find

$$3 \cos \phi = 3 \cos \chi + (\epsilon_1 + \epsilon_2 + \epsilon_3) \cos \chi - (\theta_1 + \theta_2 + \theta_3) \sin \chi,$$

or, using the above relations, we have

$$\cos \phi = \cos \chi,$$

or ϕ is the mean value of ϕ_1, ϕ_2, ϕ_3.

Collecting formulæ we thus have

$$\left.\begin{aligned}
&\phi_1 = \phi + \theta_1, \qquad \phi_2 = \phi + \theta_2, \qquad \phi_3 = \phi + \theta_3, \\
&A_1 = A(1+\epsilon_1), \; A_2 = A(1+\epsilon_2), \; A_3 = A(1+\epsilon_3); \\
&\theta = \phi + \theta_0, \qquad\quad \theta_1 + \theta_2 + \theta_3 = 0, \\
&A_0 = A(1+\epsilon_0), \qquad \epsilon_1 + \epsilon_2 + \epsilon_3 = 0.
\end{aligned}\right\} \quad . \quad (25)$$

Now certain relations can be found between the quantities ϵ and θ for the current triangle shown in fig. 6, by equating the ratios of the sides to the sines of the opposite angles and reducing.

These relations prove to be

$$\left.\begin{aligned}
&3^{\frac{1}{2}} \epsilon_1 = \theta_2 - \theta_3, \qquad -3^{\frac{1}{2}} \theta_1 = \epsilon_2 - \epsilon_3, \\
&3^{\frac{1}{2}} \epsilon_2 = \theta_3 - \theta_1, \qquad -3^{\frac{1}{2}} \theta_2 = \epsilon_3 - \epsilon_1, \\
&3^{\frac{1}{2}} \epsilon_3 = \theta_1 - \theta_2, \qquad -3^{\frac{1}{2}} \theta_3 = \epsilon_1 - \epsilon_2.
\end{aligned}\right\} \quad . \quad (26)$$

From these equations it can be shown, with the help of the last relations of (25) and equivalent relations such as

$$2\,\epsilon_2\epsilon_3 = \epsilon_1{}^2 - \epsilon_2{}^2 - \epsilon_3{}^2,$$

that

$$\left.\begin{array}{c} \epsilon_1{}^2 + \theta_1{}^2 = \epsilon_2{}^2 + \theta_2{}^2 = \epsilon_3{}^2 + \theta_3{}^2 = 2\,\epsilon^2, \\ 3\,\epsilon^2 = \epsilon_1{}^2 + \epsilon_2{}^2 + \epsilon_3{}^2\,; \end{array}\right\} \quad \ldots \quad (27)$$

where

or ϵ is the square root of mean square of the quantities $\epsilon_1, \epsilon_2, \epsilon_3$, and may be said to measure the extent to which the load-currents are "out of balance." For small values, ϵ is essentially the same as the arithmetic mean of $\epsilon_1, \epsilon_2,$ and ϵ_3. Since algebraically the sum of these quantities is zero, it follows that the greatest is equal numerically to the sum of the other two, and thus ϵ is two thirds of the greatest of the quantities $\epsilon_1, \epsilon_2, \epsilon_3$.

Now, if we equate to A_0 each of the three ratios of (22) and multiply up and reduce, we obtain the three equations

$$\left.\begin{array}{c} \epsilon_0 \cos\phi - \theta_0 \sin\phi = \epsilon_1 \cos\phi - \theta_1 \sin\phi, \\ \epsilon_0 \cos(\phi-60) - \theta_0 \sin(\phi-60) = \epsilon_2 \cos(\phi-60) - \theta_2 \sin(\phi-60), \\ \epsilon_0 \cos(\phi+60) - \theta_0 \sin(\phi+60) = \epsilon_3 \cos(\phi+60) - \theta_3 \sin(\phi+60). \end{array}\right\} (28)$$

If we subtract the third of these equations from the second and simplify with the aid of (25) and (26), we get

$$\epsilon_0 \sin\phi + \theta_0 \cos\phi = -\left[\epsilon_1 \sin\phi + \theta_1 \cos\phi\right]. \quad \ldots \quad (29)$$

By squaring this equation and adding the square of the first of (28) we get

$$\epsilon_0{}^2 + \theta_0{}^2 = \epsilon_1{}^2 + \theta_1{}^2 = 2\,\epsilon^2 \text{ by (27)}; \quad \ldots \quad (30)$$

or θ_0 is necessarily less than $\epsilon 2^{\frac{1}{2}}$.

If we choose two angles β_0 and β_1 such that

$$\epsilon_0 = \theta_0 \tan\beta_0, \qquad \epsilon_1 = \theta_1 \tan\beta_1,$$

we can use (30), the first equation of (28), and (29) to prove

$$\sin(\beta_0 - \phi) = \sin(\beta_1 - \phi),$$
$$\cos(\beta_0 - \phi) = -\cos(\beta_1 - \phi)\,;$$

these simultaneous equations involve the relation

$$\beta_0 - \phi + \beta_1 - \phi = \pi$$

or

$$\beta_0 = \pi + 2\,\phi - \beta_1.$$

But from the above we have

$$\left.\begin{array}{ll} & \theta_0 = \epsilon 2^{\frac{1}{2}} \cos \beta_0, \\ \text{or} & \theta_0 = -\epsilon 2^{\frac{1}{2}} \cos (2\phi - \beta_1), \\ \text{or} & \theta_0 = -[\theta_1 \cos 2\phi + \epsilon_1 \sin 2\phi], \end{array}\right\} \quad . \quad . \quad (31)$$

where

$$\tan \beta_1 = \frac{\epsilon_1}{\theta_1} = -3^{\frac{1}{2}} \frac{\epsilon_1}{\epsilon_2 - \epsilon_3} = +3^{\frac{1}{2}} \frac{\epsilon_2 + \epsilon_3}{\epsilon_2 - \epsilon_3},$$

and the fractional error made in reading $\cos \phi$, which by (24) is $-\theta_0 \tan \phi$, becomes

$$\frac{\Delta \cos \phi}{\cos \phi} = \epsilon 2^{\frac{1}{2}} \cos (2\phi - \beta_1) \tan \phi, \quad . \quad . \quad (32)$$

in which the quantities ϵ and β_1 are determined solely by the divergences of the load-currents A_1, A_2, A_3 from their arithmetical mean value.

The same formula can be obtained by equating any two of the ratios (22) without bringing the quantities A_0 and ϵ_0 into the equations, but the working is not any shorter and the information yielded is less.

It will be apparent, on inspection of the error formula (32), that the phasemeter may give very erroneous readings when the load-currents are badly out of balance. This can be seen best by considering a numerical case. Suppose the three load-currents are 21, 22, and 17 amperes. The mean current is 20, and the errors ϵ_1, ϵ_2, ϵ_3 from the mean are 5, 10, and 15 per cent. respectively. It follows that ϵ is 10·8 per cent. and $\epsilon 2^{\frac{1}{2}}$ is 15·3 per cent. Disregarding for the moment the factor $\cos (2\phi - \beta_1)$, the above error has to be multiplied by $\tan \phi$ to get the percentage error in reading $\cos \phi$. The error will thus be much reduced on circuits of high-power factor, but for power-factors below 0·71 the value of $\tan \phi$ becomes greater than unity, and the percentage error in reading $\cos \phi$ will be correspondingly increased, though it must be noted that if the scale is graduated to read $\cos \phi$ the absolute error of the reading in the case assumed is never greater than 0·153 $\sin \phi$.

Now, a circuit having three currents proportional to those assumed would be considered badly out of balance, and as a rule a much better state of things obtains, yet the conditions instanced are quite possible in practice. The influence of the factor $(\cos 2\phi - \beta_1)$ must also be considered. This factor, though it must always decrease the magnitude of the error, can alter its value in a striking manner, and is the quantity

which determines whether the instrument reads high or low. The quantity ϵ in (32) depends merely on the average square of the fractional divergences ϵ_1, ϵ_2, ϵ_3 of the load-currents A_1, A_2, A_3 from their mean value. The quantity β_1, on the other hand, does not depend upon ϵ but upon the ratios between ϵ_1, ϵ_2, ϵ_3. Whatever the value of ϵ, β_1 may have any value in accordance with these ratios, so that $\cos(2\phi - \beta_1)$ varies sinuously with β_1 and will be numerically equal to unity for a particular relation between β_1 and ϕ. Even for the same three currents in the mains, β_1 may have no less than six values, since we can select any of the three currents for the line A_1, and after this selection is made we may choose either of the two remaining currents for the line A_2. The six possible values of β_1 for the currents assumed in the above case can be shown to be $\pm 19°$, $\pm 41°$, and $\pm 79°$, and the percentage errors for these values of β_1 and for a value of ϵ equal to 10 per cent. are shown in Table III. for loads of different power-factor. Another column is added for the case $\beta_1 = 0$, and in the last column the maximum error $\epsilon 2^{\frac{1}{2}} \tan \phi$ is shown.

TABLE III.—Percentage Error of Phasemeters
(for a load 10 per cent. out of balance).

Power-factor.	Values of β_1 in degrees.							Maxi-mum.
	−79.	−41.	−19.	0.	+19.	+41.	+79.	
1·0 ...	0	0	0	0	0	0	0	0
·9 ...	6·1	6·7	5·7	4·2	2·2	− ·3	−4·4	6·9
·8 ...	10·6	8·9	6·1	3·0	− ·5	−4·5	−9·4	10·6
·7 ...	14·1	9·2	4·3	− ·3	−5·0	−9·6	−14·2	14·4
·6 ...	16·9	8·0	·8	−5·3	−10·9	−16·0	−18·9	18·9
·5 ...	18·4	4·6	−4·6	−12·2	−18·4	−23·1	−23·6	24·5

It will be seen that the error may be quite serious. Thus a 14 per cent. error on a load of power-factor 0·7 means that the instrument may read either 0·8 or 0·6 according as the error increases or decreases the reading. Of course it must be remembered that the error is always proportional to ϵ, which may be called the "out-of-balance" of the load; so that if this out-of-balance is 2 per cent. instead of 10 per cent., all the above numbers must be divided by 5. In all cases the reading for a power-factor $\cos \phi$ fluctuates, according

to the way the load is out of balance, between the values

$$\cos \phi \pm \epsilon 2^{\frac{1}{2}} \sin \phi,$$

so that for a load 2 per cent. out of balance the reading of the power-factor cannot vary from the true value by more than $\pm \cdot 028$ as extreme limits. The above formula applies to an instrument having three coils in one system and a single coil in the other and the error is independent of the structure of the instrument. The error is largely controlled by the value of $\tan \phi$, where ϕ is the angle which we have assumed to represent both the power-factor, and the phase-difference of the moving-coil current, in reference to the current in one of the fixed coils. As any voltage of the multi-phase system may be chosen for the moving coil, provided the instrument has been correspondingly calibrated, it might at first sight appear possible to select such a voltage for the moving system as to make the value of $\tan \phi$ small for the particular power-factors the instrument is most required to read, and thus render the error under practical conditions negligible on unbalanced loads. A careful examination will, however, show that this is not the case.

As already shown, an instrument having three current-coils and three voltage-coils, all symmetrically arranged, will read correctly whether the load-currents are balanced or not. But such perfect symmetry is easier to assume in a mathematical investigation than to ensure in the structure of so complicated an instrument. If one of the moving coils is dispensed with, as in the other symmetrical case considered, the instrument is simpler to construct and its symmetry easier to secure. Such a five-circuit instrument is not quite accurate on unbalanced loads; but (19) shows that the error is only one-third as great as in the four-circuit instrument to which Table III. applies; and it will be seen that for all load-currents such as are likely to occur in practice, the error is small enough in this case to be neglected. The six-circuit instrument consists essentially of three single-phase instruments combined into one. The best solution of the phase-meter problem would no doubt consist of three single-phase instruments used one on each circuit. There are difficulties, not yet overcome, in constructing good instruments for single-phase circuits. But such instruments, even if they existed, would at present be used mostly on multi-phase circuits, and on such circuits no difficulty arises. All that is needed is for the instrument to have one of its systems of coils arranged multi-phase for two or more of the voltages, and to have its other system single-phase for the current whose power-factor

is required. The fixed phase relation of the voltages will be unaffected by the variation of the load-currents, and thus the instrument will indicate the true power-factor of the particular current chosen. Indeed, most actual phasemeters are constructed and used in precisely this manner, and the error on unbalanced loads is given by a formula like (24) with θ_1 substituted for θ_0. Reference to (30) and (31) will show that the error made in assuming that the reading indicates the average power-factor of the load is essentially the same as in the case considered, and to which Table III. applies.

In conclusion, the main results of the foregoing investigation may be thus summarised :—

Summary.

1. Phasemeters for multi-phase circuits are all equally accurate on balanced loads provided they have been correctly calibrated and possess no faults due to purely mechanical causes. Their accuracy is not affected by variations in waveform or in current-frequency. The calibration of the scale is affected by the number of coils used in the instrument, by the ratios of the ampere-turns used with these coils, by the distribution of the windings, and by the magnetic nature and properties of the magnetic circuits, especially if these contain iron ; but the accuracy of the indications is not dependent upon any of these considerations.

2. Phasemeters can be simply and accurately calibrated for balanced loads by means of a direct-current method of test.

3. The error of phasemeters on unbalanced circuits is generally serious for loads which are badly out of balance. The error, like that of a wattmeter, increases rapidly as the power-factor of the load diminishes. It can only be reduced at the expense of complication in the instrument, by increasing the number of coils used in the fixed and moving systems, and by arranging the coils and magnetic circuits to be symmetrical in regard to one another. If the true power-factor of the load is $\cos \phi$, the reading of the instrument is

$$\cos \phi + \theta \sin \phi,$$

where θ is the phase error due not to the instrument but to the unbalanced load, and is the product of two factors one of which is the maximum value of θ determined by the amount the load is out of balance, and the other may have any value between $+1$ and -1 and is the factor which determines whether the instrument reads high or low. The maximum value of θ is as follows :—

(i) For an instrument consisting of a single coil for one-coil system, and of either two or three coils for the other system,

$$\theta = \epsilon 2^{\frac{1}{2}}.$$

(ii) For a symmetrical instrument containing three coils in one system and two in the other,

$$\theta = \frac{1}{3}\epsilon 2^{\frac{1}{2}}.$$

(iii) For a mathematically symmetrical six-coil instrument

$$\theta = 0,$$

where ϵ is the square root of mean square of the fractions ϵ_1, ϵ_2, ϵ_3 by which the three load-currents differ from their mean value. It is also very approximately the arithmetical mean of the quantities ϵ_1, ϵ_2, ϵ_3 or two-thirds of the greatest of these three quantities, so that $\epsilon 2^{\frac{1}{2}}$ is essentially the same as the greatest of the quantities ϵ_1, ϵ_2, and ϵ_3.

VI. *Winding Ropes in Mines.*
By Professor JOHN PERRY[*].

[Plate II.]

IN Section G I have just read a short paper in which I give a practical formula for the strength of a winding rope. I point out the difficulty of dealing mathematically with the real problem—" When a cage is descending and the upper end of the rope is stopped, what occurs in the rope?" The problem taken up in this paper assumes no internal friction in the rope, and it is only towards the end that I refer to the effect of a yielding in the attachments at its upper or lower ends.

The solution of the present problem was given me by Dr. Love; I have gone over his work; one of my students, Mr. A. R. Richardson, has carried the work a good deal further than was done in Dr. Love's hasty letter to me. Mr. Richardson has taken a practical example and drawn curves to show how the pull alters at the two ends of the rope and how the speed of the cage alters, and on the same sheets we have what pull and speed would be if motions were simple harmonic, all parts of the rope moving in the same phase. It will be seen, therefore, that my part of the credit of producing this paper is very small.

We assume that the rope with its attached cage is moving downwards with a velocity V when the upper end of it is suddenly stopped and held fixed.

Let $-w$ be the displacement downwards of a point in the rope distant z from its lower end,

α the area of cross section of the rope,

M the mass of cage,

ρ the density of the rope,

l the length of the rope,

E Young's Modulus.

Then equating the mass-acceleration of a small element of the rope to the difference of the pulls on its two ends we get

$$\frac{\partial^2 w}{\partial t} = \frac{E}{\rho} \frac{\partial^2 w}{\partial z^2},$$

or putting

$$\frac{E}{\rho} = a^2,$$

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial z^2},$$

of which the solution is

$$w = f(at - z) + F(at + z). \quad \cdots \quad (1)$$

Again, equating the mass-acceleration of the cage to the pull in the rope at its lower end, we obtain

$$M \frac{\partial^2 w}{\partial t^2} = E\alpha \frac{\partial w}{\partial z}, \quad \text{when} \quad z = 0,$$

or using (1)

$$f''(at) + F''(at) = \frac{E\alpha}{Ma^2} \{F'(at) - f'(at)\},$$

or putting

$$\frac{E\alpha}{Ma^2} = \frac{1}{ml}, \quad \text{where} \quad m = \frac{\text{Mass of Cage}}{\text{Mass of Rope}},$$

$$f''(at) + F''(at) = \frac{1}{ml} \{F'(at) - f'(at)\}. \quad \cdots \quad (2)$$

The conditions of the problem show that for all values of $t < 0$ we may put

$$f(at - z) = -\tfrac{1}{2}V\left(t - \frac{z}{a}\right)$$

$$0 < z < l$$

and

$$F(at + z) = -\tfrac{1}{2}V\left(t + \frac{z}{a}\right)$$

Hence

$$f(y) = -\frac{V}{2a}y \qquad y < 0$$

$$\left.\begin{matrix}\end{matrix}\right\}\quad .\quad .\quad .\quad .\quad (3)$$

$$F(y) = -\frac{V}{2a}y \qquad y < +l$$

Also since the top end remains at rest for all values of $t > 0$,

$$\therefore\quad f(at - l) + F(at + l) = 0 \qquad t > 0,$$

$$i.\,e.,\quad F(y) = -f(y - 2l) \qquad y > l. \quad . \quad . \quad (4)$$

<div align="center">Consider the Interval $0 < y < l$.</div>

From (3) we see that in this interval

$$F'(y) = -\frac{V}{2a} \qquad \text{and} \qquad F''(y) = 0.$$

$\therefore$ equation (2) becomes

$$f''(y) + \frac{1}{ml}f'(y) = -\frac{V}{2mal},$$

$$i.\,e. \qquad f'(y) = -\frac{V}{2a} + Ce^{-\frac{y}{ml}}.$$

To determine the constant C we notice that the velocity of the cage does not alter suddenly.

Now when $t < 0$ velocity of cage $= -V$. Since our positive direction is upwards, and

$$\therefore\quad -V = a\{f'(y) + F'(y)\}_{y=0}$$

$$\therefore\quad -V = a\left\{-\frac{V}{2a} + Ce^{-\frac{y}{ml}} - \frac{V}{2a}\right\}_{y=0}$$

$$\therefore\quad C = 0.$$

$$\therefore f'(y) = -\frac{V}{2a}$$

$$\left.\begin{matrix}\end{matrix}\right.\quad 0 < y < l \left.\right\}. \quad . \quad . \quad (5)$$

$$\text{and} \qquad F'(y) = -\frac{V}{2a}$$

<div align="center">Consider the Interval $l < y < 3l$.</div>

From (4) we have

$$F'(y) = -f'(y - 2l),$$

$$F''(y) = -f''(y - 2l),$$

and (2) may be written

$$f''(y) + \frac{1}{ml}f'(y) = f''(y - 2l) - \frac{1}{ml}f'(y - 2l). \quad . \quad (6)$$

Using the value for $f(y-2l)$ found in (5) we have

$$f''(y) + \frac{1}{ml}f'(y) = \frac{V}{2\,mal}$$

$$\therefore \qquad f'(y) = \frac{V}{2a} + Ce^{-\frac{y}{ml}}$$

To determine the constant C we notice that the velocity of the cage does not alter suddenly.

$$\therefore \quad a\{f'(y) + F'(y)\} = a\{f'(y) + F'(y)\}$$
$$\qquad\qquad y = l - 0 \qquad\qquad\qquad y = l + 0$$

$$\therefore \qquad -V = V + Cae^{-\frac{1}{m}}$$

$$\therefore \qquad C = -\frac{2V}{a}e^{\frac{1}{m}}$$

$$\therefore \qquad f'(y) = \frac{V}{2a} - \frac{2V}{a}e^{-\frac{1}{ml}(y-l)}$$

and from (4)
$$F'(y) = \frac{V}{2a},$$

$$\therefore \quad \underline{l < y < 3l}$$

$$\left.\begin{array}{l} f'(y) = \dfrac{V}{2a} - \dfrac{2V}{a}e^{-\frac{1}{ml}(y-l)} \\[2mm] F'(y) = \dfrac{V}{2a}. \qquad\qquad\qquad \end{array}\right\} \quad \cdots \quad (7)$$

By a similar method we may proceed to find the value of the functions $f'(y)$ and $F'(y)$ for the interval $3l < y < 5l$ by using (6) and the values for $f'(y-2l)$ and $f''(y-2l)$ from (7), and assigning the constant of integration from the fact that the velocity of the cage does not suddenly alter.

We notice that the functions $f'(y)$ have a discontinuity of amount $-\dfrac{V}{a}$, and the functions $F'(y)$ a discontinuity of amount $\dfrac{V}{a}$ as we pass through the values of y which are odd multiples of l.

To study the motion, in detail, at a point z in the rope, we have when $t < \dfrac{l-z}{a}$

$$\frac{\partial \omega}{\partial t} = -V, \qquad \frac{\partial \omega}{\partial z} = 0,$$

and this state holds until the wave reaches the point.

After this, that is when $t > \dfrac{l-z}{a}$ and so long as $t < \dfrac{l+z}{a}$,

then
$$F'(at+z) = \frac{V}{2a}$$

$$f'(at-z) = -\frac{V}{2a}$$

$\therefore \qquad \dfrac{\partial \omega}{\partial t} = 0,$

$$\frac{\partial \omega}{\partial z} = \frac{V}{a}.$$

Hence a wave of extension sets out from the upper end. When the wave reaches a point the point comes to rest and the extension of the rope at the point is $\dfrac{V}{a}$.

This state continues so long as $t < \dfrac{l+z}{a}$, that is until the return wave from the cage reaches the point.

When
$$t > \frac{l+z}{a} \quad \text{but} \quad < \frac{3l-z}{a}$$

$$at - z > l, \quad \text{and} \quad at + z < 3l,$$

$\therefore \qquad F'(at+z) = \dfrac{V}{2a},$

$$f'(at-z) = \frac{V}{2a} - \frac{2V}{a} e^{-\frac{1}{ml}(at-z-l)}$$

$\therefore \qquad \dfrac{\partial \omega}{\partial t} = V\left\{ 1 - 2e^{-\frac{(at-z-l)}{ml}} \right\}$

$$\frac{\partial \omega}{\partial z} = \frac{2V}{a} e^{-\frac{(at-z-l)}{ml}}.$$

These results show a wave running back from the cage. Every point as the wave reaches it starts to move downwards with the velocity V, but its velocity rapidly diminishes.

The extension of the rope at a point when the wave reaches it becomes $\dfrac{2V}{a}$, but this also rapidly diminishes.

In this manner we may interpret our results from the values of $f'(y)$ and $F'(y)$.

Interval.	$f'(y)$.	$F'(y)$.
$0 < y < l$	$-\dfrac{V}{2a}$	$-\dfrac{V}{2a}$
$l < y < 3l$	$\dfrac{V}{2a} - \dfrac{2V}{a} e^{-\frac{(y-l)}{ml}}$	$\dfrac{V}{2a}$
$3l < y < 5l$	$-\dfrac{V}{2a} + \dfrac{2V}{a} e^{-\frac{(y-l)}{ml}} + \dfrac{4V(y-3l)}{mla} e^{-\frac{(y-3l)}{ml}}$	$-\dfrac{V}{2a} + \dfrac{2V}{a} e^{-\frac{1}{ml}(y-3l)}$
$5l < y < 7l$	$\dfrac{V}{2a} - \dfrac{2V}{a} e^{-\frac{(y-l)}{ml}} + \dfrac{4V(y-3l)}{mla} e^{-\frac{(y-3l)}{ml}} - \dfrac{2V}{am^2l^2}\left[2(y-5l)^2 - 2ml(y-5l) + m^2l^2\right] e^{-\frac{(y-5l)}{ml}}$	$\dfrac{V}{2a} + \dfrac{2V}{a} e^{-\frac{1}{ml}(y-3l)} - \dfrac{4V(y-5l)}{mal} e^{-\frac{(y-5l)}{ml}}$
$7l < y < 9l$	$-\dfrac{V}{2a} + \dfrac{2V}{a} e^{-\frac{(y-l)}{ml}} + \dfrac{4V(y-3l)}{mal} e^{-\frac{(y-3l)}{ml}} - \dfrac{2V}{am^2l^2}\left[2(y-5l)^2 - 2ml(y-5l) + m^2l^2\right] e^{-\frac{(y-5l)}{ml}} + \dfrac{8}{3}\dfrac{V}{am^3l^3}\left[(y-7l-ml)^3 + m^3l^3\right] e^{-\frac{(y-7l)}{ml}}$	$-\dfrac{V}{2a} + \dfrac{2V}{a} e^{-\frac{1}{ml}(y-3l)} - \dfrac{4V}{mal}(y-5l)e^{-\frac{(y-5l)}{ml}} + \dfrac{2V}{am^2l^2}\left[2(y-7l)^2 - 2ml(y-7l) + m^2l^2\right] e^{-\frac{(y-7l)}{ml}}$

Pull at Upper End $= pr.$

Interval.	
$0 < at < 2l$	$\dfrac{V}{a}$.
$2l < at < 4l$	$-\dfrac{V}{a} + \dfrac{4V}{a} e^{-\frac{(at-2l)}{ml}}$.
$4l < at < 6l$	$\dfrac{V}{a} + \dfrac{4V}{a} e^{-\frac{(at-2l)}{ml}} - \dfrac{8V}{mal}\left(at-4l\right)e^{-\frac{(at-4l)}{ml}}$.
$6l < at < 8l$	$-\dfrac{V}{a} + \dfrac{4V}{a} e^{-\frac{(at-2l)}{ml}} - \dfrac{8V}{mal}\left(at-4l\right)e^{-\frac{(at-4l)}{ml}} + \dfrac{4V}{am^2l^2}\left[2(at-6l)^2 - 2ml(at-6l) + m^2l^2\right]e^{-\frac{(at-6l)}{ml}}$.
$8l < at < 10l$	$\dfrac{V}{a} + \dfrac{4V}{a} e^{-\frac{(at-2l)}{ml}} - \dfrac{8V}{mal}\left(at-4l\right)e^{-\frac{(at-4l)}{ml}} + \dfrac{4V}{am^2l^2}\left[2(at-6l)^2 - 2ml(at-6l) + m^2l^2\right]e^{-\frac{(at-6l)}{ml}} - \dfrac{16}{3}\dfrac{V}{am^3l^3}\left[(at-8l-ml)^3 + m^3l^3\right]e^{-\frac{(at-8l)}{ml}}$.

Velocity of Cage $= V_0$. $y = at$.

Interval.	
$0 < y < l$	$-V$.
$l < y < 3l$	$V - 2Ve^{-\frac{(y-l)}{ml}}$.
$3l < y < 5l$	$-V - 2Ve^{-\frac{(y-l)}{ml}} + \frac{2V}{ml}\left\{2(y-3l)+ml\right\}e^{-\frac{(y-3l)}{ml}}$.
$5l < y < 7l$	$V - 2Ve^{-\frac{(y-l)}{ml}} + \frac{2V}{ml}\left\{2(y-3l)+ml\right\}e^{-\frac{(y-3l)}{ml}} - \frac{2V}{m^2l^2}\left[2(y-5l)^2+m^2l^2\right]e^{-\frac{(y-5l)}{ml}}$.
$7l < y < 9l$	$-V - 2Ve^{-\frac{(y-l)}{ml}} + \frac{2V}{ml}\left\{2(y-3l)+ml\right\}e^{-\frac{(y-3l)}{ml}} - \frac{2V}{m^2l^2}\left[2(y-5l)^2+m^2l^2\right]e^{-\frac{(y-5l)}{ml}}$ $+ \frac{2V}{3m^3l^3}\left[4(y-7l)^3 - 6ml(y-7l)^2 + 6m^2l^2(y-7l) + 3m^3l^3\right]e^{-\frac{(y-7l)}{ml}}$.

Pull at Lower End $= p_0$.

Interval.	
$0 < at < l$	$0.$
$l < at < 3l$	$\dfrac{2V}{a} e^{-\frac{(at-l)}{ml}}$.
$3l < at < 5l$	$\dfrac{2V}{a} e^{-\frac{(at-l)}{ml}} - \dfrac{2V}{mal}\left\{2(at-3l) - ml\right\} e^{-\frac{(at-3l)}{ml}}$.
$5l < at < 7l$	$\dfrac{2V}{a} e^{-\frac{(at-l)}{ml}} - \dfrac{2V}{mal}\left\{2(at-3l) - ml\right\} e^{-\frac{(at-3l)}{ml}} + \dfrac{2V}{am^2l^2}\left[2(at-5l)^2 - 4ml(at-5l) + m^2l^2\right] e^{-\frac{(at-5l)}{ml}}$.
$7l < at < 9l$	$\dfrac{2V}{a} e^{-\frac{(at-l)}{ml}} - \dfrac{2V}{mal}\left\{2(at-3l) - ml\right\} e^{-\frac{(at-3l)}{ml}} + \dfrac{2V}{am^2l^2}\left[2(at-5l)^2 - 4ml(at-5l) + m^2l^2\right] e^{-\frac{(at-5l)}{ml}} - \dfrac{2}{3}\dfrac{V}{am^3l^3}\left[4(at-7l)^3 - 18(at-7l)^2ml + 18m^2l^2(at-7l) - 3m^3l^3\right] e^{-\frac{(at-7l)}{ml}}$.

The values of $f'(y)$ and $F'(y)$ have been calculated so far as the interval $7l < y < 9l$, and from them the values of the stress at the upper end of the rope, the velocity of the cage, and the stress at the cage, have been tabulated.

The discontinuities in the values for $f'(y)$ and $F'(y)$ can be removed by considering a yielding at the upper end.

If we consider a yielding at the cage and at the upper end, we find that the stresses are greatly reduced.

Consider the following case :—

Weight of Head Gear = 4600 lbs.

At the upper end the spring is such that 400,000 lbs. deflects it 1 foot. At the lower end the spring is such that 160,000 lbs. deflects it 1 foot, and $l = 1000$ feet. (These round numbers are taken for convenience of calculation.)

Then as far $t = \dfrac{4l}{a}$ we find that instead of the maximum stress at the upper end being $3\dfrac{V}{a}$ it is only $1\cdot4\dfrac{V}{a}$. And as far as $t = \dfrac{5l}{a}$ the maximum stress at the lower end is $1\cdot6\dfrac{V}{a}$ instead of $2\dfrac{V}{a}$.

Mr. Richardson's example was this :—Weight of cage 4 tons, cross section of rope 5·33 sq. inches, weight of rope per cubic inch 0·14 lb. Young's modulus of the material $14\cdot5 \times 10^6$ lb. per sq. inch [*]. He gives three cases in which the lengths of rope out were respectively 107 feet, 1000 feet, and 2000 feet.

Explanation of the Diagrams (Pl. II.).

The scale of time in each case is such that the distance marked OT is at/l, or the time required for a pulse to travel from one end of the rope to the other. Thus in fig. 1, where the length of rope is only 107 feet, it is 1/156 of a second ; in fig. 2, where $l = 1000$ feet, it is 0·06 second ; and in fig. 3, where $l = 2000$ feet, it is 0·12 second.

The full line curve V shows how the speed of the cage alters as time goes on ; the height OV represents 40 feet per second. The dotted curve V shows the same thing on the assumption that after imagining one-third of the mass of the rope to be added to that of the cage, the rope is regarded as massless. This I shall call the approximate solution.

[*] From some rough measurements upon wire rope since the calculations were made I am inclined to think that what is equivalent to the Young's modulus in the rope ought to be taken as about one-twentieth of that of the steel itself, or about one-tenth of what Mr. Richardson assumed ; also that the internal friction on the rope must be very large indeed.

The full line curves Q and P show the stress at the bottom and top ends of the rope respectively. The height OP represents 36,000 lb. per sq. inch. The dotted curve P shows what both of those would be according to the approximate solution. In all cases gravitational forces are neglected.

As was to be expected, the approximate solution is quite different from the other when the rope is long, and seems to agree with the other better and better as the rope is shorter. Also it is evident that if internal friction in the rope or any other cause is likely to destroy the discontinuities which we observe at the times of reflexion, then the approximate solution is probably more correct than the other.

VII. *On Electrical Vibrations and the Constitution of the Atom.*
By Lord RAYLEIGH, *O.M., F.R.S.*[*]

IN illustration of the view, suggested by Lord Kelvin, that an atom may be represented by a number of negative electrons, or negatively charged corpuscles, enclosed in a sphere of uniform positive electrification, Prof. J. J. Thomson has given some valuable calculations[†] of the stability of a ring of such electrons, uniformly spaced, and either at rest or revolving about a central axis. The corpuscles are supposed to repel one another according to the law of inverse square of distance and to be endowed with inertia, which may, however, be the inertia of æther in the immediate neighbourhood of each corpuscle. The effect of the sphere of positive electrification is merely to produce a field of force directly as the distance from the centre of the sphere. The artificiality of this hypothesis is partly justified by the necessity, in order to meet the facts, of introducing from the beginning some essential difference, other than of mere sign, between positive and negative.

Some of the most interesting of Prof. Thomson's results depend essentially upon the finiteness of the number of electrons; but since the experimental evidence requires that in any case the number should be very large, I have thought it worth while to consider what becomes of the theory when the number is infinite. The cloud of electrons may then be assimilated to a fluid whose properties, however, must differ in many respects from those with which we are most familiar. We suppose that the whole quantities of positive and negative are equal. The difference between

them is that the positive is constrained to remain undis-placed, while the negative is free to move. In equilibrium the negative distributes itself with uniformity throughout the sphere occupied by the positive, so that the total density is everywhere zero. There is then no force at any point; but if the negative be displaced, a force is usually called into existence. We may denote the density of the negative at any time and place by ρ, that of the positive and of the negative, when in equilibrium, being ρ_0. The repulsion between two elements of negative $\rho\,dV$, $\rho'\,dV'$ at distance r is denoted by

$$\gamma \, . \, r^{-2} . \, \rho\,dV \, . \, \rho'\,dV' \quad . \quad . \quad . \quad . \quad (1).$$

The negative fluid is supposed to move without circulation, so that a velocity-potential (ϕ) exists; and the first question which presents itself, is as to whether there is " condensation." If this be denoted by s, the equation of continuity is, as usual [*],

$$\frac{ds}{dt} + \nabla^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Again, since there is no outstanding *pressure* to be taken into account, the dynamical equation assumes the form

$$\frac{d\phi}{dt} = R \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where R is the potential of the attractive and repulsive forces. Eliminating ϕ, we get

$$\frac{d^2s}{dt^2} = -\nabla^2 R \quad . \quad . \quad . \quad . \quad . \quad (4).$$

In equilibrium R is zero, and the actual value depends upon the displacements, which are supposed to be small. By Poisson's formula

$$\nabla^2 R = 4\pi\gamma\rho_0 s \quad . \quad . \quad . \quad . \quad . \quad (5),$$

so that

$$\frac{d^2s}{dt^2} + 4\pi\gamma\rho_0 s = 0 \quad . \quad . \quad . \quad . \quad . \quad (6).$$

This applies to the interior of the sphere; and it appears that any departure from a uniform distribution brings into play forces giving stability, and further that the times of oscillation are the same whatever be the character of the

* 'Theory of Sound,' § 244.

disturbance. It is worthy of note that the constant $(\gamma\rho_0)$ of itself determines a *time*.

In considering the significance of the vibrations expressed by (6), we must remember that when s is uniform no external forces having a potential are capable of disturbing the uniformity.

We now pass on to vibrations not involving a variable s, that is of such a kind that the fluid behaves as if incompressible. An irrotational displacement now requires that some of the negative fluid should traverse the surface of the positive sphere (a). In the interior $\nabla^2 R = 0$.

To represent simple vibrations we suppose that ϕ, &c. are proportional to e^{ipt}. By (3) $\nabla^2\phi = 0$; and we take (at any rate for trial)

$$\phi = e^{ipt} r^n S_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where S_n is a spherical surface harmonic of the nth order. The velocity across the surface of the sphere at $r = a$ is

$$d\phi/dr = n\, a^{n-1} e^{ipt} S_n\; ;$$

and thus the quantity of fluid which has passed the element of area $d\sigma$ at time t is

$$\rho \int \frac{d\phi}{dr}\, dt \,.\, d\sigma = \frac{\rho_0 n a^{n-1}}{ip} e^{ipt} S_n d\sigma \quad . \quad . \quad . \quad (8).$$

The next step is to form the expression for R, the potential of all the forces. In equilibrium the positive and negative densities everywhere neutralize one another, and thus in the displaced condition R may be regarded as due to the surface distribution (8). By a well-known theorem in Attractions we have

$$R = -\frac{4\pi\gamma\rho_0 n r^n S_n e^{ipt}}{ip(2n+1)} \quad . \quad . \quad . \quad . \quad (9).$$

But by (3) this is equal to $d\phi/dt$, or $ip\, e^{ipt} r^n S_n$. The recovery of $r^n S_n$ proves that the form assumed is correct; and we find further that

$$p^2 = \frac{4\pi\gamma\rho_0 \,.\, n}{2n+1} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This formula for the frequencies of vibration gives rise to two remarks. The frequency depends upon the density ρ_0, but not upon the radius (a) of the sphere. Again, as n increases, the pitch rises indeed, but approaches a finite limit given by $p^2 = 2\pi\gamma\rho_0$. The approach to a finite limit as we

advance along the series is characteristic of the series of spectrum-lines found for hydrogen and the alkali metals, but in other respects the analogy fails. It is p^2, rather than p, which is simply expressed; and if we ignore this consideration and take the square root, supposing n large, we find

$$p \propto 1 - 1/2n,$$

whereas according to observation n^2 should replace n. Further, it is to be remarked that we have found only one series of frequencies. The different kinds of harmonics which are all of one order n do not give rise to different frequencies. Probably the simplicity of this result would be departed from if the number of electrons was treated merely as great but not infinite.

The principles which have led us to (10) seem to have affinity rather with the older views as to the behaviour of electricity upon a conductor than with those which we associate with the name of Maxwell. It is true that the vibrations above considered would be subject to dissipation in consequence of radiation, and that this dissipation would be very rapid, at any rate in the case of n equal to unity[*]. But this hardly explains the difference between the two views .

The problem of the vibration of electricity upon a conducting sphere has been considered by Prof. Thomson[†], but his solution does not appear to me to have the significance usually attributed to it. For the vibration of order 1, the value of p (with the same meaning as above) is

$$p = \frac{V}{a} \left\{ \frac{i}{2} + \frac{\sqrt{3}}{2} \right\} \quad . \quad . \quad . \quad . \quad . \quad (11).$$

But the solution corresponding thereto is

$$\frac{e^{ipt} e^{i\lambda r}}{\lambda r} \left(i + \frac{1}{\lambda r} \right),$$

where $\lambda = p/V$, V is the velocity of light, and a the radius of the sphere. Considering only the exponential factors, we have

$$e^{ipt} e^{i\lambda r} = e^{-\frac{1}{2}(1 + i\sqrt{3})(Vt - r)/a} \quad . \quad . \quad . \quad . \quad (12),$$

including the non-periodic factor $e^{\frac{1}{2}r/a}$. Thus, although (12) diminishes exponentially with the time and represents a

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[*] In this case we should have to consider how the positive sphere is to be held at rest.

[†] Proc. Lond. Math. Soc. xv. p. 197, 1884; 'Recent Researches,' § 312, 1893.

motion in a sense divergent, the disturbance *increases* exponentially with r; and thus (12) cannot apply to a problem where the disturbance is supposed to originate in the neighbourhood of the sphere.

The analysis of the electrical problem is necessarily rather elaborate, and in illustration it may be well to consider the analogous question for sound. For the term of order zero, the velocity potential ψ_0 of a divergent wave takes the form[*]

$$\psi_0 = \frac{S_0}{r} e^{ik(at-r)} \quad \ldots \quad \ldots \quad (13),$$

where a denotes the velocity of sound. In the usual theory the divergent vibration is supposed to be maintained by forces operative at $r=0$ with a prescribed frequency. At present we regard (13) as applicable to the space outside a certain sphere of radius c, whose surface remains at rest, so that the case is that of air vibrating round a solid ball of radius c. The condition to be satisfied at $r=c$ is $d\psi_0/dr=0$; so that

$$1 + ikc = 0 \quad \ldots \quad \ldots \quad (14),$$

and

$$\psi_0 = \frac{S_0}{r} e^{-(at-r)/c} \quad \ldots \quad \ldots \quad (15).$$

In like manner for the term of order unity we have

$$r\psi_1 = A \cos\theta \, e^{-ikr}\left(1 + \frac{1}{ikr}\right) e^{ikat} \quad \ldots \quad (16),$$

and the surface condition gives

$$ikc + 2 + \frac{2}{ikc} = 0 \quad \ldots \quad \ldots \quad (17),$$

whence

$$ikc = -1 \pm i \quad \ldots \quad \ldots \quad (18).$$

When r is great, (16) becomes accordingly

$$\psi_1 = \frac{A \cos\theta}{r} e^{(-1\pm i)(at-r)/c} \quad \ldots \quad (19).$$

Both in (15) and (19) ψ diminishes exponentially with the time but increases exponentially with the distance r. The case is not mended if we start with $e^{ik(at+r)}$. Instead of (15) we then find

$$\psi_0 = \frac{S_0}{r} e^{(at+r)/c} \quad \ldots \quad (20),$$

increasing exponentially with r as before and now also with t.

It does not appear that any solution exists of the kind

[*] 'Theory of Sound,' § 325.

sought, unless we introduce another reflecting surface. For the enclosed space thus defined, we find of course undissipated vibrations, and p becomes wholly real.

In the calculation of frequencies given above for a cloud of electrons the undisturbed condition is one of equilibrium, and the frequencies of radiation are those of vibration about this condition of equilibrium. Almost every theory of this kind is open to the objection that I put forward some years ago[*], viz. that p^2, and not p, is given in the first instance. It is difficult to explain on this basis the simple expressions found for p, and the constant differences manifested in the formulæ of Rydberg and of Kayser and Runge. There are, of course, particular cases where the square root can be taken without complication, and Ritz[†] has derived a differential equation leading to a formula of this description and capable of being identified with that of Rydberg. Apart from the question whether it corresponds with anything mechanically possible, this theory has too artificial an appearance to inspire much confidence.

A partial escape from these difficulties might be found in regarding actual spectrum lines as due to *difference tones* arising from primaries of much higher pitch,—a suggestion already put forward in a somewhat different form by Julius.

In recent years theories of atomic structure have found favour in which the electrons are regarded as describing orbits, probably with great rapidity. If the electrons are sufficiently numerous, there may be an approach to steady motion. In case of disturbance, oscillations about this steady motion may ensue, and these oscillations are regarded as the origin of luminous waves of the same frequency. But in view of the discrete character of electrons such a motion can never be fully steady, and the system must tend to radiate even when undisturbed[‡]. In particular cases, such as some considered by Prof. Thomson, the radiation in the undisturbed state may be very feeble. After disturbance oscillations about the normal motion will ensue, but it does not follow that the frequencies of these oscillations will be manifested in the spectrum of the radiation. The spectrum may rather be due to the upsetting of the balance by which before disturbance radiation was prevented, and the frequencies will correspond (with modification) rather to the original distribution of electrons than to the oscillations. For example,

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[*] Phil. Mag. xliv. p. 362, 1897; 'Scientific Papers,' iv. p. 345.
[†] Drude, *Ann.* Bd. xii. p. 264, 1903.
[‡] Confer Larmor, 'Matter and Æther.'

if four equally spaced electrons revolve in a ring, the radiation
is feeble and its frequency is four times that of revolution.
If the disposition of equal spacing be disturbed, there must
be a tendency to recovery and to oscillations about this dis-
position.   These oscillations may be extremely slow ; but
nevertheless frequencies will enter into the radiation once,
twice, and thrice as great as that of revolution, and with
intensities which may be much greater than the original
radiation of fourfold frequency.

An apparently formidable difficulty, emphasised by Jeans,
stands in the way of all theories of this character.   How
can the atom have the definiteness which the spectroscope
demands ?   It would seem that variations must exist in (say)
hydrogen atoms which would be fatal to the sharpness of the
observed radiation ; and indeed the gradual change of an
atom is directly contemplated in view of the phenomena of
radioactivity.   It seems an absolute necessity that the large
majority of hydrogen atoms should be alike in a very high
degree.   Either the number undergoing change must be
very small or else the changes must be sudden, so that at any
time only a few deviate from one or more definite conditions.

It is possible, however, that the conditions of stability or
of exemption from radiation may after all really demand this
definiteness, notwithstanding that in the comparatively simple
cases treated by Thomson the angular velocity is open to
variation.   According to this view the frequencies observed
in the spectrum may not be frequencies of disturbance or of
oscillations in the ordinary sense at all, but rather form an
essential part of the original constitution of the atom as
determined by conditions of stability.

Terling Place, Witham, Nov. 3.

VIII. *On the Constitution of Natural Radiation.*
By Lord RAYLEIGH, *O.M.*, *F.R.S.*[*]

THE expression of Prof. Larmor's views in his paper [†]
"On the Constitution of Natural Radiation" is very
welcome.   Although it may be true that there has been no
direct contradiction, public and private communications have
given me an uneasy feeling that our views are not wholly
in harmony ; nor is this impression even now removed.   It
may conduce to a better understanding of some of these im-
portant and difficult questions if without dogmatism I

endeavour to define more clearly the position which I am disposed to favour on one or two of the matters concerned.

On p. 580, in comparing white light and Röntgen radiation, Prof. Larmor writes: "Both kinds of disturbance are resolvable by Fourier's principle into trains of simple waves. But if we consider the constituent train having wave-length variable between λ and $\lambda + \delta\lambda$, *i. e.* varying irregularly from part to part of the train within these limits, a difference exists between the two cases. In the case of the white light the vibration-curve of this approximately simple train is in appearance steady ; it is a curve of practically constant amplitude, but of wave-length slightly erratic within the limits $\delta\lambda$ and therefore of phase at each point entirely erratic. In the Fourier analysis of the Röntgen radiation the amplitude is not regular, but on the contrary may be as erratic as the phase." This raises the question as to the general character of the resultant of a large number of simple trains of approximately equal wave-length. In what manner will the resultant amplitude and phase vary ? In several papers * I have considered particular cases of approximately simple waves, showing how they may be resolved into absolutely simple trains of approximately equal wave-lengths. But now the question presents itself in the converse form. What are we to expect from the composition of simple trains, severally represented by

$$a_1 \cos \{(n + \delta n_1)t + \epsilon_1\} \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where δn_1 is small, while the amplitude a_1 and the initial phase ϵ_1 vary from one train to another ?

In virtue of the smallness of δn_1 we may appropriately regard (1) as a vibration of speed n and of phase $\epsilon_1 + \delta n_1 t$, variable therefore with the time. The amplitude and phase may be represented in the usual way by the polar coordinates of a point ; and the point representing (1) accordingly lies on the circle of radius a_1 and revolves uniformly with small angular velocity. For the present at any rate I suppose that the amplitudes a_1, a_2, &c. are all equal (1), in which case the points lie all upon the same circle. The radius from the centre O to any of the points P upon the circumference is a vector fully representative of the vibration, and the resultant of the vectors represents the resultant of the vibrations.

After the lapse of a time t the points have moved from their initial positions P to other positions Q, and the aggregate of the vectors OP is replaced by the aggregate of OQ.

* See especially Phil. Mag. vol. l. p. 135 (1900).

The *difference* is the aggregate of PQ. Now we suppose that t is so related to the greatest δn that all the arcs PQ are small fractions of the quadrant, and the question before us is the amount of the difference between the resultants of the OP's and the OQ's, *i. e.* of the PQ's. There are certain cases where we can say at once that the difference of resultants is small, small that is relatively to the whole. This happens when all the P's are rather close together, *i e.* when the component vibrations have initially nearly the same phase. It is then certain that at the end of the time t the amplitude and phase are but little altered from what they were at the beginning. Over this range the vibration is approximately simple, and the range is inversely as the greatest departure from the mean frequency n.

But in general the distribution of initial phases ϵ causes the resultant to be much less than if the phases were in agreement, and it may even happen that the initial resultant is zero. At the end of the time t the resultant will probably not be zero, so that in this case the change is relatively large. The proposition that small changes in the phases of the components can lead only to relatively small changes in the resultant is thus not universally true ; and we must inquire further as to the conditions under which the conclusion is probable.

The most important case for our purpose is when the initial phases are distributed *at random*, as they would presumably be when Röntgen radiation is concerned. If the components are very numerous (and of equal amplitude unity), the problem is one which I have considered on former occasions *.

It appears that the probability of a resultant amplitude lying between r and $r+dr$ is

$$\frac{2}{m} e^{-r^2/m} r\, dr \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where m is the number of components. Or the probability of an amplitude exceeding r is $e^{-r^2/m}$. The *mean* intensity (when the phases are redistributed at random a great many times) is m, corresponding to the amplitude $\sqrt{m}$.

When r is great compared with $\sqrt{m}$, the probability of an amplitude exceeding r becomes vanishingly small. When on the other hand r is small, the probability of a resultant less than r is approximately r^2/m. It appears that the chance of the resultant lying outside the range from say $\frac{1}{4}\sqrt{m}$ to $2\sqrt{m}$ is comparatively small.

* Phil. Mag. vol. x. p. 73 (1880) ; 'Scientific Papers,' vol. i. p. 491 ; 'Theory of Sound,' 2nd ed. vol. i. § 42 *a*.

We have next to consider the resultant of the components PQ. Here again the phases are distributed in all directions. The amplitudes, however, are no longer equal, but they are small relatively to unity. Although the contrary is not impossible, it would seem that in all probability the resultant amplitude of the PQ's is small in comparison with that of the OP's, from which it follows that, exceptional cases apart, the amplitude and phase of the resultant remain but little changed at the end of a time t, such that the changes of phase of the individual components are small.

From the above discussion I am disposed to infer that a Fourier element of radiation necessarily possesses in large degree the characteristic which (if I rightly understand him) Prof. Larmor associates with white light in contrast to Röntgen radiation. Of course, after the lapse of a sufficient time the final phases of the components lose all simple relation to the initial phases. The final phase of the resultant is then without relation to the initial phase, and the amplitudes may differ finitely, but in all probability within somewhat restricted limits. From this variation it seems to me white light cannot be exempt.

In the above and, so far as I remember, in what I have written previously, the question is purely kinematical. In saying that Fourier's theorem is competent to answer any question that may be raised respecting the action of a dispersive medium, I take for granted that the law of dispersion is given in its entirety. I quite admit that if there are any wave-lengths for which the behaviour of the medium is unknown, a corresponding uncertainty must attach to the fate of any aggregate in which these are included. Doubtless a complete statement of the law of dispersion may involve the case of wave-lengths for which the medium is not transparent.

As regards the passage quoted from Sir G. Stokes, his object was, I think, to explain the absence of refraction when Röntgen rays traverse matter. Taking light of ordinary and absolutely definite wave-length incident upon transparent matter, he contemplates the lapse of 10,000 periods before harmony is established between the ætherial and molecular vibrations, that is, as I understand it, before regular refraction is possible. At this rate the light from a soda flame would be incapable of regular refraction, for the vibrations are certainly not regular for more than 500 periods. Indeed Stokes's argument appears better adapted to prove that Röntgen rays could not traverse material media at all in a regular manner, than that they would do so without change of wave-velocity.

I must confess that I have never fully understood Stokes's position in this matter. A medium is non-refractive and nearly transparent for the pulses constituting Röntgen rays. What reception would it give to simple waves of half wave-length equal to the thickness of the pulses? I should suppose that it would be non-refractive and transparent for these also, but Stokes's argument seems to imply the contrary. The paradox would then have to be met that the medium treats simple waves less simply than compound ones.

Nov. 16, 1905.

IX. *On an Instrument for Compounding Vibrations, with application to the drawing of Curves such as might represent White Light.* By Lord RAYLEIGH, O.M., F.R.S.[*]

[Plate III.]

IN discussions respecting the character of the curve by which the vibrations of white light may be expressed, I have often felt the want of some ready, even if rough, method of compounding several prescribed simple harmonic motions. Any number of points on the resultant can of course always be calculated and laid down as ordinates; but the labour involved in this process is considerable. The arrangement about to be described was exhibited early in the year during lectures at the Royal Institution. As it is inexpensive to construct and easily visible to an audience, I have thought that such a description might be useful, accompanied with a few specimens of curves actually drawn with its aid.

A wooden batten say 1 inch square and 5 feet long is so mounted horizontally as to be capable of movement only along its length. For this purpose it suffices to connect two points near the two ends, each by means of two thin metallic wires, with four points symmetrically situated in the roof overhead. This mounting, involving four constraints only, allows also of a rotatory or rolling motion, which could be excluded, if necessary, by means of a fifth wire attached to a lateral arm. In practice, however, this provision was not used or needed. The movement of the batten along its length is controlled by a piece of spring-steel against which the pointed extremity of the batten is held by rubber bands. Any force acting in

the direction of the length of the batten produces a displacement proportional to the force[*]. The tracing point, by which the movements are recorded, is at the other end, as nearly as possible in the line joining the two points of attachment of the four suspending wires.

The longitudinal forces are due to the vibrations of pendulums hanging from horizontal cross-pieces attached to the batten at their centres. The two ends of a wire or cord are attached to the extremities of a cross-piece, the bob of the pendulum being a mass of lead (perhaps half a pound) carried at the middle of the cord. When set swinging the movements of the pendulums are thus parallel to the batten and tend to displace it along its length. In my apparatus the length of the longest pendulum is $3\frac{1}{2}$ feet.

Under the influence of one pendulum the tracing point describes a small simple harmonic motion along the length of the batten. In order to draw a curve of sines the smoked glass destined to receive the record should move vertically in its own plane. I found it more convenient and sufficient for my purpose to substitute a movement of rotation. A disk (like the face-plate of a lathe) revolves freely in a vertical plane round a horizontal axis. To this disk a piece of smoked glass is cemented and the tracing is taken near the circumference, the axis of rotation being at the same level as the tracing point, so that the movement of vibration is radial.

The disk must be made to revolve slowly and with uniform angular velocity. To effect this I employed a sand-clock, a device which works better than would be expected [†]. The sand, carefully sifted and dried, is contained in a vertical metal tube of about 1 inch diameter, and escapes below through a small aperture of size to be determined by trial. On the sand rests a weight, of such diameter as to fit the tube easily; and this in its descent rotates the disk by means of a thread, of which the free part is vertical while the remainder engages a circumferential groove. The descent of the weight is practically independent of the quantity of sand remaining at any time [‡]. It is scarcely necessary to say that the revolving parts must be so weighted as to keep the thread tight.

The advantages of the apparatus depend of course upon the facility with which a number of vibratory movements

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[*] In strictness this presupposes the fulfilment of a condition involving the period of the force and that of free vibration under the influence of the spring, which it is scarcely necessary to enter upon.

[†] It was used by H. Draper to drive an equatorially mounted telescope.

[‡] See Note at end of paper.

can be combined *. It is as easy to record the effect of a
number of pendulums as of a single one, the contribution in
each case being proportional to the amplitude of vibration.
In my instrument there are six pendulums, the shortest of
such length as to vibrate about twice as quickly as the
longest. The frequencies are in fact somewhat as the numbers
5, 6, 7, 8, 9, 10. No precise adjustment was attempted,
the object being in fact rather to avoid anything specially
simple.

The lengths of the pendulums were chosen so as to afford
an illustration of the vibrations constituting white light. Of
course a complete physical representation of light from the
sun or from the electric arc would need a much larger range
of frequency. But we may suppose this light filtered through
media capable of sensibly absorbing the ultra-red and ultra-
violet, while still remaining white so far as the eye could tell,
even with the aid of a prism. The range of an *octave*, for
which provision is made, then amply suffices.

The number of pendulums may seem, and perhaps is, rather
small. The frequency, *e. g.* 7, given by one of the pendulums
must be taken to represent a range from $6\frac{1}{2}$ to $7\frac{1}{2}$, with an
error therefore up to 1 in 14. Such an error will be serious
after 7 vibrations, but not so for 3 or 4 vibrations. Hence if
we limit ourselves to sequences of 3 or 4 waves, the repre-
sentation is about good enough.

Connected with the above is the question what amplitudes
of vibration are to be assigned to the various pendulums. It
would not be difficult to give effect to an assigned law of
spectrum intensity whether suggested by theory or found in
observation. It is to be remembered, however, that such
laws relate to averages, and do not give the relative ampli-
tudes at any particular time, which will indeed vary fortuitously
over a rather large range. I thought it therefore unnecessary
to be very particular in this respect. The vibrations of the
shorter pendulums die down more rapidly than the slower
ones. By giving the former an advantage at starting a some-
what wide range is covered.

The tracings presented no general features that might not
have been anticipated. A few specimens are reproduced—
one showing the operation of the longest and shortest pen-
dulums alone, the others the effect of all the pendulums.

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* The principle of mechanical addition is employed in an instrument
devised by Michelson.

Note on the Principle of the Sand-Clock.

The difficulty of propelling a column of sand, occupying a tube, by forces pushing at one end is well known ; but I do not remember to have seen any discussion of the question on mechanical principles. A similar phenomenon occurs in the storage of grain, the weight of which, when contained in tall bins, is found to be taken mainly on the sides and but little on the bottom of the bin *.

The unexpectedness of these effects depends upon a half unconscious comparison with fluids which in a state of rest are exempt from friction. In the present case, when the sand is moving, the tangential force at the wall is reckoned at μ times the normal force. We may suppose, as a rough approximation, that there is something like a fluid pressure p. If a be the radius of the tube and dx an element of length along the axis, the tangential force acting upon the surface $2\pi a\,dx$ is $\mu p \,.\, 2\pi a \,.\, dx$. This is to be equated to the difference of the forces upon the two faces of the slice, viz. $-\pi a^2\,dp$. Accordingly

$$\frac{dp}{p\,dx} = -\frac{2\mu}{a},$$

or

$$p = p_0 e^{-2\mu x/a},$$

the pressure diminishing as a increases. Hence a powerful pressure at $x=0$ is unable to overcome a very feeble one acting in the opposite direction at a section many diameters away. The case is similar to that of a rope coiled round a post, as used to check the motion of steamers coming up to a pier.

As regards numbers, it will not be out of the way to suppose $\mu = \frac{1}{10}$. When $x = 10a$, $p/p_0 = e^{-2} = \cdot 14$.

Nov. 22, 1905.

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* I. Roberts, Proc. Roy. Soc. vol. xxxvi. p. 226, 1884. "In any cell which has parallel sides, the pressure of wheat upon the bottom ceases when it is charged up to twice the diameter of the inscribed circle."

X. *The Absorption of α Rays.* By R. K. McClung,
M.A., *Senior Demonstrator in Physics, McGill University,
Montreal* *.

THE question of the absorption of the α rays from radium
by matter has been discussed theoretically in a most
admirable manner by Prof. Bragg in a paper which appeared
in a recent number of the Philosophical Magazine †. The
theoretical considerations put forward in that paper were also
verified experimentally by Bragg and Kleeman ‡. From the
results set forth in these papers it was deduced that the
absorption of the α rays in air is mainly due to the fact that
their energy is used up in producing ionization, and it was
shown that these rays possess the power of ionizing the gas
only within a limited distance from the ionizing source.
This limiting distance was shown to be fairly sharply defined,
and was found to depend upon the nature of the source from
which the rays came.

The source from which Bragg and Kleeman obtained their
α rays was a very thin layer of radium obtained by crystalli-
zation from solution. This crystalline layer was, as they
clearly pointed out, not a single homogeneous source of
α rays, but contained several transformation products of
radium, each of which gave out α particles and some of
which gave out β and γ rays as well. In addition to this,
although the layer of radiating matter was very thin, still
the velocity of the rays proceeding from the lower parts of
the layer would be appreciably decreased by absorption before
emerging into the air, and consequently the source would be
giving out rays of different velocities.

As the source was not homogeneous the curves which they
obtained, as they clearly show, were somewhat more com-
plicated than if a single homogeneous source had been used.
Prof. Bragg suggested in his paper that polonium, which
gives out only α rays, might have been used if it had been
available.

It was therefore suggested by Prof. Rutherford that it
would be of interest to investigate this question, using as the
source of α radiation the radioactive matter deposited on a
wire which had been suspended in a vessel containing the
emanation from radium. Although the deposit on the wire
contains the three products radium A, B, and C, still the

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* Communicated by Prof. E. Rutherford, F.R.S.
† Phil. Mag. p. 719, Dec. 1904.
‡ Phil. Mag. p. 726, Dec. 1904.

radium A disappears in a short time after removal from the emanation, and radium B does not give out any rays at all. We have therefore practically a single homogeneous source of radiation in radium C.

Another advantage in using this source of radiation is that the layer of radiating material is infinitely thin, and consequently there is no absorption by the radiating matter itself. In addition, the source of radiation can be made as small as desired simply by using a very thin wire on which to concentrate the excited activity.

This source of radiation has proved very satisfactory, and the results which have been obtained are quite in accordance with those obtained by Bragg and Kleeman.

Although there are several advantages in using the excited activity as a source of radiation, there is one obvious disadvantage in its use, namely, that the activity dies away with time. However, in all the experiments described in this paper the rate at which the excited activity decayed was measured at the same time as the other observations were made, and the necessary corrections were easily made for the decrease in the activity of the source. This decay of the activity was therefore not at all a serious difficulty.

The method of experiment and arrangement of apparatus employed was very similar to that used by Bragg and Kleeman. A diagram showing the arrangement and details of the apparatus is given in fig. 1.

Description of Apparatus.

A B C D was a metal box which inclosed the plates between which the ionization was to be measured and also the source of the ionization. E was a zinc plate connected to one pair of quadrants of a Dolezalek electrometer in the usual way and was surrounded by a guard-ring, F F, which was connected to earth. H H was a wire gauze which was about ·5 cm. distant from the plate E, and was connected to one pole of a battery of accumulators, the other pole being to earth.

K was a brass block made in two sections and contained the source of radiation. The lower half was a solid piece of brass in the centre of the upper surface of which was cut a narrow groove about ·5 mm. in depth. The upper half consisted of a similar solid brass block about 5 mm. in thickness, having a hole 2·1 mm. in diameter passing vertically through the centre. The faces of these blocks were made true so as to fit tightly together. The wire on which was deposited the

excited activity was laid in the groove in the lower section of the block, and the upper half was then placed over it so that the radiations from the wire could only emerge through the hole in the upper block. A well-defined cone of rays was

Fig. 1.
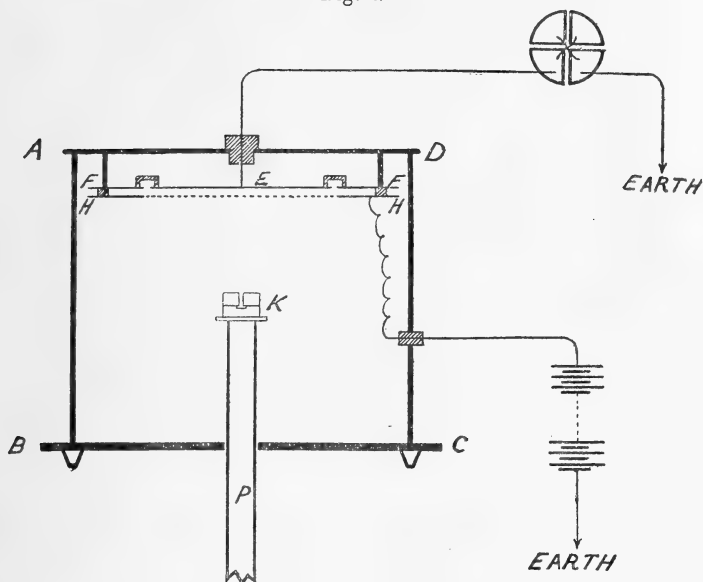
thus obtained. This brass block was placed in a definite position on an upright support, P, which could be adjusted vertically and the amount of adjustment measured on a fixed scale.

The plate E was made sufficiently large to include the entire cross-section of the cone of rays throughout the whole range of the adjustment of K.

Method of Observation.

The wire on which the radioactive matter was deposited was a thin copper one about 0·5 mm. in diameter. About an inch of this wire was attached to the negative pole of a battery of 400 volts, and suspended in the emanation from about 20 milligrams of radium bromide and allowed to remain there for a couple of hours, or longer, so as to get the maximum amount of excited activity upon it. It was then removed from the emanation and cut into short pieces and placed in the groove K. The rate of leak produced between the plates E and H was then measured for different distances of the source from the gauze H H.

During the time taken to make these measurements, the activity of the excited wire gradually decreased. It takes radium C, as has been shown by Rutherford, about 55 minutes to fall to half its maximum value after a long exposure in the emanation. This falling off in intensity of the source was corrected for as follows. The first observation after placing the excited wire in position was made at a definite distance of the source from the gauze, and the time carefully noted at which the observation was made. Then three or four observations were made at other distances, noting carefully the time at which each one was made, and then the source was replaced at the original position and the rate of leak measured again, noting the time. Then, after three or four more observations at various distances, the source was again brought back to the original position; and this routine was followed throughout the whole series of observations. The time at which each observation was made was carefully noted.

By thus observing the amount of ionization at intervals when the radioactive matter was placed at the fixed position, a decay curve, extending over the whole time of the series of observations, could be plotted. By observing the time at which each measurement was made at the other distances, the necessary correction could easily be made by referring to the corresponding time on the decay curve. The amount of ionization produced for each position of the source was thus obtained for a constant source.

Absorption of α Rays by Air.

The absorption of the α rays by air at atmospheric pressure was first investigated, and curves were obtained showing the amount of ionization produced for different distances of the source from the electrode H H. These curves were found to be quite in agreement with the results obtained by Bragg and Kleeman. One of these curves is shown in fig. 2, where the ordinates represent the distance in millimetres of the upper surface of the radiating wire from the lower surface of the wire gauze H H. The abscissæ represent the ionization as measured by the rate of leak in arbitrary scale-divisions per second.

This curve shows the same characteristics as were pointed out by the previous experimenters. It shows that, as the source recedes from the electrode, there is a very slow increase in the ionization until a distance of about 4 cms. is reached. Beyond this distance the ionization increases quite rapidly until a maximum is reached at a distance of about 5·8 cms.

thus showing, as Bragg has pointed out, that as the α particle moves farther from the source within a certain limit it becomes a better ionizer, although the velocity has become less after passing through a certain thickness of air.

Fig. 2.
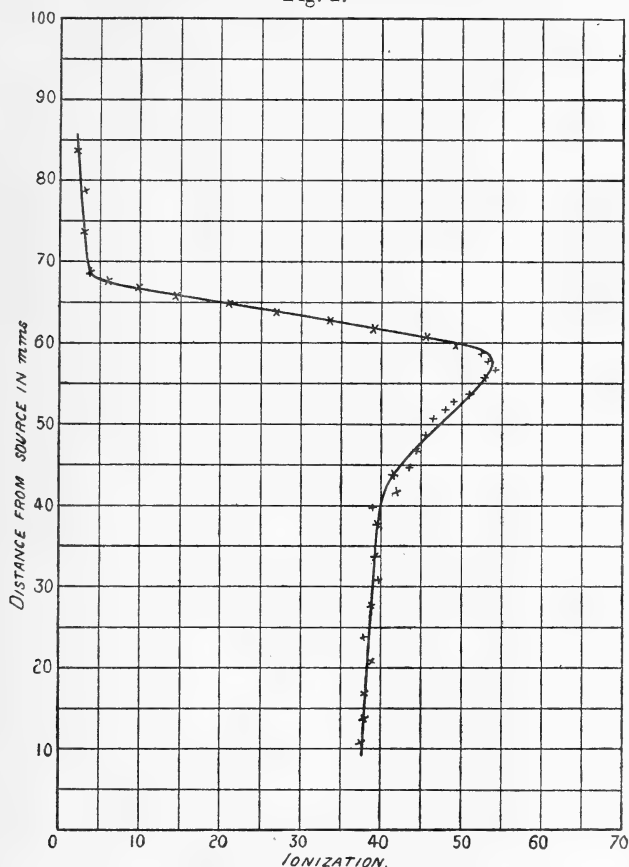
After the maximum point is reached and the distance increased, the ionization falls off extremely rapidly for a small increase in the distance until a distance of about 6·8 cms. is reached, when the ionization ceases altogether.

The part of the curve which rises rapidly from this point is due almost entirely to the natural leak of the vessel and the β and γ ray effects which are present.

The range in air over which the α ray is capable of producing ionization is therefore about 6·8 cms. This agrees very well with the result obtained by Bragg and Kleeman, who obtained a distance of 6·7 cms.

Rutherford, in a recent paper in the Phil. Mag.*, has also investigated the question of the distance at which the photographic and phosphorescent actions of the α rays cease; and he has found almost the same distances in those cases as the author found for the ionizing action.

The results shown by this curve indicate that as the velocity of the α particle from radium C decreases, the ionizing power of the particle very slowly increases over a certain range; then for a further decrease in the velocity the efficiency of the α particle as an ionizer increases quite rapidly until a maximum ionization is reached. When the velocity decreases beyond this point, the ionizing power of the α particle falls off extremely rapidly and very soon ceases altogether.

Rutherford (*loc. cit.*), in investigating this question of the velocity of the α particle from a photographic and phosphorescent point of view, has shown that when both the photographic and phosphorescent actions of the α rays cease, the particles still possess a considerable proportion of their original velocity of projection, and consequently quite a large amount of their kinetic energy. He also showed that the distance from the source at which these actions ceased corresponded very closely with the distances obtained by Bragg and by the writer for the cessation of the ionizing effect. These results therefore all appear to indicate that the α particle is not capable of producing either ionization, nor photographic nor phosphorescent effects when its velocity falls below a certain definite amount. This limiting velocity appears to be very well defined for the rays produced by a definite type of radioactive product.

We should expect from this point of view that when the point, at which the maximum ionization takes place, is reached a further increase of the distance between the source and the electrode should cause the ionization to cease abruptly. The curve, however, beyond the maximum point shows a decided upward slope instead of being quite horizontal. This upward slope of the curve is at least partly due no doubt to the fact that the angle of the cone of rays is of considerable size. When the distance between the source of the rays and the electrode is gradually increased, the air between the electrodes gets beyond the range of the extreme rays of the cone before it gets beyond the range of the central rays, as the extreme rays have to travel slightly farther, before reaching the electrodes, than the central rays. Therefore, as soon as this volume of air gets beyond the range of the extreme outside rays of the cone, the ionization will fall off, due to the fact that the width of the ionizing cone is decreasing and the volume of air acted upon is decreased.

* Phil. Mag. July 1905.

Absorption of α Rays by Aluminium.

The absorption produced by thin aluminium-foil was also investigated. The foil used was about 0·00031 cm. in thickness. The absorption of different thicknesses of aluminium was determined by measuring, in a manner similar to that for air, the ionization produced by the α rays after

Fig. 3.
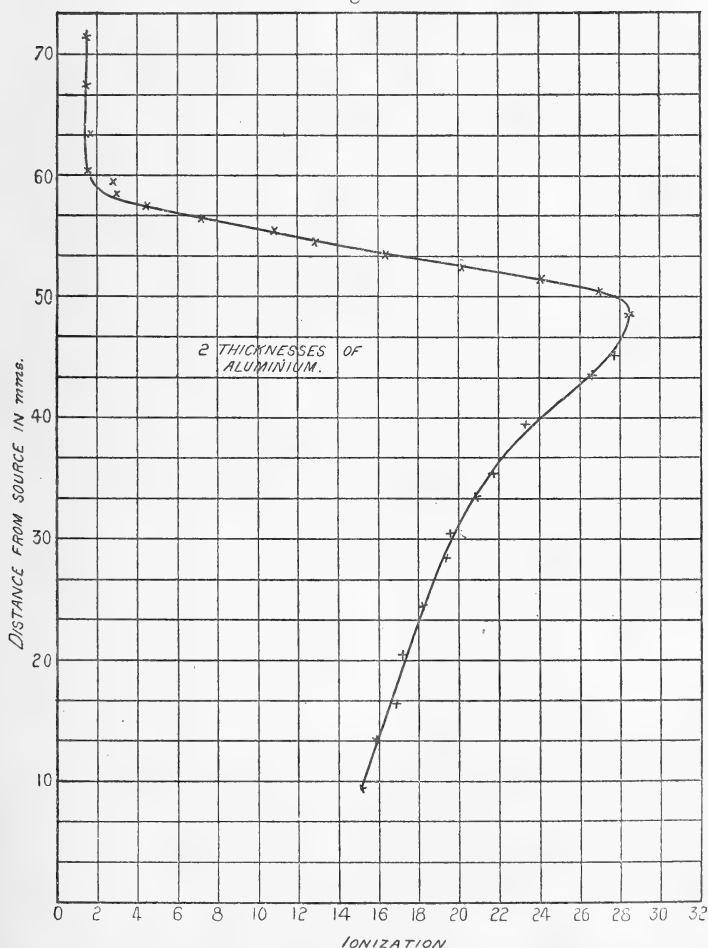
passing through a definite number of layers of the thin foil. Separate curves corresponding to the curve for air were obtained when various thicknesses of aluminium were placed in the path of the rays. These curves are shown in figs. 3, 4,

5, and 6. As will be observed, they are all exactly similar in character to the one for air.

Fig. 4.
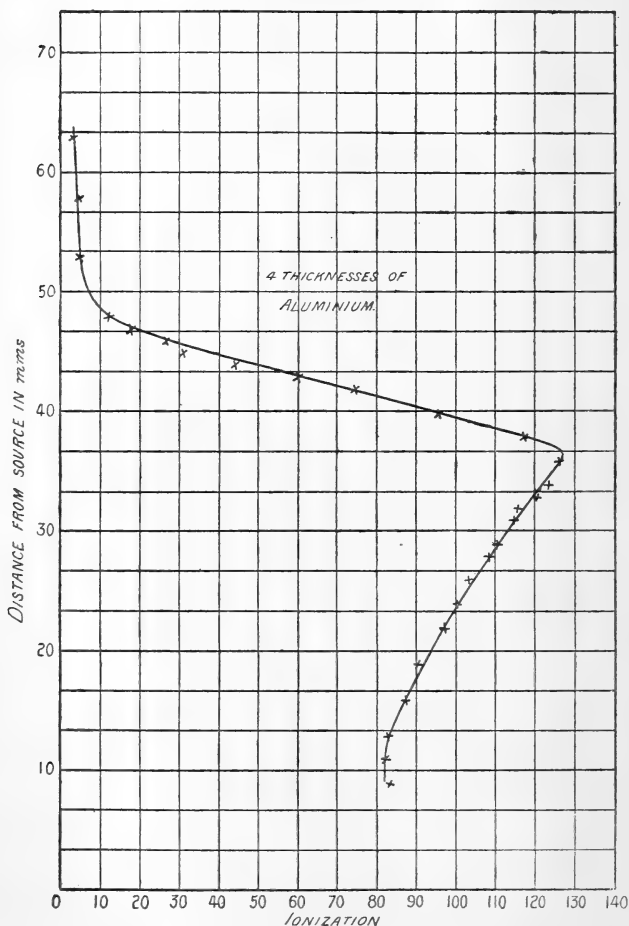
In each of these curves, as in the case of air, there are two characteristic points, namely, that at which the maximum ionization occurs and that at which the α particles cease to ionize. The range for each of these points is, of course, less than for the corresponding points for air, and the range naturally decreases the greater the absorbing thickness of aluminium. The distances at which the maximum ionization

occurs and at which the ionization ceases for different thick-
nesses of aluminium are given in Table I.

Fig. 5.
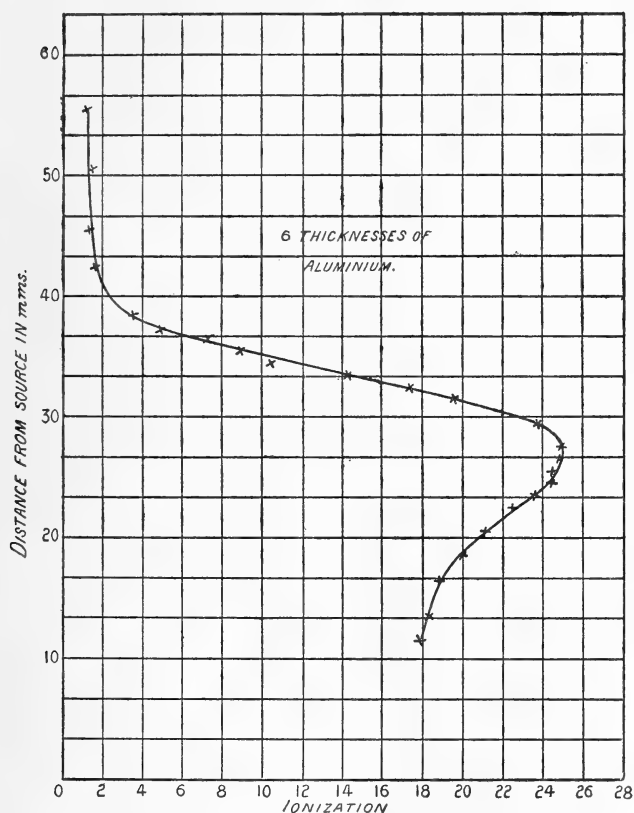
Table I.

Thickness of Aluminium (1 layer = ·00031 cm.).	Distance between Source and Gauze at which maximum ionization occurs.	Distance between Source and Gauze at which ionization ceases.
2 layers.	4·8 cms.	5·8 cms.
4 ,,	3·6 ,,	4·8 ,,
6 ,,	2·7 ,,	3·9 ,,
8 ,,	1·5 ,,	2·6 ,,

As seen from the diagram, only a very small portion of the curve below the maximum point can be obtained for a thickness of aluminium greater than eight layers, since the maximum point for a thickness beyond this occurs so close to

Fig. 6.
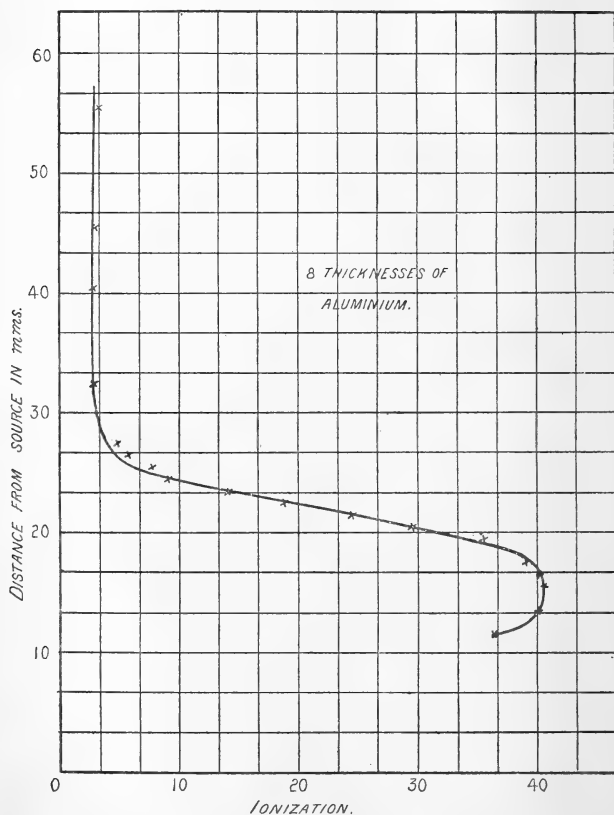
the gauze. It is not practicable to place the source of activity closer to the gauze than about 6 mm., as the hole through which the rays emerge must be at least about 5 mm. in depth in order that the angle of the cone of rays may not be too large.

Comparison of the Absorption by Aluminium and Air.

From these curves for air and aluminium the absorption of the α rays by these substances can easily be compared by

considering either the point of maximum ionization, or the point at which the ionization ceases. Consider, for instance, the point at which the ionization ceases in the case of air and in the case when the rays pass through two layers of aluminium. The distance between the source and the gauze at which the ionization ceases is decreased in the second instance. This, of course, is due to the simple fact that when the rays emerge from the aluminium their velocity has already been reduced to such an extent by absorption that they do not now require to travel so far in air before their velocity is reduced to the point at which they cease to ionize. The differences in the air-spaces in the two cases must be equivalent to the two layers of aluminium in absorbing-power. Referring to the curves in figs. 2 and 3, the maximum ranges in the two cases are 6·8 cms. and 5·8 cms. respectively. A thickness of air of 1·0 cm. must therefore absorb as much of the α rays as two layers of aluminium. Therefore, assuming that each layer produces the same amount of absorption, one layer of aluminium ·00031 cm. in thickness absorbs as much α radiation as ·5 cm. of air.

The same result ought to be obtained if we consider the point of maximum ionization in the two cases. This point in air is at a distance of about 5·8 cms., while when two layers of aluminium are interposed the maximum point is at a distance of 4·8 cms. The difference in the two cases divided by 2 is ·5, which agrees with the result obtained from considering the point at which the ionization ceases. Similar results are obtained when the other curves for the various thicknesses of aluminium are considered. These results are given in Tables II. and III.

TABLE II.

Absorbing Medium.	Distance at which Ionization ceases.	Thickness of Air corresponding to given thickness of Aluminium.	Thickness of Air corresponding to one layer of Aluminium.
Air	6·8 cms.		
2 layers of Al. ...	5·8 ,,	1·0 cm.	·50 cm.
4 ,, ,, ...	4·8 ,,	2·0 cms.	·50 ,,
6 ,. ,, ...	3·9 ,,	2·9 ,,	·48 ,,
8 ,, ,, ...	2·6 ,,	4·2 ,,	·52 ,,

TABLE III.

Absorbing Medium.	Distance at which maximum Ionization occurs.	Thickness of Air corresponding to given thickness of Aluminium.	Thickness of Air corresponding to one layer of Aluminium.
Air	5·8 cms.		
2 layers of Al. ...	4·8 ,,	1·0 cm.	·50 cm.
4 ,, ,, ...	3·6 ,,	2·2 cms.	·55 ,,
6 ,, ,, ...	2·7 ,,	3·1 ,,	·51 ,,
8 ,, ,, ...	1·5 ,,	4·3 ,,	·53 ,,

From these curves we see that we can readily compare the absorptive power of aluminium with that of air, and obtain the thickness of air which is equivalent in absorbing-power to any given thickness of aluminium.

These results are in very fair agreement with those obtained by Rutherford, using the photographic and the phospho-rescent-screen methods. He used specimens of aluminium-foil similar to those used by the writer, and therefore the results of the two investigations may be compared. He found by the photographic and the phosphorescent-screen methods that one layer of aluminium-foil was equal in absorbing-power to ·54 cm. of air. He obtained this result by measuring the distance from the source at which the α particle ceased to affect a photographic plate or a phos-phorescent screen after passing through various thicknesses of aluminium.

The close agreement of the results obtained in the two separate investigations is an indirect confirmation of the hypothesis put forward by Rutherford in his paper (*loc. cit.*) in regard to the nature of the action produced by the α rays from radium on a photographic plate or a phosphorescent screen. These two sets of results taken together appear to point strongly to the conclusion that the ionizing, photographic, and phosphorescent effects of the α rays from radium are all due to the same cause, and it is very probable that in each case the effect produced is primarily one of ionization.

In conclusion I wish to express my indebtedness to Prof. Rutherford, at whose suggestion the subject of this research was undertaken, for his valuable suggestions during the progress of the investigation.

McDonald Physics Building,
    McGill University,
        Montreal, July 21st, 1905.

XI. *The Effect of High Temperatures on the Rate of Decay
of the Active Deposit from Radium.* By HOWARD L.
BRONSON, *Ph.D.**

A PRELIMINARY account of the method and some of
the results obtained have already been given in the
American Journal of Science for February and July 1905.
As the method is new and has proved very satisfactory in
obtaining the decay curves of radioactive substances, it is
here given in some detail.

Apparatus and Method.

In the ordinary method of comparing ionization currents
by an electrometer, the rate of movement of the needle is
taken as a measure of the current. In doing this, it is
assumed that the capacity of the system and the lag of the
needle behind the potential is the same for different rates.
These assumptions may, in some cases, be warranted, but
they certainly are not when the currents compared differ
greatly in magnitude, or when the needle is moving rapidly.
In the latter case capacities may be added in parallel, which
reduces the rate of movement, but creates the difficulty of
comparing capacities. This takes time and is never entirely
satisfactory. A still greater difficulty is experienced with
the " rate " method, when rapidly changing ionization cur-
rents are to be measured, because, from the nature of the
method, the observations can be neither instantaneous nor
taken in rapid succession. The desirability, therefore, of a
more direct and rapid method of measurement is evident.

The following method, suggested by Professor Rutherford,
has been found by the writer to be very convenient and
satisfactory for all radioactive measurements thus far
attempted. The theory of the method is as follows :—

The arrangement of the apparatus, as seen in fig. 1 (p. 144),
differs only from that ordinarily used in having one pair of
quadrants of the electrometer connected not only to the testing
vessel, but also to earth through a very high resistance. It is
evident in this case that, when the key is closed, the ioniza-
tion current from the testing vessel will continue to increase
the potential-difference of the quadrants until the conduction
current through the high resistance becomes equal to the
ionization current. Since this conduction current is propor-
tional to the potential-difference, it follows that, when equi-
librium is reached, the ionization current also will be pro-
portional to it. If the electrometer is well adjusted, the

deflexions, when small, will be proportional to the same potential-difference of the quadrants. This makes a simple direct-reading instrument.

Fig. 1.
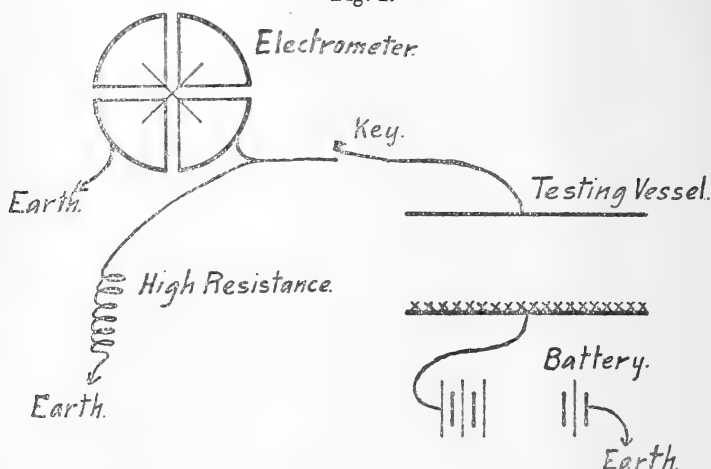
Resistances of the order of 10^{11} ohms made of amyl alcohol and carbon on glass were tried, and gave results which compared favourably with those obtained by the " rate " method. The results, however, were not satisfactory, because the resistances did not remain sufficiently constant.   This was probably due in one case to polarization, and in the other to temperature changes.   Professor Rutherford then suggested the possibility of using an ionization current in place of the conduction current through a high resistance.   The possibility of doing this is due to the fact that the ionization current through a gas, subject to a constant source of ionization, is proportional to the potential-difference between the plates when this potential-difference is small.   Any strongly radioactive substance which will remain approximately constant during the time of a single experiment can be used as the source of ionization.   In the present case a very radioactive bismuth plate from Dr. Sthamer of Hamburg was used.   The activity of this plate was due to a deposit of the so-called radio-tellurium of Marckwald, and has the advantage of not giving out any penetrating rays.   If, for example, radium had been used, it would have been necessary to surround the vessel in which it was placed with a shield of lead of considerable thickness, in order to prevent any disturbance by the penetrating rays.

The advantages of this " constant deflexion " method are obvious: deflexions are independent of the capacity, and testing vessels may therefore be interchanged without error, measurements can be made over a wide range without any difficulty, and observations can be taken instantaneously and in as rapid succession as desirable. In some cases they were taken at five-second intervals.

The precision of an individual measurement is, however, limited by a slight oscillation of the needle. Repeated attempts have been made to eliminate this by better shielding from external electrostatic action, and by careful attention to contacts, but all to no purpose. The most probable explanation seems to be that it is due to exceedingly small and rapid changes in the ionization current itself. This difficulty, however, is not serious, for it seldom causes an error in a single observation of as much as one per cent.

In the actual carrying out of the experiments the following arrangement of apparatus was found to be the most convenient and satisfactory :—

Fig. 2.
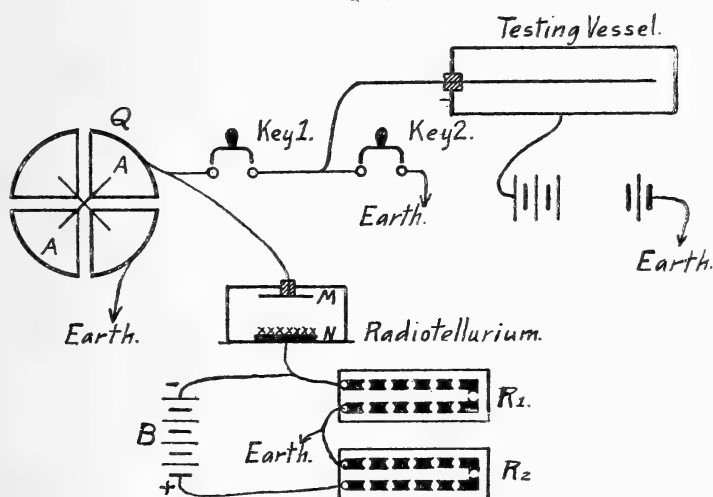
The electrometer was of the ordinary quadrant type, except that it was fitted with a very light needle of silvered mica, suspended by a fine phosphor-bronze wire. The needle was permanently connected to a battery of small accumulators, which kept its potential constant. As ordinarily used the needle was kept at a potential of 85 volts, and gave a deflexion of 15 cm. for a potential-difference of 1 volt

between the quadrants. By the use of a finer suspension-wire and a higher voltage on the needle, it was possible to increase this sensitiveness nearly one-hundred fold.   The scale was 100 cm. in length and only 180 cm. from the electrometer, but it was possible, by turning it at a slight angle, to make the deflexions over its whole length proportional to the potential-difference of the quadrants, with a maximum error of not more than one half of 1 per cent.   In case the current to be measured produced a deflexion larger than 100 cm., a modification of this arrangement was adopted. By means of the battery B and the two resistances R_1 and R_2, the potential of the radio-tellurium plate was brought below that of the earth.   This reduces the potential-difference between the quadrants by the same amount, and thus makes the deflexions measurable on the scale.   In this way ionization currents large enough to produce a deflexion of 150 cm. could be measured.   At this point a limit was reached, because larger currents than this through the radio-tellurium vessel were not proportional to the potential-difference between the plates. No doubt with a more active substance, such as radium bromide, the currents would remain proportional through a still larger range.

Both the radio-tellurium and the plate M above it were covered with very thin aluminium-foil, to eliminate as far as possible any contact potential-difference ; and the whole was enclosed in a metallic vessel to prevent electrostatic or air disturbances.   In order to correct for the small contact potential-difference, which always remained, an extra key was introduced between the testing vessel and the electrometer.   When both keys were closed the testing vessel and all four quadrants of the electrometer were connected to earth ; when key 2 was open the earth connexion was broken and the testing vessel was connected with the electrometer ; when key 1 was open the earth connexion of quadrants A A was broken, and the reading then obtained was the true zero, and eliminated any error due to contact potential-differences. The entire apparatus was carefully surrounded by a metallic screen to eliminate any electrostatic disturbance, the keys being mechanically worked from without.

Results obtained by Curie and Danne.

In the course of a careful investigation of the rate of decay of the active deposit from radium, an attempt was made to see whether it was permanently changed by high temperatures. The conclusion arrived at, contrary to that of Curie and Danne[*],

* *Comptes Rendus,* cxxxviii. p. 748 (1904).

is that temperatures between 700° and 1100° C. do not permanently alter the rate of decay of the active deposit.

Miss Gates * showed that when a platinum wire covered with the active deposit from radium is raised to a white heat, the active matter is removed from the platinum wire and deposited on cooler bodies in the neighbourhood. Curie and Danne still further extended our knowledge by showing that the constituents of the active deposit were not equally volatile. In addition, they also found that the rate of decay had apparently been permanently altered by the high temperature. The following table gives some of their results :—

t.	*θ.*
630°	29·3
830	24·6
1000	21·0
1100	20·3
1250	24·1
1300	25·4

Here t is the temperature in degrees centigrade to which the active deposit was raised, and θ is the " period," that is the time in minutes required for the activity to fall to half value. Curie and Danne also state that the curves were all exponential. They conclude from this that the rate of decay has been permanently altered, and that, as the temperature is raised, the period reaches a minimum, at about 1100° C., and then begins to increase again. It was not unnatural to expect that high temperatures should decrease the period of the active deposit, but it seemed very remarkable that the period should reach a minimum and then increase again when still higher temperatures were used.

Results of the present Research.

It seemed possible that all the results of Curie and Danne might be accounted for by the difference in volatility of the constituents of the active deposit. In order to see whether this were the case, a copper wire on which the active matter had been deposited was sealed in a piece of glass combustion-tubing, which entirely prevented the escape of any volatile products. By exhausting the tube before sealing, it was found that it would stand temperatures high enough to melt the copper wire inside, that is, temperatures of at least 1100° C.

In all cases where high temperatures were employed the

* Physical Review, May 1903.

wires were heated in a small electric furnace, made by Dr. C. A. Timme of Berlin. A calibration curve for the furnace had previously been made by the use of a platinum-rhodium thermo-junction. The wires were left in the furnace different lengths of time in different experiments, but this seemed to have very little effect on the result. On removal from the furnace, the glass tube containing the wire was covered with thin aluminium-foil and placed in the bottom of a cylindrical testing vessel with a central electrode. In this case the ionization was produced by the β rays alone. In those cases where the wire was heated without being sealed in a glass tube, the wire itself was made the central electrode of the testing vessel. Sometimes it was covered with a sufficient thickness of lead-foil to cut off all the α rays, but generally it was left uncovered, in which case the ionization was largely produced by the α rays. As the curves were the same in either case, it will be unnecessary to mention which method was taken in any particular experiment.

B, fig. 3, is the logarithmic decay curve, obtained when

Fig. 3.
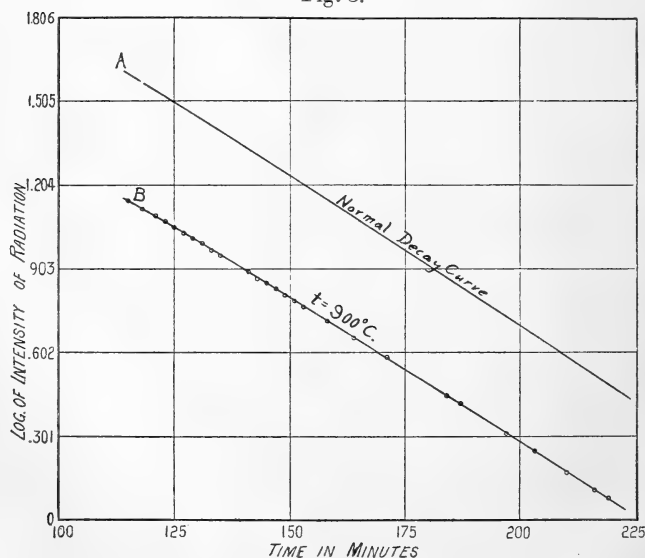
the active deposit was sealed in a glass tube and heated to 900° C., and A is the decay curve for the active deposit which has not been subjected to a high temperature. These curves are approximately parallel, showing that the rate of decay was not measurably changed by a temperature of 900° C.

A large number of experiments were made in this manner using temperatures varying from 700° to 1100° C., but in no case did the period fall below twenty-six minutes.

If then, as the above experiments seem to show, the results of Curie and Danne are to be explained by the volatilization of one or more of the constituents of the active deposit, we should not expect to find the period of that part of the active matter remaining on the wire a function merely of the temperature, for other conditions besides temperature ought to affect the amount of volatilization, and this was found to be the case. For example, the removal of the volatile products, by blowing a current of air through the furnace or by inserting a cold copper rod for a few seconds before removing the platinum wire, always diminished the period of the matter remaining on the wire.

Fig. 4 is a fair example of the results obtained when the active matter is deposited on a platinum wire and heated in the furnace.

Fig. 4.
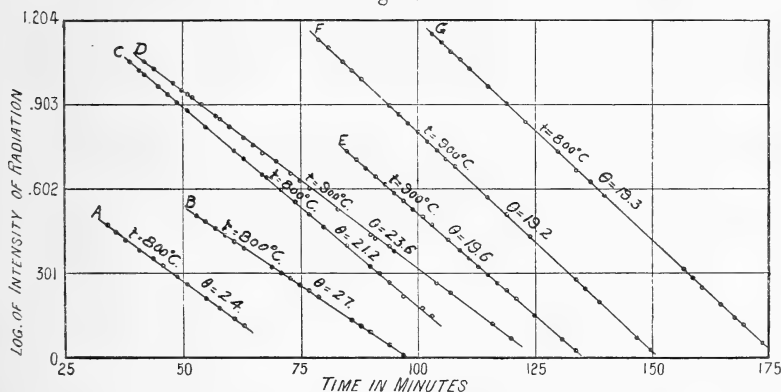
A, B, C, and G are four decay curves of the active deposit, which had been previously heated to about 800° C. In the case of the other three curves, the temperature used was about 900° C. The time is reckoned from the removal of the wire from the emanation. The first points on curves F and G should be at about 200 and 300 minutes respectively, as in these cases the wires were not placed in the furnace for several hours after their removal from the emanation. As was expected, the period was found to vary considerably, but it was always smallest when care was taken to remove the volatile matter from the furnace before removing the wire. A large number of other curves were taken both at lower

and higher temperatures, but always with similar results. There was, however, one fact very noticeable among all the curves taken, especially among those where care was taken to remove the volatile products, namely, that a large number of them had periods between nineteen and twenty minutes. The only explanation of this seemed to be that there was left on the wire in these cases a simple radioactive substance, which gave out "rays" and decayed to half value in about nineteen minutes.

Now Rutherford * has shown that, neglecting the first half-hour, the decay curve of the active deposit from radium is satisfactorily explained by assuming two successive products, radium B and radium C : the matter B giving rise to no rays, and the matter C to α, β, and γ rays. Taking twenty-eight minutes as the period of one of these, he calculated that the period of the other must be twenty-one minutes. Theoretically it makes no difference whether the longer period belongs to the matter B or C, but the above-mentioned experiments of Curie and Danne supplied the evidence that caused him to decide that the longer period belonged to radium C.

The results given in this paper, on the contrary, seem to furnish conclusive evidence that radium B has the longer period. In this case radium becomes analogous to thorium and actinium, each of which has a rayless change of longer period than the change immediately following.

There is still one point to be explained. Curie and Danne state that their curves are exponential, and the curves in fig. 4 would seem to confirm this. Now if the different periods obtained are due to a mixture of two substances in various proportions, one of which gives rays and has a period of nineteen minutes, and the other of which gives no rays and has a period of twenty-six minutes, then it is evident that the rate of decay of the mixture must keep decreasing, because the matter having the longer period decays more slowly, and therefore the proportion of it present in the mixture must keep increasing. In order to settle this point, the decay of the activity of the heated deposit was measured over a long period.

Fig. 5 shows the result of two experiments of this kind. 1 A, 1 B, and 1 C are three sections of the same curves, obtained after heating the active deposit to about 650° C. 1 A was taken immediately after heating, 1 B after about two hours, and 1 C after about four hours. The respective values of the period were **22·4**, **23·6**, and **25·5** minutes. In the case

* Philosophical Transactions, vol. cciv. p. 196.

of curve 2, the temperature was about 800° C. 2A was taken immediately after heating, 2 B after about one hour, and 2 C after about two and a half hours. The values obtained for

Fig. 5.
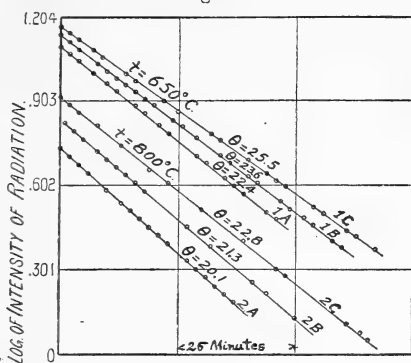
the period were respectively 20·1, 21·3, and 22·8 minutes. Thus it is seen that the curves obtained after heating the deposit were not exponential but had a continually increasing period.

Decay Curve of the Excited Activity from Radium.

Although 19·2 and 19·3 minutes, calculated from F and G (fig. 4), are the lowest values obtained for the period of radium C, yet even in these cases it would seem probable that all of radium B had not been removed, and therefore that the true period of C was not over 19 minutes. It would seem of interest to check the above value of the period of radium C by comparing the experimental decay curve with the theoretical curve, calculated on the assumption that the period of radium C is nineteen minutes. The period of the final part of the decay curve of the active deposit from radium is usually taken as about twenty-eight minutes. This is about the correct value for the period two and a half hours after the removal of the active deposit from the presence of the emanation. If, however, the period is measured after five or six hours, it will be found to be considerably less.

Fig. 6 shows the final part of four decay curves. The first point on each curve is more than four and a half hours after the removal of the active deposit from the emanation; yet even in these cases the curves are not exactly exponential. These curves give 26·1 minutes as the mean value of the period of the active deposit between five and seven hours after its removal from the emanation. From this it can be

easily calculated that the final period of decay, that is the period of radium B, is about 25·8 minutes.   The experimental results are not accurate enough to make the last figure

Fig. 6.

LOG OF INTENSITY OF RADIATION.
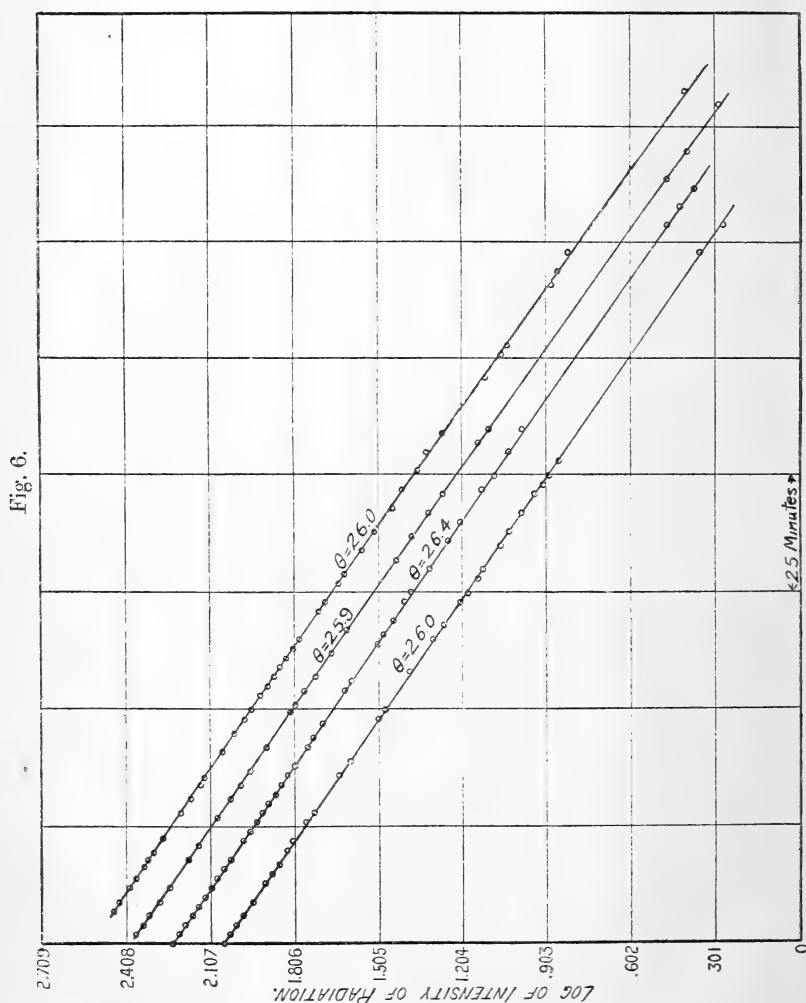
certain, and a mean of a large number of other values obtained would indicate that 25·8 minutes may be a trifle too small. It seems certain, however, that the period of radium B is very nearly 26 minutes.

The theoretical values in the following table were obtained by assuming that only two changes take place in the active

deposit, and that the periods of these are twenty-six and nineteen minutes respectively.

Time.	Long Exposure.		Short Exposure.	
	Observed Value.	Calculated Value.	Observed Value.	Calculated Value.
50	52·9	53·1	3·57	3·61
60	43·4	43·7	3·19	3·17
70	35·7	35·7	2·75	2·73
80	28·9	28·8	2·31	2·29
100	18·5	18·5	1·56	1·54
120	11·6	11·6	1·01	1·00
140	7·2	7·2	·64	·63
160	4·4	4·3	·395	·393
180	2·63	2·63	·241	·242
200	1·59	1·58	·148	·147
220	·95	·95	·089	·089
240	·56	·57	·054	·054
260	·33	·33	·032	·032
280	·19	·20	·0186	·0191

The early part of the curve, which is especially affected by the first rapid change in the active deposit, is still under investigation. When this is completed it may be found necessary to take a value for the period of radium C slightly smaller than nineteen minutes, in order to make the theoretical agree as well as possible with the experimental curve. In any case, the above close agreement between the experimental and theoretical curves, added to the separate measurement of both periods, would seem to furnish conclusive evidence that twenty-six and nineteen minutes are approximately the correct values for the periods of radium B and C respectively.

Conclusions.

The above experiments would seem to prove :—

(1) That temperatures between 700° and 1100° C. do not permanently affect the rate of decay of the active deposit from radium.

(2) That radium B, and not radium C, has the longer decay period.

(3) That the previous values of twenty-eight and twenty-one minutes are both too large for the decay periods of radium B and C respectively, and that twenty-six and nineteen minutes are much closer to the true values.

In conclusion, I take pleasure in expressing my indebtedness to Professor Rutherford for his many valuable suggestions and kind supervision of this work.

Macdonald Physics Building,
McGill University, Montreal, Sept. 25, 1905.

XII. *On the Diffraction Theory of Microscopic Vision.* By ALBERT B. PORTER [*].

1. A LTHOUGH thirty-two years have passed since Professor Ernst Abbe [†] proposed his diffraction theory of microscopic vision, it is still to some extent a matter of controversy among microscopists [‡], and is perhaps less familiar to physicists than its importance warrants. The theory may be briefly stated in the following form. If a lens is to produce a truthful image of an illuminated object, it must have an aperture sufficient to transmit the whole of the diffraction pattern produced by the object ; if but part of this diffraction pattern is transmitted, the image will not truthfully represent the object, but will correspond to another (virtual) object whose whole diffraction pattern is identical with that portion which passes through the lens ; if the structure of the object is so fine, or if the aperture of the lens is so narrow, that no part of the diffraction pattern due to the structure is transmitted by the lens, then the structure will be invisible no matter what magnification is used. Abbe and others have devised a number of interesting experiments [§] to illustrate the theory, but the complete mathematical development has never been published [||].

2. The particular case in which the object is a transmission grating consisting of alternate opaque and transparent lines may, however, be treated by means of a simple application of Fourier's theorem. Let a and b be respectively the widths of the transparent and opaque lines on the gratings, and let A be the amplitude of the (monochromatic) light which will be assumed to fall upon the grating at perpendicular incidence ; then the distribution of amplitude in the light passing through the grating will be as shown in fig. 1. The function represented by this curve may be developed in a cosine series by means of the formula

$$f(x) = \tfrac{1}{2}b_0 + b_1 \cos \frac{\pi x}{c} + b_2 \cos \frac{2\pi x}{c} + \ldots + b_m \cos \frac{m\pi x}{c} + \ldots .$$

in which $\qquad b_m = \dfrac{2}{c} \displaystyle\int_0^c f(x) \cos \frac{m\pi x}{c}\, dx.$

Fig. 1.
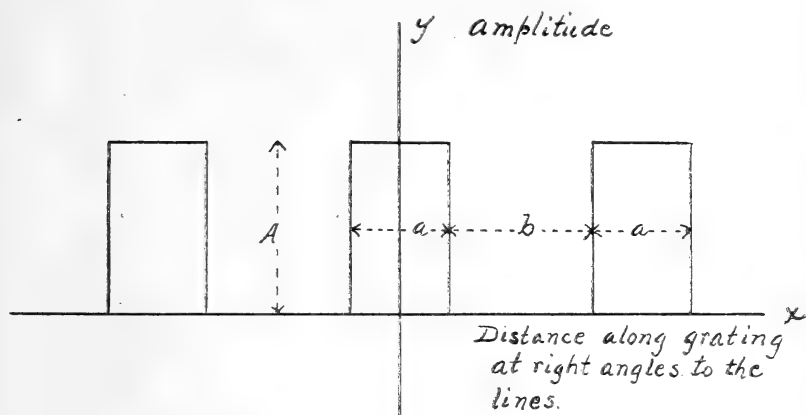
In the case under consideration

$$f(x) = \text{A from } x = 0 \text{ to } x = \frac{a}{2},$$

$$= 0 \text{ from } x = \frac{a}{2} \text{ to } x = \frac{a}{2} + b,$$

$$= \text{A from } x = \frac{a}{2} + b \text{ to } x = a + b,$$

and $c = a + b.$

Hence

$$b_m = \frac{2}{a+b} \left[\int_0^{\frac{a}{2}} \text{A} \cos \frac{m\pi x}{a+b} dx + 0 + \int_{\frac{a}{2}+b}^{a+b} \text{A} \cos \frac{m\pi x}{a+b} dx \right]$$

$$= \frac{2}{a+b} \int_0^a \text{A} \cos \frac{m\pi x}{a+b} dx = \frac{2\text{A}}{m\pi} \sin \frac{m\pi a}{a+b}.$$

It follows that

$$b_0 = \frac{2\text{A}a}{a+b}, \quad b_1 = \frac{2\text{A}}{\pi} \sin \frac{\pi a}{a+b}, \quad b_2 = \frac{2\text{A}}{2\pi} \sin \frac{2\pi a}{a+b}, \text{ \&c.} \quad . \quad (1)$$

and

$$f(x) = \frac{\text{A}a}{a+b} + \frac{2\text{A}}{\pi} \sin \frac{\pi a}{a+b} \cdot \cos \frac{\pi x}{a+b} + \frac{2\text{A}}{2\pi} \sin \frac{2\pi a}{a+b} \cdot \cos \frac{2\pi x}{a+b} + \ldots$$

$$+ \frac{2\text{A}}{m\pi} \sin \frac{m\pi a}{a+b} \cdot \cos \frac{m\pi x}{a+b} + \ldots \quad (2)$$

Now the ordinary theory of the transmission grating shows that, in the case under consideration, the amplitude of the light in the central image is

$$\frac{\mathrm{A}a}{a+b}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \quad \cdot \quad \cdot \quad (3)$$

and that the amplitude in the mth spectrum is

$$\frac{\mathrm{A}}{m\pi} \sin \frac{m\pi a}{a+b}. \quad \cdot \quad \cdot \quad \cdot \quad \quad \cdot \quad (4)$$

A comparison of (2) with (3) and (4) shows that the first term in (2) represents the amplitude in the central image, while the coefficient of the cosine in each succeeding term represents the sum of the amplitudes in the two spectra of corresponding order. It thus appears that a diffraction-grating performs a double process of harmonic analysis. In the first place it analyses the incident radiation according to wave-length in the well-known manner, distributing the colours in order in each spectrum ; in the second place, as shown in equations (2), (3), and (4), it analyses the distribution of wave-amplitude in its own plane, distributing the Fourier components of the amplitude curve in order among the successive spectra.

3. We may also look at the matter from another point of view. The curve drawn in fig. 1 not only represents the distribution of wave-amplitude in the plane of the grating, but is also the curve which shows the distribution of *transparency* over the surface of the grating itself, the axis of x representing zero transparency, *i. e.* complete opacity, and the height A perfect transparency. (To avoid circumlocution, transparency is defined throughout this paper in terms of the *amplitude* of the transmitted light, not in terms of intensity.) Equation (2) is evidently the development of this transparency curve as a series of harmonic distributions of transparency. The non-periodic term corresponds to a surface of uniform, but imperfect, transparency ; while each periodic term represents a surface covered with equidistant parallel bands of varying transparency. Supposing these bands to run perpendicular to the plane of the paper, the distribution of transparency in them will be indicated by the ordinates in fig. 2, where the axis of x again represents perfect opacity, the ordinate B a certain degree of transparency, and the ordinate $-$B represents an equal degree of what we may term negative transparency, *i. e.*, transparency coupled with a half-period change of phase in the transmitted light. A surface of this sort may be called a simple harmonic grating or,

briefly, a *simple grating*.   A simple grating would give spectra
of the first order only, and no central image.   Such a grating

Fig. 2.
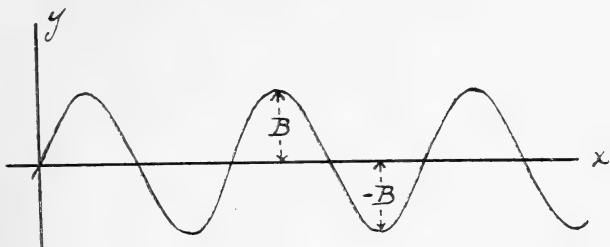
could perhaps be realized by covering the alternate transparent
bands on a harmonically shaded plate with strips of a trans-
parent film giving a half wave-length retardation.

4. It now follows immediately from equation (2) that the
grating of fig. 1, and, indeed, in the more general case, any
opacity grating ruled with n lines per unit length, may be
considered to be formed by the superposition, on an imper-
fectly transparent surface, of a series of simple gratings ruled
with $n, 2n, 3n, 4n$, &c. lines per unit length.   The imperfectly
transparent surface is responsible for the central image, while
each of the simple gratings gives rise to the spectra of one
order, the amplitude of the light in each spectrum being
proportional to the amplitude of the transparency curve of
the corresponding simple grating.   Furthermore, it is easily
seen that the necessary and sufficient condition for the absence
of the spectra of any given order is the absence of the corre-
sponding simple grating.

5. The theory here briefly indicated may be extended to
include retardation gratings, both transparent and reflecting,
but this development is beyond the scope of this paper.   One
deduction from the theory may, however, be mentioned
because of the ease with which it can be verified experimentally.
The sharply ruled grating of fig. 1 consists of an infinite
series of simple gratings which give a fan of spectra on each
side of the central image.   If the sharp corners in fig. 1 are
rounded, that is to say if the lines of the grating are blurred,
Fourier's theorem indicates the absence of the higher
harmonic terms in equation (2), and hence also the absence
of the more finely spaced simple gratings and the corresponding
spectra of higher orders.   When the lines are much blurred,

the shading will approximate that indicated in fig. 3, which represents the transparency curve of a grating formed by

Fig. 3.
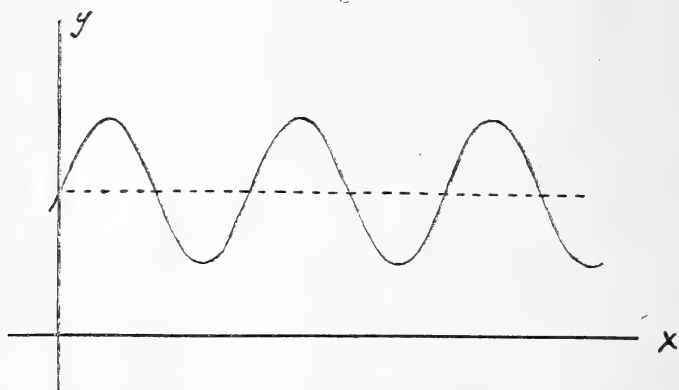
superposing a simple grating on an imperfectly transparent surface. Such a grating should evidently give a central image and spectra of the first order only. This point was tested in the following manner. A glass grating by Max Levy, with 400 very sharply ruled black lines to the inch, was held against a photographic dry plate so that one edge of the grating was in contact with the film, while the opposite edge was separated from the film by interposing a strip of thick paper. During exposure to the light of an incandescent lamp, the grating and plate were kept oscillating about an axis parallel to the lines of the grating. When examined under a microscope, the lines near one edge of the developed plate are seen to be only moderately blurred, while there is much more blurring toward the other edge of the plate. On looking at an incandescent lamp through that part of the plate where the lines are sharpest, the spectra of the first three orders are seen. As the plate is moved across the eye the second and third order spectra rapidly fade away, so that only the first order spectrum is seen where the lines are most blurred. The original Levy grating gives many spectra, the first thirty-five orders being plainly visible when a sodium flame is used as the source.

6. When a lens forms a real image of a grating, it does so by adding together in the focal plane the harmonic components of the diffracted light. If the illumination is central and the aperture of the lens is so narrow that it cannot pass the light represented by the second and succeeding terms of equation (2), *i.e.* if it passes only the non-periodic first

term, or central beam, the illumination in the focal plane is uniform and no image of the lines of the grating is formed. If the first spectrum, *i. e.* the first periodic term in (2), is transmitted by the lens, an image is formed having a periodic structure corresponding to that of the grating, but in which the lines are much blurred. In paragraph 4 it was shown that an actual grating may be considered to be composed of an imperfectly transparent surface on which are superposed a number of simple gratings each of which is responsible for the whole of the light in the spectra of one order. The structure of the image can now be readily explained. The lens can, so to speak, image only what it sees ; if the aperture is so narrow that it does not gather in the spectra of the second and higher orders, it receives no light from any of the simple component gratings except the first one, and hence can only image that one. Hence the image will represent a simple grating superposed on an imperfectly transparent plate and will present precisely the structure indicated in fig. 3. When the aperture of the lens is further widened so as to admit spectra of higher and higher orders, the definition becomes sharper and sharper and the image in general approximates more and more closely to a true representation of the object, but with exceptions which will be mentioned in paragraph 9. These results may be readily verified with an ordinary microscope if central illumination is used and an iris diaphragm is placed above the objective for conveniently varying the aperture ; the diffraction spectra can be seen by removing the eyepiece and looking down the tube, and the iris can then be adjusted to intercept all spectra above those of a given order. The effects are best studied with monochromatic illumination. The shaded grating mentioned in paragraph 5 is a particularly interesting object to examine ; when the iris is wide open, the lines are seen to be considerably sharper near one edge of the grating than near the other, but when the iris is contracted so as to transmit only the spectra of the first order, the lines appear equally blurred over all parts of the plate.

7. A grating ruled with lines closer together than the wave-length of light gives no spectra with central illumination and, if the lines are closer together than half a wave-length, no spectra will be given under any circumstances. As Abbe has shown, this sets a limit to the possible resolving power of the microscope ; for the lines are imaged by the diffracted light, and when none is diffracted no image of the lines can be formed. If green light is used to illuminate the object, the limit of possible resolution will be reached at about 40,000

lines to the centimetre and, since no second order spectrum is given by rulings finer than 20,000 to the centimetre, the images of rulings between these limits will merely show blurred or harmonically shaded lines as indicated in fig. 3. The only possible method of increasing the resolving power is to use light of shorter wave-length; this may be done by utilizing the violet or ultra-violet rays, or by using an immersion-lens and filling the space between object and lens with a liquid of high refractive index in which light-waves are shortened. In the photo-micrographic outfit designed by Dr. A. Koehler * both methods of securing short wave-lengths have been utilized and the limit of resolution has been raised to about 90,000 lines to the centimetre. Although the discussion has here been limited to the visibility of periodic structures such as gratings, it is evident from the general principle involved that in no case can we expect to secure any advantage by increasing the magnifying power of a microscope much beyond the point at which the half wave-length of the light used becomes an easily visible magnitude.

8. It is sometimes assumed that there is an essential difference between microscopic and macroscopic vision and that the phenomena of diffraction play no part in the latter †. Whatever difference there may be between the two cases arises, however, from the relative size of the objects involved and from the special methods of illumination employed with the microscope. By choosing a suitable method of illumination, as in the following experiment, it can be readily demonstrated that the images of periodic structures formed by the naked eye itself are due to diffracted light. The experiment is patterned as closely as possible after one of Abbe's microscopical experiments, but with the microscope left out. Light from an arc lamp, a in fig. 4, passes through a pinhole

Fig. 4.
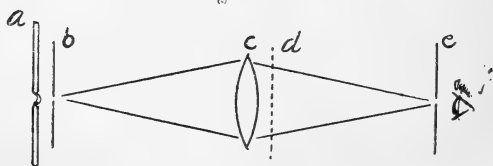
in the screen b, and is focussed by means of a photographic lens c on a cardboard screen at e, about 25 or 30 centimetres

* *Zeitschrift für wissenschaftliche Mikroskopie,* xxi. 1904, pp. 129–165 and 273–304.
† Carpenter-Dallinger, 'The Microscope,' 8th ed., 1901, p. 62.

from the lens.   At d, immediately in front of the lens, a
piece of wire gauze having about 30 wires to the centimetre
is placed.   The diffraction pattern produced on the screen e
by the wire gauze is shown much enlarged in fig. 5, where o

Fig. 5.
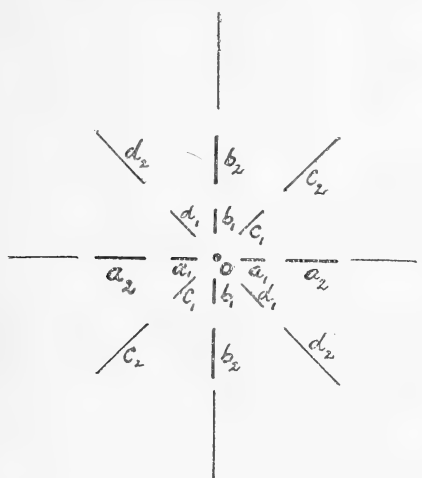
is the central image, a_1 and a_2 are the spectra due to the
vertical wires, b_1 b_2, those due to the horizontal wires ; and
c_1 c_2, d_1 d_2 are spectra due to the combined effects of both sets
of wires.   By cutting small holes in the screen e, so as to
allow but part of the diffraction pattern to pass into the eye
at f, the changes in the appearance of the gauze may be
conveniently studied.   If the screen e is pierced by a hole
only large enough to transmit the central beam o, the wire
gauze is quite invisible.   If a horizontal slit is used which
transmits only the central beam o and the spectra a_1 a_2, the
vertical wires alone are seen. If the slit is turned vertically so
as to transmit o and b_1 b_2, the horizontal wires alone are visible.
If the slit is turned at an angle of 45° so that the central
beam and the secondary spectra c_1 c_2 pass through it, neither
the vertical nor horizontal wires are seen, but a very real
looking set of wires appears running diagonally in the
direction d_1 d_2, such a set of wires as would in fact give rise
to the spectra c_1 c_2 if acting alone.   If the card is pierced
with three pinholes which transmit the central beam o and
the two second-order spectra a_2 a_2, a set of vertical lines is
seen, the lines being half as far apart as the wires in the
gauze.

9. Certain appearances noted in the image of a grating produced by a microscopic objective stopped down as described in paragraph 6 suggested the advisability of a closer study of these effects. The method outlined in paragraph 2 is sufficient to determine the nature of the image of any opacity grating given by a lens of any aperture. The curve showing the distribution of amplitude in the plane of the grating of the light which passes through it is developed in a Fourier series, both sines and cosines being used if the lines of the grating are shaded unsymmetrically. From the aperture of the lens and the spacing of the lines of the grating the number of orders of spectra transmitted by the lens is determined and, remembering that each periodic term in the Fourier series corresponds to the spectra of one given order, all terms in the series are rejected which represent spectra not transmitted by the lens. The remaining terms when summed give the amplitude curve, in the plane of the grating, of the effective light which is transmitted by the lens. The amplitude curve in the image differs from this merely in having its length increased and its height diminished in proportion to the magnification. By squaring the ordinates of this curve, the intensity curve is obtained which completely determines the structure of the image. Intensity curves were roughly drawn to scale in this way for the images of several different gratings and for a number of different apertures. Figs. 6–I to 6–XI show a set of these intensity curves giving the distribution of light in the images, formed by lenses of different apertures, of a grating whose opaque lines are twice as broad as the transparent spaces. In each case the distribution of light in two bright lines is shown, the base line of each figure representing darkness. The Roman numeral indicates in each case the highest order of spectra transmitted by the lens. Fig. 6–I shows the image formed by a lens of aperture just sufficient to transmit the spectra of the first order, and indicates the presence of a moderately bright streak *a* down the middle of each dark line in the image. Fig. 6–II shows the image formed by a lens transmitting the spectra of the first two orders ; the image is much sharper than that of fig. 6–I. The third, sixth, ninth, &c., orders are absent with the grating under consideration. Figs. 6–IV and 6–V show the nature of the images when the aperture is further widened so that spectra up to and including the fourth and fifth orders respectively are transmitted by the lens. The noteworthy point here is the appearance of a dark streak down the centre of each bright line. This is particularly interesting because it is a defect, or rather a falsification, in the image which

has been produced by improving the lens. The image pro-
duced by a lens admitting spectra of the eighth order is
shown in fig. 6–VIII. Here two dark streaks show in

Fig.6.–I
a

Fig.6.–VII

Fig.6.–II

Fig.6.–VIII

Fig.6.–IV

Fig.6.–X

Fig.6.–V
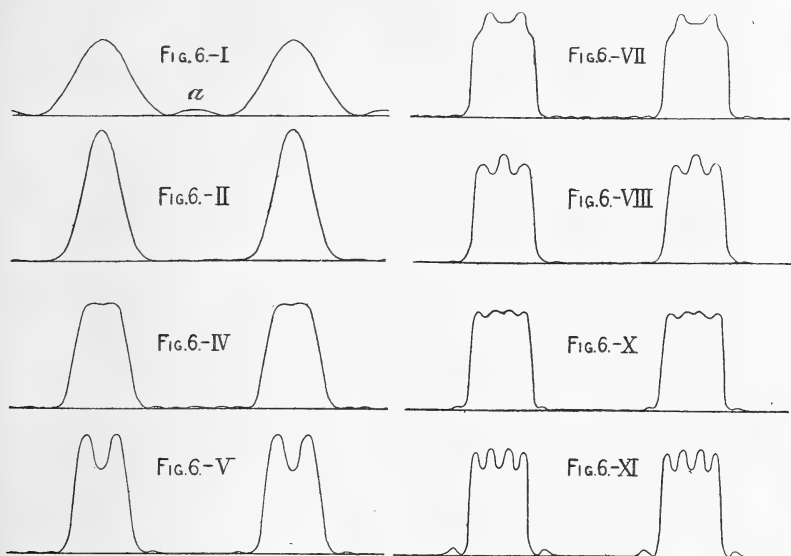
Fig.6.–XI

each bright line. Fig. 6–XI shows the presence of three
dark bands in each bright line of the image formed by a lens
admitting spectra up to the eleventh order.

Using a steel tool, a number of coarse gratings were ruled
on smoked glass, and one was finally secured which, when
examined with a wide-angled lens, showed sharp black lines
approximately twice as broad as the transparent spaces.
This grating was illuminated with monochromatic light and
examined by means of a low-power objective backed by an
iris diaphragm to secure variable aperture. The chief details
shown in figs. 6–I to 6–XI were clearly seen in the image
as the iris was slowly opened. With the iris contracted so
that the first-order spectra alone were transmitted, the bright
lines were broad and fuzzy while a distinct bright streak
showed along the centre of each black line, making each look
like a highly refracting transparent filament. When the iris
was slowly opened, these bright streaks disappeared, the bright
lines became noticeably sharper, and soon a dark streak *

* It was the appearance of these dark streaks which suggested a
detailed study of the images ; when first observed they were mistaken
for narrow lines of lampblack left by the ruling point.

appeared down the centre of each (fig. 6–V) which faded away as the iris was further opened, to be succeeded by two dark streaks (fig. 6–VII), then three fainter ones (fig. 6–XI) which soon disappeared with further enlargement of the aperture. When the grating was illuminated by white light the bright streaks shown in fig. 6–I could be seen but, on account of the overlapping of the spectra of the higher orders, the dark streaks indicated in figs. 6–IV to 6–XI did not appear in the image. These experiments suggest that some caution should be used in interpreting minute details of structure in microscopic images when monochromatic illumination is used.

10. The curves given in figs. 6–I to 6–XI show the intensity of the light in all parts of the image. They are also interesting from another point of view for, from what has been said in paragraphs 4 and 5, it will be seen that each of these curves shows the exact distribution of transparency (defined now in terms of the *intensity* of the transmitted light) over the surface of a grating which gives spectra of certain definite orders and definite intensities. Thus a grating having lines shaded as indicated in fig. 6–IV would give a central image and spectra of the first, second, and fourth orders only, and these spectra would have exactly the same intensities as those given by a sharply ruled grating whose opaque lines are twice as broad as the transparent spaces. This application of Fourier's theorem enables one, in fact, to design an opacity grating presenting any desired anomaly. The realization of such gratings is quite a different matter, but it is hoped that some results may be presented in a later paper.

11. Transparent objects which are subjected to microscopic examination are always of finite thickness, and it is of some interest to inquire whether light diffracted by portions of the object lying outside the focal plane may not modify the image. The mathematical analysis of this question promising to be troublesome and unprofitable, an experimental illustration of the suspected effect was sought and one was finally found which is extremely striking. It is well known that when two transmission gratings are laid face to face with lines parallel and ruled surfaces slightly separated, the spectra formed by the double grating are intersected by transverse shadow-bands whose spacing depends upon the distance between the faces of the grating. Two photographic gratings with 3000 lines to the inch thus arranged with faces about a millimetre apart, showed a black band in each of the two spectra of the first order and, by changing the angle of

incidence, these bands could be brought into any desired part of the spectra. The double grating was placed upon the stage of a microscope supplied with an iris diaphragm above the objective, and the instrument was sharply focussed on the upper grating. The field of view was then illuminated by monochromatic light obtained by focussing an arc spectrum on the slit of a collimator. The arrangement of the apparatus is shown in fig. 7, where a is the arc, S is a spectroscope

Fig. 7.
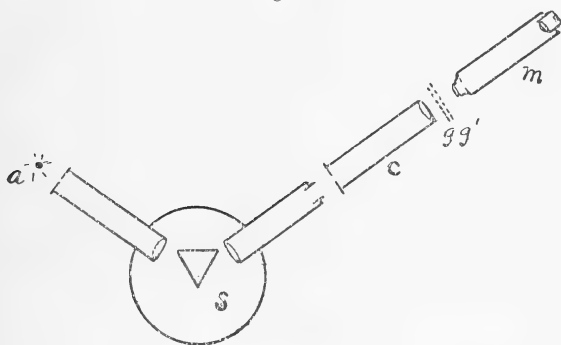
with eyepiece removed, c is a collimator, $g\,g'$ are the two gratings, and m is the microscope. By rotating the prism, the gratings could be illuminated by light of every wave-length in succession. Using yellow light, the eyepiece of the microscope was removed so that the diffraction spectra could be seen, the iris was closed until all spectra above the first order were cut out, and the angle of incidence of the light on the gratings was then varied until the shadow-bands reached the yellow part of the spectra and caused the disappearance of the two yellow dots of light which repre-sented the two first-order spectra. The eyepiece of the microscope was now replaced, and the prism was slowly turned so as to secure illumination with different colours of mono-chromatic light, while the image of the upper grating was observed. As had been anticipated, the image was normal and the lines were clearly seen when the field of view was illuminated with any colour except yellow, but when yellow light was used the lines were completely obliterated from the larger part of the field. The obliteration of the lines would doubtless be everywhere complete with two gratings of perfectly uniform and equal brilliancy. The iris was now opened so as to admit the spectra of the second order. Again the image of the lines was normal except when the field of

view was illuminated with yellow light ; in this colour the lines were seen to be doubled in number in those parts of the field where, in the former experiment, they were obliterated. Both of these effects are immediately explained by the absence of yellow light from both spectra of the first order.

12. In the experiments which have been described, the microscope was used under what would be considered normal working conditions, with central illumination, and circular diaphragms centred on the optic axis. Nevertheless, when certain relations existed between the aperture of the lens and the coarseness of structure of the object, images were formed which were utterly false in their smaller details, and other images were profoundly modified by the presence of structure lying entirely beyond the focal plane. It therefore seems that a working knowledge of the phenomena and laws of diffraction might well form a part of the equipment of everyone who uses the microscope and attempts to interpret its indications.

XIII. *Some Properties of the α Rays from Radium*. *By* E. RUTHERFORD, *F.R.S., Macdonald Professor of Physics, McGill University, Montreal* *.

[Plate IV.]

(Second paper.)

IN the July number of the Philosophical Magazine I described some experiments which showed that the α particles, emitted from a wire made active by exposure to the radium emanation, diminished in velocity after passing through matter.

The active deposit of radium, rather than radium itself, was chosen as a source of radiation in order to obtain a homogeneous pencil of rays. The active deposit on the wire is of extreme thinness, and consequently the rays emerge from the wire without any alteration by their passage through matter. About 15 minutes after removal from the emanation, the activity of the wire is almost entirely due to the product radium C. Using the photographic method, the rays were found to be all equally deflected by a strong magnetic field, *i. e.* they consisted of α particles projected at the same speed.

The velocities of these particles, deduced from the deflexion of a narrow beam of the rays by a magnetic field,

were all diminished by a constant amount after passing through a definite thickness of aluminium. It was observed that the photographic action of the α rays became relatively very feeble when the velocity of the α particles fell to about 60 per cent. of their maximum velocity.

Bragg and Kleeman (Phil. Mag. Dec. 1904, Sept. 1905) had previously shown by a most convincing series of experiments that the α rays emitted from radium consisted of four distinct sets expelled from the four α ray products present in radium in equilibrium, viz. radium itself, the emanation, radium A and C. Each product contained in a very thin film of radium in equilibrium emitted rays which ceased to ionize the gas after passing through a definite distance in air. This "range" of the α particles was, for example, about 3·5 cms. for the rays emitted from radium itself, and about 7 cms. for the rays emitted from radium C. In a thick layer of radium, where the α particles emerge into the gas from different depths, they concluded that each product gave out rays which had all ranges of ionization in air varying between zero and the maximum range.

In order to explain their results they supposed that the α particles diminished in velocity and consequently in their range of ionization by passing through matter. The results of my experiments indicated that the velocity of the particles was not reduced much below 60 per cent. of the velocity of the particles from radium C when they failed to ionize the gas or to affect appreciably a photographic plate.

According to the views of Bragg and myself, the rays emitted from a thick layer of radium in equilibrium were complex in character, and consisted of four sets of rays each of which consisted of particles projected over a considerable range of velocity.

As the simplest hypothesis, it was implicitly assumed that the α particles from the various radium products had the same mass and carried the same charge, and differed from each other only in the initial velocity of their projection from the radium and its products.

In a recent number of the *Comptes Rendus* (No. 11, Sept. 11, 1905), M. Henri Becquerel has taken exception to my experimental results and deductions. He reiterates his original view that the α rays from radium are homogeneous, and concludes that the retardation of velocity of the α particles in their passage through aluminium, observed by me for the α particles from radium C, does not exist in the α particles expelled from radium, or at any rate from his radium. For this reason he concludes that the theory proposed by Bragg

and Kleeman to explain their results on the absorption of the α rays of radium is untenable.

On the contrary, M. Becquerel considers that the α particles from radium all escape into the air with the same velocity, and that this velocity is not altered by their passage through matter. In order to explain the decreasing curvature of the path of the rays in their passage through air, first shown by him, M. Becquerel supposes that the mass of the α particle in some way increases in its passage through air.

There is thus a complete difference of opinion between M. Becquerel and myself on several very important points.

Before entering on a detailed discussion of these points of difference, I shall first briefly describe the experiments on which M. Becquerel* has largely based his conclusions.

After I had shown by the electric method, with the aid of the weak radium preparation then in my possession (activity 19,000), that the α particles were deflected both by a magnetic and electric field, M. Becquerel, with a very active preparation, repeated the magnetic deflexion by the photographic method. By a simple and ingenious arrangement, he was able to obtain on the one photographic plate the trace of the rays deflected in a magnetic field over a considerable distance of their path. A thin layer of radium served as a source of α rays. A narrow beam of rays, after passing through a parallel slit, fell on a photographic plate placed at right angles to the slit and inclined at a small angle with the vertical. By reversing the magnetic field at intervals two diverging lines, showing the trace of the rays in a magnetic field, were obtained on the photographic plate.

M. Becquerel states that the rays showed no appreciable dispersion in a magnetic field, for no difference in the width of the lines was observed for fields of 10,000 and 20,000 units. He consequently concluded that he was dealing with a homogeneous beam of rays. By measurement of the distance between the two diverging traces obtained on the plate, he found that the radius of curvature of the path of the rays increased with distance from the source. This was a very remarkable and important result. As an explanation M. Becquerel suggested that the mass of the α particle progressively increases with distance by accretions from the air.

In his last paper (*loc. cit.*) M. Becquerel describes another interesting experiment, the results of which, in his opinion, are directly in contradiction to my own. To quote his own words:—" Je me suis alors proposé de reprendre avec mes

* *Comptes Rendus,* cxxxvi. pp. 199, 431, 977, 1517 (1903).

anciens appareils une expérience permettant de recevoir, sur une même plaque photographique parallèle à la fente, un faisceau de rayons α du radium dévié par un champ magnétique, dont une moitié ne traversait aucun écran, et dont l'autre pouvait traverser divers écrans d'aluminium. Un écran vertical formé de lames de mica s'étendant de la source à la fente et de la fente à la plaque empêchait les deux moitiés du faisceau d'empiéter l'une sur l'autre. Les épaisseurs d'aluminium traversées dans divers expériences ont varié depuis celle d'une feuille d'aluminium battu jusqu'à l'épaisseur de 0mm,034 tout à fait comparable à celle des écrans employés par M. Rutherford.

" Les écrans d'aluminium ont été placés soit sur la source, soit sur la fente, soit à quelques millimètres de la plaque photographique. Dans tous les cas, les deux traces parallèles des deux moitiés du faisceau dévié ont été exactement dans le prolongement l'une de l'autre et n'ont pas présenté le décalage auquel on aurait dû s'attendre d'après la publication de M. Rutherford.

" Il faut donc en conclure que la propriété observée par ce savant est relative à des rayons particuliers émis par le fil activé qui lui servait, mais qu'avec le radium, et en particulier avec le sel de radium employé dans mes expériences, le ralentissement indiqué par M. Rutherford ne se produit pas.

" Cette nouvelle expérience, confirmant les conclusions que j'avais déduites de mes premières observations, conduit à rejeter les interprétations de MM. Bragg, Kleeman et Rutherford. J'ajouterai que les nombres rappelés plus haut sont relatifs à des distances de la source inférieures à celle du premier changement observé par M. Bragg."

In an account of the same paper communicated to the *Physikalische Zeitschrift* (No. 20, Oct. 15, 1905), a photograph is reproduced which bears out the description of M. Becquerel. The two sets of lines are continuations of one another[*].

At first sight, this result appears to be in direct contradiction to the view that the α particles suffer a retardation of velocity in their passage through aluminium, but, paradoxical as it may appear, this experimental result observed by M. Becquerel with a thick layer of radium is a necessary consequence of the views held by Bragg and myself.

Before discussing this interesting point, I shall first briefly consider some further experiments of my own.

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[*] *Note.* It will be noted that all of the experiments of Becquerel have been made in air at atmospheric pressure.

Retardation of the Velocity of the Particles from Radium C.

In my previous paper, I showed that in a constant magnetic field reversed at intervals, the width between the two bands obtained on the photographic plate steadily increased when successive layers of foil were placed over the active wire coated with radium C. A clear photographic impression was observed with twelve layers of foil each ·00031 cm. thick placed over the wire. At this stage, the distance between the bands indicated that the velocity of the α particles had been reduced to 62 per cent. of their maximum value. Using 13 layers of foil, I was unable to observe any photographic effect of the rays.

I have repeated these experiments, and have obtained substantially the same results. A photographic effect was observed at 12 layers, but disappeared with 13 layers.

These experiments have clearly brought out the fact that the photographic effect produced by the α rays decreases with the thickness of foil traversed, and falls off very rapidly between 11 and 13 layers of foil. Experiments are at present in progress to determine as accurately as possible the variation of the velocity of the α particles between these limits.

For the purpose of illustration of the increased width of the bands after passing aluminium, I have used the arrangement employed by M. Becquerel, described above.

The rays from the active wire were divided into two parts by mica screens. On one side the active wire was bare, and on the other covered with 8 layers of foil. The photograph obtained in a vacuum is shown in fig. 1 (Pl. IV.).

In order to have about the same intensity of photographic effect for the two sets of rays, the screens were placed so as to cover about $\frac{3}{5}$ of the length of active wire with the aluminium-foil. The greater divergence of the rays after passing through 8 layers of foil is clearly shown in the figure. Both sets of lines are well defined with fairly sharp edges *.

* In the figure, the width of the lines after traversing the aluminium foil is seen to be somewhat broader than the others. This difference is much more marked than I have obtained in other and similar experiments. This broadening resulted from an accident during the exposure of the wire to the emanation. In order to obtain a very active wire, too high a voltage was applied to the wire, and there was a discharge across the gas. This caused a black deposit near one end of the copper wire, and this portion of the wire was afterwards placed under the aluminium. Some of the rays, before escaping from the wire, had to pass through this deposit, and consequently were slightly reduced in velocity. The rays consequently were not homogeneous in the beginning. As the photograph, however, very clearly brings out the retardation of the velocity of the α particles, it was not thought necessary to repeat this experiment under more ideal conditions.

There can be no doubt that the rays from radium C are reduced in velocity by their passage through matter.

Complexity of the Rays.

We have seen that M. Becquerel considers the rays from radium to be homogeneous, *i. e.* that they consist of particles all projected at equal speeds. This conclusion is based on the observation that no appreciable broadening of the trace of the rays was observed by increasing the magnetic field from 10,000 to 20,000 c.g.s. units.

After the experimental results given by Bragg of the difference in the range of ionization of the rays from the various radium products, the evidence of the homogeneity of the rays requires to be extremely convincing before it can be accepted.

As a matter of fact, it is not difficult to show that the α rays from a thick layer of radium, far from being homogeneous, undoubtedly consist of rays unequally deflected in a constant magnetic field. Since the width of the beam of rays is, in most cases, comparable with the amount of its deviation by a strong magnetic field, the unequal deflexion of the rays is most clearly shown when a narrow beam of rays is employed.

This is clearly brought out in the photograph shown in fig. 2. One milligram of pure radium chloride (obtained from the Société Centrale de produits chimiques, Paris) was spread in a narrow groove. After passing through two narrow slits, the rays fell on a photographic plate placed 4 cms. above the radium. The apparatus was exhausted to a low pressure, and then placed between the poles of an electromagnet which was first excited by a weak current, producing sufficient field to bend away the β rays but not appreciably to deflect the α rays.

The trace of the rays is seen in the narrow line A in the photograph. After two hours, a strong constant current was passed for another two hours, and the trace B of the deflected beam of rays is shown in the figure.

The current was kept very constant during the experiment, and the strength of the magnetic field certainly did not vary at any time more than $\frac{1}{2}$ per cent.

If the rays are homogeneous, the two bands should be equal in width and their edges equally defined. The band B is, however, much broader than A, and the edges lack the definition of the latter. A close examination of the original negative shows that while the edges of A are fairly sharply defined, the band B is dark in the centre, but the photographic impression falls off gradually on either side, and it is difficult to fix with certainty the extreme edges of the deflected

band.   Actual measurement showed that the band B was at least 1·9 times the width of A.

It is not easy in these experiments to detect the presence of rays which cause a weak photographic effect ; for it must be remembered that there is always a darkening of the whole plate due to the β and γ rays from the radium when placing the apparatus in position, and to the γ rays during the whole time of the experiment.

Taking as the simplest assumption that the α particles have all the same value of e/m, the α particles escaping from a thick layer of radium will be moving at different velocities. The most penetrating rays are those from radium C.   Let V_0 be the maximum velocity of these rays.   Now I have shown that the photographic action of the rays from radium C becomes relatively very small when their velocity falls to about ·$6V_0$; and this will hold for the rays of all products when their velocity falls to the same value.   Theoretically, we might thus expect to detect by their photographic action in a vacuum the presence of rays whose velocity lies between ·$6V_0$ and V_0.   In practice, however, it is not to be expected that rays with these extreme limits would be clearly brought out on account of the finite width of the undeflected beam, and of the difficulty of determining with certainty the edges of the diffused band.   By measurement of the plate, it was found that rays producing a clear photographic effect were certainly present which had velocities varying between ·$67V_0$ and ·$95V_0$.

These results were obtained by directly comparing the deflexion of the rays from an active wire under exactly the same conditions as in the radium experiment.

The results obtained are thus in general agreement with those to be expected on the views held by Bragg and myself. There can be no doubt that the α rays from radium are complex, and consist of α particles which are unequally deflected in a magnetic field [*].

Explanation of M. Becquerel's Experiment.

We are now in a position to explain why the trace of the α rays from radium in a strong magnetic field is not altered when the radium is covered with layers of aluminium-foil.

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[*] This broadening of the pencil of rays in a magnetic field has been also experimentally observed in a recent paper by Mackenzie (Phil. Mag. Nov. 1905).

First, consider the experiments with an uncovered thick layer of radium. The outside edge of the traces obtained on the plate by reversal of the field will be due to the most deflected rays which are just able to produce a photographic impression on the plate. The inner edge of the traces will correspond to the rays which are least deflectable, *i. e.* which are expelled with the greatest speed.

Now suppose the radium is covered with any thickness of foil, provided it is less than that required to cut off completely the photographic effect of the α rays. All the α particles will be reduced in speed, and the more slowly moving ones will lose their photographic effect. The edge of the deflected trace will, however, be due as before to the most deflected rays which are just able to produce an appreciable photographic effect. These rays will have exactly the same velocity as the corresponding rays in the first experiment, and consequently the outside edge of the photographic traces will be continuations of that observed in the first experiment. This holds equally well whether the experiments are made in air or in a vacuum.

Since, however, the more penetrating rays are reduced in velocity by their passage through aluminium, the inside edge of the photographic traces will be displaced outwards. This effect will be extremely difficult to detect unless a very narrow beam of rays is employed. In addition, the undoubted scattering of the rays in air (which will be discussed later) will tend to obliterate this effect in experiments similar to those of M. Becquerel, which were made in air at atmospheric pressure.

It is thus seen that the interesting experiment recorded by M. Becquerel is satisfactorily explained on the view that the particles decrease in velocity in passing through matter; and, taken in conjunction with my own experiments on the reduction of speed of the rays from radium C in passing through matter, affords an indirect but striking proof of the general correctness of the theory of the absorption of α rays put forward by Bragg. It is not possible to take the position that the rays, emitted from an active wire coated with radium C, do not exist in radium in equilibrium; for apart from other evidence, McClung (*supra*, p. 131) has shown that the range of ionization of the rays from an active wire coated with radium C is identical with the range of the corresponding rays in radium measured by Bragg and Kleeman.

Increase of the Radius of Curvature of the Path of the Rays in Air.

We have seen that M. Becquerel early drew attention to the fact that the radius of curvature of the trace of the α rays obtained in air increased with increase of distance from the source. Assuming that the rays were homogeneous, he suggested as an explanation that the mass of the α particle increased as it passed through the air.

Proceeding on these assumptions, it is seen that the α rays from each product present in radium must increase in curvature with the distance of air traversed. In order to test this point definitely, I made the following experiment, with an active wire coated with radium C as a source of rays. Using the same apparatus employed to obtain the photograph in fig. 1, the part of the active wire on one side of the mica screen was covered with a sheet of metal of sufficient thickness to absorb all the α rays. The apparatus was filled with air at atmospheric pressure, and the trace of the rays obtained in a uniform magnetic field is seen in fig. 3 (Pl. IV.). After 40 minutes, the metal sheet was transferred to the other side of the screen and the apparatus was then exhausted. The trace of the rays then obtained in a vacuum is shown by the two narrower lines in the same figure. There is a striking difference between the two pairs of lines obtained in air and in vacuum.

The air-bands are more deflected than the latter and in addition are broader. This experiment conclusively shows that the α rays from the active wire *decrease* in radius of curvature and consequently in velocity after passing through air. The greater width and lack of definition of the air-lines have been noticed in all other experiments, and show evidence of an undoubted scattering of the rays in their passage through air. The slight shift of the vacuum-lines in the figure, relative to the air-lines, was due to an accidental displacement of the plate in transferring the metal sheets in the middle of the experiment. Experiments are in progress to see whether this scattering also occurs in the passage of the rays through a solid substance. This scattering of the rays probably also occurs with aluminium, but it would not be evident in the experiments where the layers of foil are placed over the active wire *between the source and the slit.*

Since the rays from the other products of radium in all probability behave in the same way as the rays from radium C, we see that this result of the decrease of the radius of

curvature of the path of the rays in air is in contradiction to the hypothesis of M. Becquerel that the value of e/m decreases with distance from the source.

We have seen, however, that this conclusion of M. Becquerel is based on the incorrect assumption that the rays are homogeneous. Taking into account the complexity of the α rays from radium, the results observed by M. Becquerel receive a simple explanation without the necessity of assuming that the value of e/m either increases or decreases with distance. An explanation along these lines was given last year by Bragg and Kleeman. Without entering into intricate calculations, the general reason of the increase of the radius of curvature of the path of the rays with distance of air traversed can easily be seen.

At a distance of 2 cms. from the source, for example, the outside edge of the trace is due to the most slowly moving particles, which are just able to produce a photographic impression after traversing 2 cms. of air. The inside edge is due to the most penetrating rays, viz. those from radium C, whose velocity has been reduced somewhat by passing through the 2 cms. of air.

Suppose we consider the trace of the rays distant 3 cms. from the source. As before, the outside edge of the trace is due to particles which are just able to produce a photographic effect after traversing 3 cms. of air. These particles are thus projected initially with a greater velocity than those which were only able to affect a plate for a distance of 2 cms. The average velocity of the α particle along its path is consequently greater than for the corresponding case for a distance of 2 cms. The outside edge of the trace will therefore be deflected through a smaller distance than would be expected if the average velocity were the same in the two cases.

The radius of curvature of the outside edge of the path will consequently increase with increasing distance.

Quite a contrary effect should be observed on the inside edge of the trace, which is produced by the α particles of greatest velocity escaping from the radium. This velocity is, as we have seen, reduced by the passage through air, and consequently the average velocity of the α particles, which traverse the path of 3 cms., is less than for the path of 2 cms. The inside trace should theoretically show evidence of decreasing radius of curvature. This would have the effect of contracting the natural width of the trace unless the scattering of the rays in air masks it experimentally. Quite apart from

this, it can easily be shown, from calculations based on the numbers given in my last paper on the retardation of velocity of the α particles in their passage through air, that the decrease of curvature of the outside trace more than compensates for the increase of curvature of the inside trace. The net result of these two opposing processes is to increase the radius of curvature of the path, deduced from measurements on both edges of the trace, to about the extent observed by M. Becquerel.

Summary of Results.

The following points have been brought out in this paper:—

(1) The rays from radium in radioactive equilibrium are complex and consist of α particles projected with different velocities.

(2) The α particles decrease in velocity in their passage through air and through aluminium.

(3) The absence of increased deflexion of the rays from a thick layer of radium after passing through aluminium, observed by M. Becquerel, is a necessary consequence of the complexity of the rays.

(4) The decreasing curvature of the path of the rays in air, observed by M. Becquerel, is also a necessary consequence of the complexity of the rays.

(5) There is evidence of a distinct scattering of the rays from radium C in their passage through air.

A full discussion of the connexion between the ionizing, phosphorescent, and photographic actions of the α rays will be reserved for a later paper.

Macdonald Physics Building,
McGill University, Montreal, Nov. 15, 1905.

XIV. *The Relation between the Radioactivity and the Composition of Uranium Compounds.* By HERBERT N. McCOY, *Asst. Prof. of Chemistry, University of Chicago* *.

THE early determinations † of the radioactivity of pure uranium compounds showed but a roughly approximate proportionality between the activity and the percentage of uranium. The methods used did not take into account the absorption of the alpha rays, to which ionization is almost

* Communicated by the Author. Read before the Amer. Phys. Soc., Chicago, April 21, 1905; Abstract, Phys. Rev. xx. p. 381 (1905).

† Mme. Curie, *Thèse*, Paris, 1903; also Chem. News, lxxxviii. p. 98 (1903). McCoy, *Ber. d. Chem. Ges.* xxxvii. p. 2641 (1904).

wholly due, by the uranium compound itself. If this absorption occurs according to the simple absorption law, it is readily shown * that

$$k_2 = \frac{2 \cdot 303 \; s}{w} \log \frac{1}{1-x},$$

where k_2 is the absorption coefficient for unit weight on unit area ; x is the ratio of the observed activity of any very thin film of a uranium compound to that of a film of the same substance sufficiently thick to be of maximum activity ; s is the common area of the films ; and w is the weight of the thinner film. The films were from ·001 to ·04 mm. in thickness, the thicker films showing maximum activity. k_2 is a good constant for any pure uranium compound. If the observed radioactivity of one sq. cm. of any film of maximum activity is called α it is shown † that $k_1 = 2k_2 \, \alpha$, where k_1 is the total activity of 1 g. of the uranium compound. The ratio, K, of the total activity, so determined, to the uranium content of the compound is a constant, within the limit of error of $\pm$ 1 per cent., and represents the total alpha ray activity of 1 g. of pure uranium, independently of its form of chemical combination. If α is taken as unity for pure U_3O_8, K is equal to 791. This means that the total activity of 1 g. of pure uranium is 791 times that of 1 sq. cm. of a layer of pure U_3O_8 sufficiently thick to be of maximum activity.

Preparation of a Standard of Radioactivity.

The above results were obtained by means of compounds prepared directly from " chemically pure " uranium nitrate as furnished by the firm of C. A. F. Kahlbaum. Since all uranium ores contain radium ‡, it was necessary to determine with certainty whether the material used was free from radium in amounts sufficient to appreciably alter its activity. This was done by starting with a uranium ore, which of course contained the maximum of such impurities, and continuing the process of purification until no further change of activity occurred. The activity of the U_3O_8 so prepared did not differ by a measurable amount from that of my old " standard " oxide, which had been made by direct ignition of Kahlbaum's nitrate. The measurements of the activity were made electrically in the manner previously described. The error of comparison was about 0·2 per cent.

* McCoy, J. Amer. Chem. Soc. xxvii. p. 391 (1905).

† McCoy, *loc. cit.*

‡ McCoy, *Ber. d. chem. Ges.* xxxvii. p. 2641 (1904). Boltwood, Am. J. Sci. xviii. p. 97 (1904) ; Phil. Mag. [6] ix. p. 599 (1905).

The details of the process were as follows :—29 g. of pitch-blende (from the Wood Mine, Colorado), containing 60 per cent. of uranium, and being about 2·5 times as active as pure U_3O_8, were dissolved in nitric acid and the solution evaporated to dryness. The residue was taken up in water and filtered, and the filtrate added to an excess of ammonium carbonate solution. The solution was boiled and filtered from the precipitate containing iron, &c. Copper, lead, &c. were removed from the filtrate by treatment with ammonium sulphide. The uranium was precipitated from a small portion of the filtrate by prolonged boiling. Ignition of the precipitate of ammonium uranate gave U_3O_8 which was 4·9 per cent. more active than the standard oxide. The treatment up to this point had therefore removed all but about 2 per cent. of the radioactive impurities originally present in the ore. To the main portion of the solution was added a solution of 2 g. of barium chloride. The solution was heated to boiling and a slight excess of ammonium sulphate added. The barium sulphate, so formed, was filtered out. It was freed from traces of uranium by treatment with hydrochloric acid. It was still about as active as uranium oxide. That the activity of the barium sulphate was due chiefly to radium and not to UX was shown by the facts, first, that the rays were almost wholly absorbed by a thin sheet of aluminium, and second that the activity increased in intensity with time. The treatment of the solution with barium chloride and ammonium sulphate was repeated twice. The second lot of barium sulphate was but 0·05 as active as the first, while the third lot was practically inactive. The U_3O_8 prepared from the solution after the third precipitation differed in activity from the old standard U_3O_8 by only 0·15 per cent. This difference is just about the limit of accuracy of the method of comparison. Further treatments with barium salts did not lower the activity of the uranium oxide. These results show that radium and other active bodies occurring in uranium ores are easily removed. There is therefore no doubt that the uranium oxide finally obtained was free from such active impurities, in appreciable quantities. In addition it is established that the material previously used * in determining the absorption coefficients and activities of uranium compounds was also of the same high degree of purity †.

* McCoy, J. Amer. Chem. Soc. xxvii. p. 391 (1905).

† The work on absorption coefficients of pure uranium compounds is being greatly extended by Mr. H. M. Goettsch, working in this laboratory ; the results previously announced are being fully confirmed. Attempts by Mr. Goettsch to lower the activity of the Kahlbaum uranium nitrate, used in the preparation of the standard oxide, by treatment with barium salts, were without effect.

Many persons have used a definite portion of uranium oxide as an empirical standard of radioactivity. I believe it is now possible to accurately define a standard of activity that may readily be reproduced with an accuracy of at least a few tenths of one per cent. To do this, ordinary uranium salts are first to be purified by successive treatments with ammonium carbonate, ammonium sulphide, and barium salts in the manner described. In converting the precipitated ammonium uranate into the oxide, U_3O_8, it is necessary to ignite in a stream of oxygen, since Zimmermann has shown * that ignition in air causes slow loss of oxygen and finally the formation of a lower oxide. The standard film of oxide is prepared by grinding in an agate mortar 0·8 to 1·0 g. of the substance with a little freshly distilled chloroform until the former is reduced to an impalpable powder. This is then stirred up with about 15 c.c. more of chloroform and poured into a shallow metal dish about 7 cm. in diameter †. The dish is covered until the solid has settled. The chloroform is then allowed to evaporate spontaneously. The U_3O_8 is left as a very uniform black film, which adheres well to the dish. Such films may be used daily for months without deterioration. They must obviously be protected from dust when not in actual use. The *total* activity of 1 g. of uranium is, as has been said, 791 times the *observed* or *surface* activity of each sq. cm. of such a film.

I have shown ‡ that the total activity, k_1, of any uranium compound can be found in a second way which does not involve the determination of the absorption coefficient. The ratio of the activities of two films is the inverse of the ratio of their respective times of discharge of the same electroscope through the same range of the scale ; corrected, of course, for the small natural leak. If a is this ratio for any thin film compared with a standard film of U_3O_8 of equal area s, and of sufficient thickness to be of maximum activity, and if the activity of 1 sq. cm. of the standard film be taken as unity, then the observed activity of the first film will be sa. If the first film be infinitely thin, there would be no absorption, and therefore

$$sa = \tfrac{1}{2} k_1 w$$

or

$$k_1 = 2s \left/ \frac{w}{a} \right.$$

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* *Ann. Chem.* (Liebig), ccxxxii. p. 276 (1885).
† I have used tinned lids intended for jelly glasses. These were cheaply obtained in large quantities; after being used once, they were discarded to avoid the effect of excited activity which could not be easily removed.
‡ J. Amer. Chem. Soc. xxvii. p. 402 (1905).

The theoretical value of $\frac{w}{a}$ for an infinitely thin film may be readily found by plotting the values of this ratio for thin films of various weights and finding, by a small graphical extrapolation, the zero value. This value may be designated as $\left(\frac{w}{a}\right)_0$.

Therefore
$$k_1 = 2s\Big/\left(\frac{w}{a}\right)_0.$$

The values of k_1 as found by this second method agreed well, in the case of pure uranium compounds, with those as determined by the first method.

The Radioactivity of Uranium Ores.

Having found satisfactory methods of determining accurately the total radioactivity of pure uranium compounds, these methods have now been applied to the further study of uranium ores. The theory that radium is a disintegration product of uranium demands that all uranium compounds sufficiently old should contain radium in amounts directly proportional to their uranium content. It should therefore be found that the total radioactivity of any uranium ore should be directly proportional to its percentage of uranium. This was, in fact, found to be approximately true for twelve ores from widely different localities *. The average deviation from the mean value of the ratio of activity to percentage of uranium was 7·1 per cent. The determinations of the activity, in the above cases, were made upon equal weights of ore without due allowance being made for difference of absorption. The method of determining total activity, which depends upon a determination of the absorption coefficient of the substance, can be applied only to those substances for which the absorption coefficient is a true constant. This condition will obviously be fulfilled only by substances the radiations of which are homogeneous. If we disregard the beta rays, the ionizing effect of which is negligible, the radiation of every pure uranium compound is homogeneous. For every such compound the absorption coefficient is a constant. Uranium ores all contain radium and its numerous active transformation products. The alpha rays of these bodies have quite different penetrating powers. It was actually found, as anticipated, that the absorption coefficient of ores was not constant, but varied with the thickness of the film. While

* McCoy, *Ber. d. chem. Ges.* xxxvii. p. 2654 (1905).

the first method of calculating the total activity was therefore excluded, in the case of ores, there was, on the other hand, no theoretical objection to the application of the second method.

Only the purest minerals are perfectly homogeneous. Since the ordinary method of powdering might involve a loss of some dust of greater or less activity than the average for the ore, the crushed samples were moistened with pure redistilled alcohol and ground in an agate mortar to a very smooth paste. The alcohol was then allowed to evaporate. The powder so obtained was finer and more uniform than could be obtained by dry grinding. The films were prepared in the manner previously described. As suspending liquids I used for pitchblende, chloroform ; for gummite, alcohol ; and for carnotite, water.

The uranium in pitchblende and gummite was determined as previously described [*], by treating the nitric-acid solution of the ore with an excess of sodium carbonate and precipitating the uranium from the neutralized filtrate by means of sodium hydroxide [†]. The sulphuric acid solution of the uranium precipitate was reduced with zinc and titrated with permanganate. In the analysis of carnotite, vanadium was first removed by treating the dry sample repeatedly with dry hydrogen chloride gas [‡]. Duplicate analyses were made in each case. Five different ores have been studied as follows :—

TABLE I.

Mineral.	Locality.	Per cent. Uranium.
1. Pitchblende.	Colorado.	51·1
2. Pitchblende.	Bohemia.	40·3
3. Pitchblende.	Unknown.	61·1
4. Gummite.	North Carolina.	54·7
5. Carnotite.	Colorado.	39·9

None of these ores contained more than traces of thorium.

[*] *Ber. d. chem. Ges.* xxxvii. p. 2641 (1904).
[†] Brearley, Analytical Chemistry of Uranium, 1903, states that uranium is not completely precipitated by sodium hydroxide. This is true only when the latter contains carbonate, as it invariably does unless special precautions are taken. I have used sodium hydroxide from which the small amount of carbonate originally contained was removed by the addition of a little calcium chloride solution.
[‡] Hillebrand, Amer. J. Sci. x. p. 135 (1900).

The results are given in Tables II. to VI.   Column two gives the weight, w, of the film ; column three gives the ratio, a, of the activity of the film of the mineral to that of the standard film of pure U_3O_8 of equal area.   The fourth column contains the values of $\dfrac{w}{a}$.   The accompanying curve represents Table II. and illustrates the method of determining $\left(\dfrac{w}{a}\right)_0$ graphically.
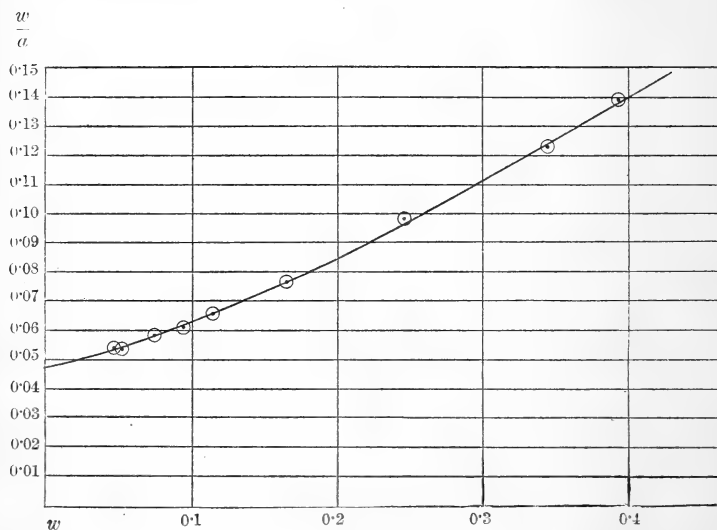
Table II.—Pitchblende, No. 1.

	$w.$	$a.$	$\dfrac{w}{a}$
1	·3922	2·827	·1387
2	·3443	2·813	·1224
3	·2458	2·504	·0982
4	·1640	2·140	·0766
5	·1135	1·730	·0656
6	·0943	1·533	·0615
7	·0741	1·276	·0581
8	·0523	·976	·0536
9	·0466	·860	·0542
			$\left(\dfrac{w}{a}\right)_0 = ·0472$

TABLE III.—Pitchblende, No. 2.

	w.	a.	$\dfrac{w}{a}$
1	·3056	2·126	·1438
2	·3026	2·220	·1363
3	·2412	2·130	·1133
4	·1488	1·673	·0889
5	·1346	1·616	·0833
6	·0837	1·170	·0716
7	·0657	·938	·0701
8	·0631	·937	·0673
9	·0470	·725	·0648
10	·0407	·637	·0638
			$\left(\dfrac{w}{a}\right)_0 = ·0594$

TABLE IV.—Pitchblende, No. 3 *.

	w.	a.	$\dfrac{w}{a}$
1	·9060	2·924	·3099
2	·5007	2·920	·1715
3	·2207	2·675	·0825
4	·1559	2·275	·0685
5	·0855	1·582	·0541
6	·0706	1·343	·0526
7	·0618	1·235	·0501
8	·0580	1·135	·0511
9	·0360	·815	·0442
10	·0332	·714	·0465
11	·0273	·584	·0467
			$\left(\dfrac{w}{a}\right)_0 = ·0400$

TABLE V.—Gummite, No. 4.

	w.	a.	$\dfrac{w}{a}$
1	·2764	2·318	·1192
2	·1810	1·958	·0924
3	·0704	1·173	·0600
4	·0686	1·143	·0600
5	·0654	1·099	·0595
6	·0570	·965	·0591
7	·0472	·862	·0548
			$\left(\dfrac{w}{a}\right)_0 = ·0446$

* The activity measurements represented by Table IV. were made by Miss Emma Carr, to whom I wish here to express my thanks.

TABLE VI.—Carnotite, No. 5.

	w.	a.	$\dfrac{w}{a}$
1	·3704	1·829	·2025
2	·2634	1·765	·1493
3	·0949	1·094	·0868
4	·0477	·644	·0741
5	·0423	·574	·0736
6	·0172	·272	·0631
			$\left(\dfrac{w}{a}\right)_0 = ·0612$

In Table VII., k_1 represents the total activity of unit mass of the mineral as calculated from

$$k_1 = 2s \Big/ \left(\frac{w}{a}\right)_0.$$

The area, s, was equal to 39·82 sq. cm.   P is the weight of uranium in 1 g. of ore.

TABLE VII.

	$\left(\dfrac{w}{a}\right)_0$	k_1.	P.	$\dfrac{10^{-1} k_1}{P}$
1	·0472	1687	·511	330
2	·0594	1341	·403	332
3	·0400	1990	·611	326
4	·0446	1786	·547	327
5	·0612	1301	·399	326
				Mean 328

The value of $\dfrac{k_1}{P}$ is very nearly constant and is equal to 3280 for uranium minerals.   The corresponding constant for pure uranium compounds is 791.   The ratio of these two quantities is 4·15.   That is, for equal uranium content the ores are 4·15 times as active as the pure compounds.

If radium and all of the other radioactive substances in uranium ores are genetically related to uranium, it should be found that the amounts of each such substance in the ore, and therefore also the total radioactivity, should be directly proportional to the uranium content, provided the mineral is sufficiently old to have reached the equilibrium condition. The results just described seem, therefore, to confirm the

conclusions drawn from my preliminary experiments, viz.: that uranium is the parent not only of radium, but also of all the other active substances which accompany it. Apparently the experimental error of the present work is about ± 1 per cent.; but it may be somewhat greater. In consequence there may be comparatively small amounts of active bodies (polonium, actinium, &c.) which are not derivatives of uranium. But as far as radium and its known products are concerned, the case is clear; for, in addition to the evidence I have presented, it has been shown by Boltwood[*] that the quantity of radium emanation obtained from various uranium ores is strictly proportional to the weight of uranium in the sample. In addition Soddy[†] has finally obtained radium emanation from a quantity of pure uranium nitrate which two years previous had been entirely freed from radium and its products. It is true, however, that the amount of emanation so obtained was much smaller than one should expect if radium is produced *directly* from uranium X[‡].

It is possible to calculate, from data here presented, the relative activity of radium and uranium, on the assumption that all of the excess activity of uranium ores is due to radium, together with its products. These bodies, collectively, are $4 \cdot 15 - 1$ or $3 \cdot 15$ times as active as that mass of uranium with which they are associated. Now Rutherford and Boltwood[§] have shown that one part of radium is in equilibrium with $1 \cdot 35 \times 10^6$ parts of uranium in ores. Therefore radium (with its products) is $3 \cdot 15 \times 1 \cdot 35 \times 10^6 = 4 \cdot 25 \times 10^6$ times as active as uranium. The value usually accepted is only half as great as this. This ratio may also be calculated from other data. Rutherford and McClung[||] have found the maximum saturation current due to 1 sq. cm. of a thick layer of U_3O_8 to be about $4 \cdot 5 \times 10^{-13}$ ampere. Therefore the total current due to 1 g. of uranium is $791 \times 4 \cdot 5 \times 10^{-13} = 3 \cdot 6 \times 10^{-10}$ ampere[¶]. The total current due to 1 g. of radium (plus products) is $1 \cdot 2 \times 10^{-3}$ ampere[**].

From these results it follows that radium is $3 \cdot 3 \times 10^6$ times as active as uranium[††]. This estimate may be too

[*] *Loc. cit.*
[†] Phil. Mag. [6] ix. p. 768 (1905).
[‡] *Cf.* Boltwood, Amer. J. Sci. xx. p. 239 (1905).
[§] Amer. J. Sci. xx. p. 55 (1905).
[||] Phil. Trans. A, 1901, p. 25.
[¶] McCoy, J. Amer. Chem. Soc. xxvii. p. 402 (1905).
[**] Rutherford, 'Radioactivity,' 1904, p. 156.
[††] The direct comparison of the activity of a thick film of pure U_3O with that of a minute (known) quantity of a pure radium salt will doubtless give a more reliable result.

low, since the activity of the radium, as given above, pro-
bably does not include that due to the slowly changing
products radium E and radium F. On the other hand, the
estimate of 4.25×10^6, made by the first method, may be too
high owing to the presence of one or more other radioactive
bodies intermediate between uranium and radium, in the
series of uranium transformation products. The existence of
such intermediate substances has been suggested by Soddy *
in explanation of the slowness with which radium emanation
is produced by uranium. Possibly emanium and its product †
actinium are among these supposed intermediate substances.

Chicago, August 1905.

XV. *Notices respecting New Books.*

Transactions of the International Electrical Congress, St. Louis, 1904.
Published under the care of the General Secretary and the
Treasurer. Printed by J. B. Lyon Company, Albany, N.Y.,
1905. Vol. I. pp. 879 ; Vol. II. pp. 984 ; Vol. III. pp. 980.

THESE handsome volumes, embodying the work of the Inter-
national Electrical Congress at St. Louis, will form a valuable
reference book alike for the pure physicist and the electrical
engineer. The scope of the Congress's activity was a very wide
one, as may be gathered from the following list of Sections:—
(A) General theory (including mathematical and experimental
papers) ; (B) General applications ; (C) Electro-chemistry ;
(D) Electric power transmission ; (E) Electric lighting and distri-
bution ; (F) Electric transportation ; (G) Electric communication ;
(H) Electrotherapeutics. As might have been expected, the
papers contributed vary a good deal as regards both length and
importance, some being little better than brief general notes,
while others are very thorough investigations of permanent value
to the scientific worker and engineer. Vol. I., which contains the
papers contributed to Sections A and B, should strongly appeal to
the pure physicist. It opens with a highly interesting paper by
Prof. H. T. Barnes on the Mechanical Equivalent of Heat, and
contains a large number of contributions to the ionic theory by
leading workers in this branch of science. The papers in Section B
are rather more technical, there being, in addition to some which
deal with hysteresis, magnetic viscosity, electrolytic rectifiers &c.,
others devoted to alternators, induction motors and single-phase
commutator motors. Volume II. contains the Transactions of
Sections C, D, and E. In Section C, we have valuable papers on

* *Loc. cit.*
† Marckwald, *Ber. d. chem. Ges.* xxxviii. p. 2264 (1905).

electrochemical theory, including one by Prof. T. W. Richards, " The Relation of the Hypothesis of Compressible Atoms to Electro-Chemistry," and others on electrolysis and various practical applications of electro-chemistry. The papers in Section D will be of great interest to engineers having to deal with high-voltage plants and transmission lines, as this subject is very fully discussed by many leading authorities. Section E is, perhaps, not quite up to the standard of those already noticed, as although it contains a few papers of real merit, there are others which deal in vague generalities without conveying any solid information. Vol. III. embraces Sections F, G, and H. In Section F, the problem of railway electrification naturally claims the first place, and the single-phase system comes in for a good deal of discussion. In Section G, we have papers on high-speed telegraphy, wireless telegraphy, and modern telephone exchanges. Section H will appeal to medical men : it contains papers on the uses of X-rays, photo-therapy and radio-therapy.

Although the work as a whole is well printed and illustrated, it seems a pity that the discussions on the various papers should have been put into very small type, so small as to make it somewhat trying to the eyes ; the larger type used for the papers might have been retained without unduly swelling the size of the volumes, and it would certainly have been less trying to the reader.

Anleitung zu wissenschaftlichen Beobachtungen auf Reisen. Herausgegeben von Professor Dr. G. VON NEUMAYER. Dritte Auflage. Lieferungen 1 & 2. Hannover : M. Jänecke, 1905.

THE first two parts of the third edition of the above work contain articles on the determination of latitude and longitude (by L. Ambronn), on topographical survey work (by P. Vogel), and on anthropology and ethnography (by F. v. Luschan). The last-named article contains many useful suggestions and is very explicit in its directions to the would-be explorer. The work is to be completed in about 12 parts.

Annuaire pour l'An 1906. Publié par le Bureau des Longitudes. Avec des Notices scientifiques. Paris : Gauthier-Villars. Prix : 1 fr. 50 c. Pp. iv + 712 + A.161 + B.18 + C.8 + D.41.

THIS wonderful little annual contains, in addition to the usual astronomical, physical, and physico-chemical tables, a special illustrated article by M. G. Bigourdan, entitled "Solar Eclipses. Brief instructions regarding the observations which may be made during them." In view of the interest manifested in the recent solar eclipse, the article is a timely and simple exposition of the subject with which it deals.

XVI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

November 8th, 1905.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

THE following communications were read :—

1. 'The Coast-Ledges in the South-West of the Cape Colony.
By Prof. Ernest Hubert Lewis Schwarz, A.R.C.S., F.G.S.

The following coast-shelves have been recognized by the author
in Cape Colony :—

Name of shelf.	Western. Feet.	Midlands. Feet.	Eastern. Feet.	Native Territories. Feet.
Cyphergat	5000–6000 ?	5000–6000 ?	5450 ?	—
Sterkstroom	3500–4000 ?	4000 ?	4406 ?	4500 ?
Kentani	......	......	2500	2500
De Vlugt	1500	1000	1500	1500
Uplands	700	463	467	600
Bamboes Bay	50–100	200	151	50–200
Sea-level	—	—	—	—
Agulhas	600			

The most striking of these is the Upland shelf, which extends from
Caledon to Port Elizabeth. It is cut by deep gorges into narrow
ridges or 'ruggens,' but at a height the level tops of these ridges
can be observed. The surface is in places covered with superficial
deposits, cemented boulder-deposits, gravels, and sandy clays,
hardened at the surface into ironstone or freshwater quartzite.
The author considers that this shelf cannot have been formed as a
peneplain, but by marine denudation. On the 150-to-200-foot plateau
there are deposits with marine shells, and in a depression on its top
the evaporation of rain-water produces a large quantity of salt.
The rock-shelf under the Cape Flats appears also to have been cut
by the sea. The Agulhas Bank seems to consist of a succession of
ledges, but it is not known whether further shelves extend beyond
its margin. Taking the ledges together, the continent would appear
to have been subject to lifts of 600 or 700 feet, with intermediate
halts and setbacks. The author introduces the term 'absolute
base-level of erosion' 'to express the ocean-floor, including the shelf
or level of erosion cut by the surf and off-shore currents that came
near the water's edge when the depression of the present land-
masses commenced.' The author compares the shelves of Cape
Colony with those described on the European and American sides of
the North Atlantic, and he places the 'absolute base-level of erosion'
at 12,000 feet in North America, 8000 feet in Europe, and 1200 feet

in South Africa. With these varying heights he correlates the topography of the bordering continents—the sharp divides, open river-valleys, permanent rivers and deltas, of Europe and America, where the movement has been downward and has almost reached bottom, in contrast with the flat undenuded divides, the steep, narrow gorges, the waterfalls, and the rocky river-gates, of South Africa, which is on the upgrade and probably near the top.

2. 'The Glacial Period in Aberdeenshire and the Southern Border of the Moray Firth.' By Thomas F. Jamieson, F.G.S.

One of the most interesting features in the Glacial geology of Aberdeenshire is the Red Clay found along the eastern coast of the county. It consists of red sediment brought by ice flowing along the coast from south to north, which also carried rocks from the coast between Montrose and Lunan Bay. The clay is sometimes finely laminated ; at other times it is mixed with stones ; and at times, again, contains esker-like mounds of gravel. It includes fragments of Crag-shells and of Mesozoic limestone. The purer masses of clay seem to have formed in a sheet of water lying in front of the ice, between it and the land, during the retreat of the Aberdeenshire ice, and at a time when the coast was submerged beneath water to a level exceeding 300 feet above the present coast-line. Evidence of the northward motion of the ice is given from striæ, the transport and removal of flints, and the bending-over of the edges of folia of gneiss. The red clay is underlain by a grey clay, and sometimes covered by a similar one. The author has recently discovered remains of a still older, dark indigo in colour, and containing small fragments of sea-shells. This has, however, been swept away in most places by subsequent ice-movement. In Banffshire a fine dark clay seems to have been formed under the same circumstances as the Red Clay of Aberdeenshire. The only evidence of warm intervals in this part of Scotland is that inferred from the melting-away of the masses of ice, which preceded and followed the deposition of the Red Clay and the shell-beds of Clava.

On the southern border of the Moray Firth the author gives examples of Glacial marking on the rocks, and refers to the transport of boulders, including a huge mass of Oolitic rocks 40 feet thick, a mass of clay once considered to be an outlier of Lias, 'pipe-rock,' and the fossiliferous Greensand débris at Moreseat, now considered to have been transported by ice. The ice appears to have been 4000 to 5000 feet thick about Inverness, in order to reach to and overflow Mormond in North-Eastern Aberdeenshire. The terraces of gravel found at decreasing heights on the Spey, near Rothes, seem to have been formed during stops in the retreat of the ice. Arctic shells of deep-water type, found

in silt-beds south of Rattray Head, give evidence of a period of considerable submergence; but the recurrence of an ice-sheet appears to have destroyed most of this evidence, as is seen in a section at King Edward.

November 22nd.—J. E. Marr, Sc.D., F.R.S., President, in the Chair.

The following communications were read:—

1. 'On a New Specimen of the Chimæroid Fish, *Myriacanthus paradoxus*, Ag., from the Lower Lias of Lyme Regis (Dorset).' By Arthur Smith Woodward, LL.D., F.R.S., F.L.S., F.G.S.

2. 'The Rocks of the Cataracts of the River Madeira and the adjoining portions of the Beni and Mamoré.' By John William Evans, D.Sc., LL.B., F.G.S.

The crystalline rocks of the cataracts of the River Madeira and the lower waters of its tributaries are part of a ridge with a north-westerly and south-easterly strike, similar to that of the Andes in the same latitudes. This strike is especially prevalent in Equatorial regions. With the exception of comparatively-recent alluvial deposits and a few pebbles of chert, pronounced by Dr. G. J. Hinde to be of marine origin but uncertain date, only crystalline rocks are met with in the falls. They all appear to be igneous, and are mostly massive in character, though some dyke-rocks occur. In places they are typical gneisses, and they are often banded, but in some cases they show no signs of foliation. The prevailing type is acid, with a considerable proportion of alkalies, especially soda; but some of the rocks are distinctly basic in character. Analyses of several of these rocks, made by Mr. G. S. Blake, are tabulated; and in one case the chemical analysis is compared with one made from the proportion of minerals washed out from the thin sections. Accounts of the megascopic and microscopic characters of all the rocks encountered are given. The more acid rocks are usually fine in grain, and are often granulitic in structure. In most cases the quartz seems to have crystallized out before the felspar. The occurrence of andalusite of chiastolitic type and of sillimanite as inclusions in a felspar is referred to, as well as the presence in one rock of an unusual type of allanite. An altered basalt is described, which contains minute concentric structures allied to those of a pyromeride. Above and below the region of the cataracts is a wide expanse of country covered with alluvium, either of recent or later Tertiary date.

3. 'The Doncaster Earthquake of April 23rd, 1905.' By Charles Davison, Sc.D., F.G.S.

The Doncaster earthquake of 1905 was a twin, with its principal

epicentre half a mile north of Bawtry, and the other about 4 miles east of Crowle and close to the centre of the disturbed area of the Hessle earthquake of April 13th, 1902. The distance between the two epicentres is about 17 miles. The disturbed area contains about 17,000 square miles, including the whole of the counties of Lincoln, Nottingham, Derby, Stafford, Leicester, and Rutland, the greater part of Yorkshire, and portions of Lancashire, Cheshire, Shropshire, Worcestershire, Warwickshire, Northamptonshire, Cambridgeshire, and Norfolk. The originating fault runs from about E. 38° N. to W. 38° S., and appears to be nearly vertical within the south-western focus and inclined to the south-east in the north-eastern focus. The first and stronger movement took place within the south-western focus. A twin-earthquake is probably due to the differential growth of a crust-fold along a fault which intersects it transversely, the first movement as a rule being one of rotation of the middle limb, accompanied by the almost simultaneous slip of the two arches, and followed soon afterwards by a shift of the middle limb. The movements, in which the Doncaster earthquake originated, presented a slight variation in this order. They consisted of successive, but continuous, displacements, first of the south-western arch, then of the middle limb, and finally of the north-eastern arch.

<div align="center">December 6th.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.</div>

The following communications were read :—

1. 'The Physical History of the Great Pleistocene Lake of Portugal.' By Prof. Edward Hull, LL.D., F.R.S., F.G.S.

The formations bordering the lower banks of the Tagus near Lisbon are arranged by the author in the following order of succession :—

6. Recent & Quaternary.	Alluvia of the Tagus.
5. Lacustrine.	{ Marls with *Lymnæa* (Lacustre superior). { Sands and gravel.
4. Post-Pliocene & Pliocene.	{ Not represented, unless by some land-glacial { beds due to elevation.
3. Miocene.	'Almada Beds.' Calcareous marls and limestones, with marine fossils.
2. Eocene (?).	Unfossiliferous sands and gravels (Lacustre inferior).
1. Upper Cretaceous.	Hippurite-Limestone.

A description is given of the Lacustre superior; the Almada Beds are considered to be Miocene, and as the Pliocene is not represented, except possibly by certain glacial deposits, the author considers that that period was one of great uplift, when the suboceanic gorge,

an extension of the present course of the River Tagus, was exca-
vated. The margin of the lake was probably formed by the granite
of Das Vargans and Cunheira. There is evidence that the general
level of the lake-bed was nearly that of the outer sea, and that the
sea-waters gained occasional access to the lake during the earlier
stage of its formation. The lake was eventually drained by the
channel cut by the Tagus at the harbour of Lisbon, upon the
elevation of the land to about its present level.

2. 'The Geological Structure of the Sgùrr of Eigg.' By Alfred
Harker, M.A., F.R.S., F.G.S.

The pitchstone which forms the Sgùrr of Eigg is a massive sheet,
some 400 feet thick, reposing with discordance upon the succession
of alternating basalts and dolerites which make up the greater part
of the island. The lower surface of the pitchstone is irregularly
undulating, and in two places fragmental accumulations are seen
immediately beneath it. The generally-received interpretation
regards the pitchstone as a lava-flow, or series of flows, occupying
an old river-valley excavated in the basalts, and the fragmental
deposits have been regarded as river-gravels of the pitchstone-age.
This is the view put forward by Sir Archibald Geikie.

After a detailed survey of the ground, the author finds it
impossible to accept this view, and he gives reasons for considering
the pitchstone to be intrusive. The form of its base, as mapped
out, does not seem to be reconcilable with that of a river-valley,
and its character is that of an intrusive junction rather than an
erosion-surface. The fragmental deposits are believed to be of
volcanic origin and of the basalt-age. The one exposed at the
seaward termination of the ridge is a volcanic agglomerate, probably
filling a small vent. The other, seen at the southerly base of the
Sgùrr, is a bedded agglomerate, partly rearranged by water-action.
The Torridonian and Oolitic sandstone-blocks which are abundant
in it are held to have been brought up from below, and fossil wood
of Oolitic age has been brought up in the same manner. The
absence of fragments of the sill-dolerites (themselves younger than
the lavas, but cut off by the pitchstone) in both accumulations
seems to assign them unequivocally to the age of the basalt, and
their conjunction with the pitchstone must then be considered
accidental.

The conclusions arrived at bring the rock of the Sgùrr of Eigg
into relation with the other British Tertiary pitchstones, which are
all intrusive. Thus also is avoided the difficulty of assuming a
great erosion in inter-volcanic times, a hypothesis for which the
supposed river-valley was the sole evidence.

Table III.

Abscissae (ωt)	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
Given ords f(t)	101	280	479	609	634	602	535	399	213	77	33	-4	-88	-153	-159	-202	-377	-606	-720	-659	-495	-313	-156	-29
													88	153	159	202	377	606	720	659	495	313	156	29
2(H₁+H₃+H₅+)	189	433	638	811	1011	1208	1255	1058	708	390	189	25												
2(H₁+H₂+H₃+H₄+)	202	560	958	1218	1268	1204	1070	798	426	154	66	-8												
2(H₂+H₄+H₆+H₇)	15	127	320	407	257	-4	-185	-260	-282	-236	-123	-33												
	185	260	282	236	123	35																		
4(H₂+H₆+)	198	387	602	643	380	29																		
4(H₂+H₄+H₆)	26	254	640	814	514	-8																		
4 H₄	-172	-133	38	→																				

$$H_4 = 45 \sin 4(\omega t - 42)$$

4(H₂+H₆)	198	387	602	643	380	29																		
	380	29																						
	578	416																						
	602	643																						

12 H₆	-24	→	→	8	-76	-8	-76	→																
4 H₆	-8	-76																						
4 H₂	206	463	594	567	388	105	→																	

$$H_6 = -19 \sin 6(\omega t - 14)$$

$$H_2 = 151 \sin 2(\omega t - 5°)$$

2(H₁+H₃+H₅-)	189	433	638	811	1011	1208	1255	1058	708	390	189	25												
	708	390	189	25																				
	897	823	827	836																				
	1011	1208	1255	1058																				

6 H₃	-114	-585	-428	-222	→	38	143	74	-38	-128	-143	-74												
2H₃	-38	-128	-143	-74		128	143																	
2(H₁+H₅)	227	561	781	885	973	1080	1112	984	746	518	332	99												

$$H_3 = -74 \sin 3(\omega t - 10°)$$

H₁ + H₅   [10 e.s. ords from plot of]

ωt	15°	33°	51°	69°	87°	105°	123°	141°	159°	177°
H₁+H₅	113	307	413	467	524	557	472	324	210	79
					524	557	472	324		
					210	79				
					847	943	885	791		
					885	791				
5 H₅	-38	152	→	→		-8	30	8	-30	-8	30
H₅	-8	30	8	-30	30					
H₁	121	277	405	497	532	527	464	354	218	49

$$H_5 = 31 \sin 5(\omega t - 18°)$$

$$H_1 = 540 \sin(\omega t - 2°)$$

$$f(t) = 540 \sin(\omega t - 2°) + 151 \sin 2(\omega t - 5°) - 74 \sin 3(\omega t - 10°) + 45 \sin 4(\omega t - 42°) + 31 \sin 5(\omega t - 18°) - 19 \sin 6(\omega t - 14°)$$

Table 1.

Abscissae (ωt)	12°	24°	36°	48°	60°	72°	84°	96°	108°	120°	132°	144°	156°	168°	180°	192°	204°
H₁+H₇+H₁₁	35	250	435	597	752	881	955	976	968	933	847	706	542	372	182	−35	−250
−H₅	43	4	−39	−43	−4	39	43	4	−39	−43	−4	39	43	4	−39		
−C₃	−73	35	135	182	153	73	−35	−135	−182	−153	−73	35	135	182	153		
Given ords=f(t)	65	211	339	458	603	769	947	1107	1189	1129	924	632	364	186	68		

	924	632	364	186	68
	989	843	703	644	671
	769	947	1107	1189	1129
3C₃	220	−104	−404	−545	−458
3C₁₅	1	−1	1	−1	1
5(H₃+H₃)	219	−103	−405	−544	−459

	65	211	339	458	603	769
	947	1107	1189	1129	924	63?
	364	186	68			
	1376	1504	1596	1587	1527	1401
	1587	1527	1401			
5C₅	−211	−23	195			
5C₁₅	2	−2	0	= ⅙[−211+23+195]		
5H₅	−213	−21	193			

Amp. H₁ / Phase H₁ − α

	Sines	angles	12°−α
306	·0355	2° 2′	2° 2′
1562·6			
47·300	·2513	14°·40	2° 40′
·301·0	·4407	26° 9	2° 9
·41·370	·6048	37° 13	1 13
194·000			
221·00			
258·144			
234·58	·7152	45°·40	3 20
217·00	·5491	33°·18	2 42
170·752	·3768	22° 6	1 52
·12·003	·854	10°·41	16 17
73·441			16 17
34·596		mean = 2° 2′
8281		
18260·07		
·5 36551·24		α = 9°·58′
243·475		
h₁ = 987		

Amp. H₃ / Phase H₃

	Sines	angles	3(24°−β)
11·990	·1907	11° 0	11° 0
2·635			
41·006	·7500	48°·35	12° 40
3·20	·8500	58 12	13 48
52·470	·9050	23 35	18 4
57·54·200			49 22
7899·7		12 22
5h₃ = 540		24°−β = 4°·7′
h₃ = 180		β = 19°·53′
		crossing down

Amp. H₅ / Phase H₅

	Sines	angles	5(36°−γ)
11·342	·8212	55°·12′	55°·12′
1·10			
33·12			
2076·4	·0894	5° 6′	54°·52′
3·1·332		mean = 55°
·3693		
5h₅ = 235		γ = 25°
h₅ = 47		crossing up

$$f(t) = 987 \sin(\omega t - 9°58′) - 180 \sin 3(\omega t - 19°53′) + 47 \sin 5(\omega t - 25°).$$

Table II.

Abscissae (ωt)	12°	24·9°	37·7°	50·6°	63·4°	76·3°	89·1°	102°	114·9°	127·7°	140·6°	153·4°	166·3°	179·1°	192°
H₁	34	253	459	641	793	905	969	987	951	871	749	587	597	186	−34
−H₇	−1	−11	1	11	−1	−11	1	11	−1	−11	1	11	−1	−11	
H₁+H₇	35	264	458	630	794	916	968	976	952	882	748	576	398	197	

	794	916	968	976
	952	882	748	576
	398	197		
	2179	2259	2174	2182
	2174	2182		
7 H₇	5	77		

Amp. H₇ = $\frac{1}{7}\sqrt{5^2 + 77^2}$ = 11

Phase H₇ = δ

$\sin 7(12° − δ) = 5/77$
$7(12° − δ) = 3°43′$
δ = 11°·28′

Corrected Amp. H₁ = $\sqrt{987^2 − 11^2}$ = 987

Corrected Phase H₁

Sines	angles	
·0344	1° 58′	1° 58′
·2563	14° 51′	2° 51
·4650	40° 30	2° 30
·6494		1° 38
·7508	49°·21′	
·5947	36°·30	2°·4′
·4042	25°·33	2°·6
·1684	10° 51	2°·0
	mean = 2°	
	α = 10°.	

$$f(t) = 987 \sin(\omega t - 10°) - 180 \sin 3(\omega t - 19°53′) + 47 \sin 5(\omega t - 25°) + 11 \sin 7(\omega t - 11°28′)$$

Table III.

Abscissae (ωt)	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
Given ords f(t)	101	280	479	609	634	602	535	399	213	77	33	−4	−88	−153	−159	−202	−377	−606	−720	−659	−495	−313	−156	−29
	88	153	159	202	377	606	780	659	495	313	156	29												
2(H₁+H₅+H₆+	189	433	638	811	1011	1208	1255	1058	708	390	189	25
2(H₁+H₂+H₃+H₆)	202	560	938	1218	1268	1204	1070	798	486	154	66	−8
2(H₂+H₄+H₆)	13	127	320	407	257	−4	−185	−260	−282	−236	−123	−33
	185	260	282	236	123	33
4(H₂+H₆+	198	387	602	643	380	29
4(H₂+H₄+H₆)	26	254	640	814	514	−8
4 H₄	−172	−133	38	→		→ H₄=45 sin 4(ωt−42)
4(H₂+H₆)	198	387	602	643	380	29
	380	416
	378	416
	608	643
12 H₆	−24	227		→	H₆ = −19 sin 6(ωt−14)
4 H₆	−8	−76	8	76	−8	−76
4 H₂	206	463	594	567	388	105	→ H₂=151 sin 2(ωt− 5°)
2(H₁+H₃+H₅−)	189	433	638	811	1011	1208	1255	1058	708	390	189	25
	708	390	189	25
	897	823	827	836
	1011	1208	1255	1058
6 H₃	−114	−388	−428	−822
2 H₃	−38	−128	−143	−74	38	128	143	74	→ H₃=−74 sin 3(ωt−10°)
2(H₁+H₅)	227	361	781	885	973	1080	1112	984	746	518	332	99

H₁+H₅ [10 e.s ords from plot of]

ωt	15°	33°	51°	69°	87°	105°	123°	141°	159°	177°
H₁+H₅	113	307	413	467	524	557	472	324	210	79
	524	557	472	324						
	210	79								
	847	943	885	791						
	885	791								
5 H₅	−38	152		→ H₅ = 31 sin 5(ωt −18°)
H₅	−8	30	8	−30	−8	30	8	−30	−8	30
H₁	121	277	405	497	532	527	464	354	218	49

H₁= 540 sin (ωt−2°)

$$f(t) = 540 \sin(\omega t - 2°) + 151 \sin 2(\omega t - 5°) - 74 \sin 3(\omega t - 10°) + 45 \sin 4(\omega t - 42°) + 31 \sin 5(\omega t - 18°) - 19 \sin 6(\omega t - 14°).$$

FIG 3
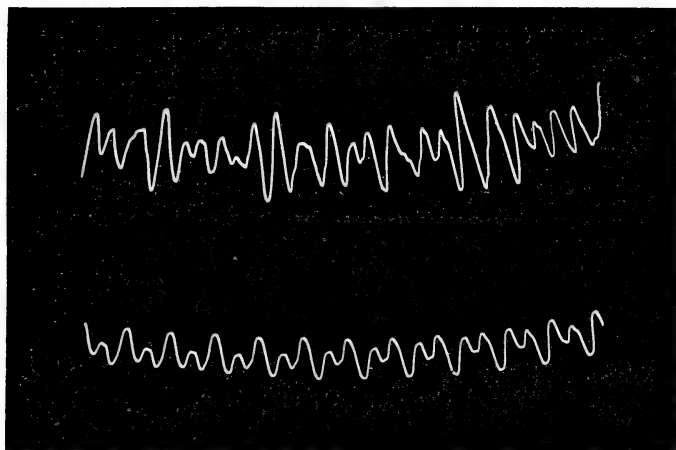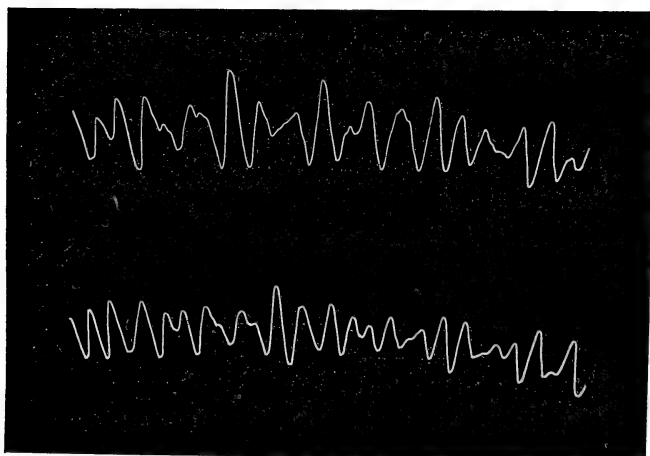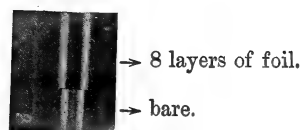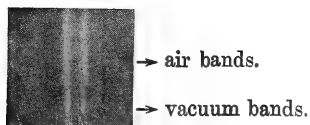
Fig. 1

Fig. 2

Fic 3

Fig. 1.

Fig. 2.

Fig. 1.

→ 8 layers of foil.

→ bare.

Magnification 1·4.

Fig. 2.

A   B

Magnification 1·4.

Fig. 3.

→ air bands.

→ vacuum bands.

Magnification 1·4.

XVII. *On the Diffraction of Short Waves by a Rigid Sphere.*
By J. W. NICHOLSON, *B.A., M.Sc., Scholar of Trinity College,*
Cambridge[*].

THE effect of a rigid sphere upon incident plane sound-
waves has been examined by Lord Rayleigh[†], who has
obtained a complete solution of the case in which the waves
have a length great in comparison with the dimensions of
the sphere. In the more general case, Lord Rayleigh has
expressed the disturbed motion in the form of an infinite
series, whose terms consist of products of surface zonal
harmonics and functions derived from Bessel functions.
The summation of this series cannot be effected; but, as
Lord Rayleigh has shown, when the waves are very long, it
practically reduces to its first two terms. In this paper, a
close approximation to the diffracted waves will be obtained
when the waves are very short in comparison with the radius
of the spherical obstacle. The method adopted, before the intro-
duction of the approximations, is that used by Lord Rayleigh.
The final result appears in the form of a definite integral
which may be evaluated to the same order of approximation.

The motion external to the sphere, whether due to the
passage of sound, or to the presence of small waves in a
frictionless fluid, is supposed to possess a velocity potential.

Taking the centre of the sphere as origin of polar coordinates, and putting $\cos \theta = \mu$, then the velocity potential in an incident plane wave travelling along x may be expressed as

$$\phi = e^{ik(r\mu - at)}. \qquad \cdots \cdots \quad (1)$$

Omitting the time factor, this may be written[*]

$$\phi = A_0 + A_1 P_1(\mu) + A_2 P_2(\mu) + \ldots + A_n P_n(\mu) + \ldots \quad (2)$$

where

$$A_n = (2n+1) \cdot i^n \left(\frac{\pi}{2kr} \right)^{\frac{1}{2}} \cdot J_{n+\frac{1}{2}}(kr). \qquad \cdots \quad (3)$$

The diffractive effect is derived from a velocity potential ψ, where

$$(\nabla^2 + k^2)\psi = 0,$$

or

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + k^2 r^2 \psi = 0;$$

since ψ must, like ϕ, be a function of zonal character. We deduce at once

$$\psi = \Sigma_0^\infty \psi_n P_n(\mu), \qquad \cdots \cdots \quad (4)$$

where

$$\frac{\partial^2}{\partial r^2}(r\psi_n) + \left(k^2 - \frac{n \cdot n+1}{r^2} \right) r(\psi_n) = 0. \qquad \cdots \quad (5)$$

This is satisfied by the Bessel function $\psi_n = r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(kr)$, but, for the present purpose, we require a solution finite at infinity, corresponding to a very large value of k. The ordinary asymptotic expansions for Bessel functions cannot be used, as they take no account of the cases in which n is very large.

Make the substitution [†]

$$r\psi_n = e^{\pm ik\omega}\phi_n, \qquad \cdots \cdots \quad (6)$$

where (ω, ϕ_n) are functions of r and n.

Substituting in the differential equation, and equating terms of different order in k separately to zero,

$$\phi_n = \left(1 - \frac{n \cdot n+1}{k^2 r^2} \right)^{-\frac{1}{4}}, \qquad \cdots \quad (7)$$

$$\omega = \int^r \left(1 - \frac{n \cdot n+1}{k^2 r^2} \right)^{\frac{1}{2}} dr. \qquad \cdots \quad (8)$$

where a term ϕ_n'' has been neglected in comparison with k.

The last equation gives

$$k\omega = (k^2 r^2 - n \cdot n + 1)^{\frac{1}{2}} - (n \cdot n+1)^{\frac{1}{2}} \cos^{-1}\left(\frac{n \cdot n+1}{k^2 r^2} \right)^{\frac{1}{2}}. \quad (9)$$

  * Lord Rayleigh, *loc. cit.*
  † Webb, Proc. Roy. Soc. vol. lxxiv. p. 315 (1904).

The general solution of (5) is of the form

$$\psi_n = \frac{Ae^{\iota k\omega} + Be^{-\iota k\omega}}{(k^2r^2 \cdot k^2r^2 - n \cdot n+1)^{\frac{1}{4}}}.$$

We now proceed to determine the values of A and B making this identical with

$$\frac{J_{n+\frac{1}{2}}(kr)}{(kr)^{\frac{1}{2}}}.$$

By Lommel's theorem, when z is large, we have

$$J_n(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos\left(z - \frac{\pi}{4} - n\frac{\pi}{2}\right) \cdot \left\{1 - \frac{1^2 - 4n^2 \cdot 3^2 - 4n^2}{2\,!\,(8z)^2} + \ldots\right\}$$
$$+ \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin\left(z - \frac{\pi}{4} - n\frac{\pi}{2}\right) \cdot \left\{\frac{1^2 - 4n^2}{8z} - \frac{1^2 - 4n^2 \cdot 3^2 - 4n^2 \cdot 5^2 - 4n^2}{3\,!\,(8z)^3} + \ldots\right\} \quad (10)$$

whence, retaining only lowest powers of $\frac{1}{z}$, our solution must be identical with

$$\int\frac{J_{n+\frac{1}{2}}(kr)}{(kr)^{\frac{1}{2}}} = \left(\frac{2}{\pi}\right)\cos\left(kr - n+1 \cdot \frac{\pi}{2}\right) \cdot \left\{\frac{1}{kr} - \frac{n-1 \cdot n \cdot n+1 \cdot n+2}{8(kr)^3}\right\}$$
$$- \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cdot \sin\left(kr - n+1 \cdot \frac{\pi}{2}\right) \cdot \left\{\frac{n \cdot n+1}{2\,(kr)^2}\right\}.$$

Since

$$\frac{1}{(k^2r^2 - n \cdot n+1)^{\frac{1}{4}}} = \frac{1}{(kr)^{\frac{1}{2}}}\left\{1 + \frac{n \cdot n+1}{4k^2r^2} + \ldots\right\}$$

and

$$k \cdot \left(1 - \frac{n \cdot n+1}{k^2r^2}\right)^{\frac{1}{2}}dr = kr + \frac{n \cdot n+1}{2kr} + \ldots.$$

Therefore

$$e^{\pm \iota k\omega} = e^{\pm \iota kr}\left\{1 \pm \frac{\iota n \cdot n+1}{2kr} + \ldots\right\};$$

whence on substitution

$$A = \sqrt{\frac{2}{\pi}}\,e^{-\frac{\iota\pi}{4}}, \qquad B = \sqrt{\frac{2}{\pi}}\,e^{\frac{\iota\pi}{4}}; \quad \cdot \quad \cdot \quad (11)$$

and finally, when kr is large, and no restriction is laid upon the value of n, except that it be less than kr,

$$\frac{J_{n+\frac{1}{2}}(kr)}{(kr)^{\frac{1}{2}}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cdot \frac{\cos\left\{(k^2r^2 - n \cdot n+1)^{\frac{1}{2}} - (n+\frac{1}{2})\cos^{-1}\dfrac{(n \cdot n+1)^{\frac{1}{2}}}{kr} - \dfrac{\pi}{4}\right\}}{\{(k^2r^2)(k^2r^2 - n \cdot n+1)^{\frac{1}{4}}\}}$$
$$\cdot \quad \cdot \quad \cdot \quad (12)$$

O 2

If n be neglected in comparison with kr, we get the well-known asymptotic expansion of the Bessel function.

For the diffracted waves diverging from the sphere, the appropriate solution is

$$\psi_n = \frac{C_n e^{-\iota k\omega}}{(k^2 r^2 \cdot k^2 r^2 - n \cdot n + 1)^{\frac{1}{4}}} \cdot \quad \cdot \quad \cdot \quad \cdot \quad (13)$$

Let c be the radius of the obstacle. Since there cannot be motion normal to it, C_n is determined by

$$\frac{\partial}{\partial r}(A_n + \psi_n) = 0 \text{ when } r = c; \quad \cdot \quad \cdot \quad (14)$$

or

$$\frac{\partial}{\partial (kc)}\left\{ (2n+1)\iota^n \left(\frac{\pi}{2kc}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(kc) \right\} = -C_n \frac{\partial}{\partial (kc)}\left\{ \frac{e^{-\iota k\omega_c}}{(k^2 c^2 \cdot k^2 c^2 - n \cdot n + 1)_{\frac{1}{4}}} \right\}.$$

Since

$$\left(\frac{\pi}{2kc}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(kc) = \frac{\cos\left(k\omega_c - \frac{\pi}{4}\right)}{(k^2 c^2 \cdot k^2 c^2 - n \cdot n + 1)^{\frac{1}{4}}} \text{ by } (12),$$

and

$$\frac{\partial (k\omega_c)}{\partial (kc)} = \frac{(k^2 c^2 - n \cdot n + 1)^{\frac{1}{2}}}{kc} \text{ by } (8),$$

we obtain

$$C_n = -(2n+1)\iota^n \frac{\partial}{\partial (kc)}\left\{ \frac{\cos\left(k\omega_c - \frac{\pi}{4}\right)}{(k^2 c^2 \cdot k^2 c^2 - n \cdot n + 1)^{\frac{1}{4}}} \right\} \bigg/ \frac{\partial}{\partial (kc)}\left\{ \frac{e^{-\iota k\omega_c}}{(k^2 c^2 \cdot k^2 c^2 - n \cdot n + 1)^{\frac{1}{4}}} \right.$$

Writing w for kc, and H for $k^2 c^2 - n \cdot n + 1$, the denominator becomes

$$\frac{e^{-\iota k\omega_c}}{w \cdot H^{\frac{1}{2}}}\left\{ -\iota w^{-\frac{1}{2}}H^{\frac{2}{4}} - \frac{H^{\frac{1}{4}}}{2}w^{-\frac{1}{2}} - \frac{w^{\frac{1}{2}}}{4}H^{-\frac{3}{4}}2w \right\},$$

or

$$-\frac{e^{-\iota k\omega_c}}{w^{\frac{3}{2}}H^{\frac{5}{4}}}\left\{ \frac{w^2}{2} + \frac{H}{2} + \iota H^{\frac{3}{2}} \right\}.$$

The numerator has the value

$$\frac{-(2n+1)\iota^n}{w \cdot H^{\frac{1}{2}}}\left\{ \left(-w^{-\frac{1}{2}}H^{\frac{3}{4}}\right)\sin\left(kw_c - \frac{\pi}{4}\right) \right.$$

$$\left. - \frac{(H^{\frac{1}{4}}w^{-\frac{1}{2}} + w^{\frac{3}{2}}H^{-\frac{3}{4}})}{2} \cdot \cos\left(kw_c - \frac{\pi}{4}\right) \right\};$$

or, if θ_1 be the argument of the trigonometric functions, the numerator is

$$+\frac{(2n+1)\iota^n}{2w^{\frac{3}{2}}H^{\frac{1}{4}}}\{(H+w^2)\cos\theta_1+\iota H^{\frac{3}{2}}\sin\theta_1\};$$

and therefore, returning to the old notation,

$$C_n=-(2n+1)\,\iota^n\,.\,e^{+\iota k\omega c}\left\{\frac{(2k^2c^2-n\,.\,n+1)\cos\theta_1+\iota(k^2c^2-n\,.\,n+1)^{\frac{3}{2}}\sin\theta_1}{2k^2c^2-n\,.\,n+1+\iota(k^2c^2-n(n+1))^{\frac{3}{2}}}\right\},$$

$$\cdots\cdots\quad(15)$$

and the velocity potential of the diffracted waves is

$$\sum_0^{\infty}\psi_n\,P_n(\mu),$$

or

$$\psi=\sum_0^{\infty}\frac{C_n\,.\,e^{-\iota k\omega}\,P_n(\mu)}{(k^2r^2\,.\,k^2r^2-n\,.\,n+1)^{\frac{1}{4}}}\quad\cdots\quad(16)$$

at the point defined by $(r,\,\mu)$ referred to the centre of the sphere as origin, and the direction of the incident waves as initial line.

The approximate formulæ above are only strictly true if kc is infinite, but are very accurate if kc is only a fairly large quantity. The summation with respect to n from zero to infinity may with great accuracy be replaced by a summation from zero to an integer close to the quantity kc. Owing to the fact that kc has been really treated as an infinite quantity, all the values of n, which can give rise to an appreciable effect in the diffracted wave, will be included in the new range from zero to kc. Terms near $n=kc$ are reserved for future treatment.

Now $\qquad n(n+1)=(n+\tfrac{1}{2})^2-\tfrac{1}{4}.$

Denoting $\dfrac{n+\frac{1}{2}}{kc}$ by x, we note that x is a quantity rising by increments $\dfrac{1}{kc}$ from $\dfrac{1}{2kc}$ to 1, or, to a sufficient order of approximation, from zero to unity.

Accordingly we write

$$\frac{(n\,.\,n+1)^{\frac{1}{2}}}{kc}=\left\{\left(\frac{n+\frac{1}{2}}{kc}\right)^2-\frac{1}{4k^2c^2}\right\}^{\frac{1}{2}}$$

$$=\frac{n+\frac{1}{2}}{kc}\text{ to a sufficient order.}$$

Also
$$\iota^n = \exp.\left(\iota kcx\,\frac{\pi}{2} + \frac{\iota\pi}{4}\right)$$

$$\omega = (r^2 - c^2 x^2)^{\frac{1}{2}} - cx\,\cos^{-1}\frac{cx}{r}$$

$$\omega_r = c(1-x^2)^{\frac{1}{2}} - cx\,\cos^{-1}x,$$

and
$$C_n = -2kcx\left\{\frac{(2-x^2)\cos\theta_1 + \iota kc(1-x^2)^{\frac{3}{2}}\sin\theta_1}{2-x^2+\iota kc(1-x^2)^{\frac{3}{2}}}\right\}$$
$$\times e^{\frac{\iota\pi}{4}}.\exp.\iota kc\left\{(1-x^2)^{\frac{1}{2}} + x\sin^{-1}x\right\} \quad . \quad (17)$$

where
$$\theta_1 = kc\left\{(1-x^2)^{\frac{1}{2}} - x\cos^{-1}x\right\} - \frac{\pi}{4}, \quad . \quad . \quad (18)$$

Now the sum of a finite series $\Sigma S(x)$ of the type (16) may be shown to be given by *
$$\Sigma S(x) + \Sigma R = \frac{kc}{2\pi\iota}\int_C S(x)\,.\,L\,dx,$$

where kc is large, and the large integer less than kc is r, where $r = kc(1-\epsilon)$, in which ϵ is small. L denotes the principal value of
$$\log x/(x-1+\epsilon).$$

R is a residue of $S(x)$ at any pole which it may have in the contour C, a unit circle round the origin.

But the poles of $S(x)$ in this case are all complex, being the roots of the equation $2-x^2+\iota kc(1-x^2)^{\frac{3}{2}}=0$. The corresponding residues cannot contain a factor of the form exp.$(kc\phi)$, where ϕ is real and positive, from the form of the integral and from physical considerations. The imaginary portion of the root corresponding to a pole will therefore introduce a real exponential factor of large negative argument into R, which may accordingly be neglected.

Thus the velocity potential of the secondary disturbance is

$$\dot{\psi} = -2\frac{kc^2}{r}e^{\frac{\iota\pi}{4}}\int_C \frac{Lx\,dx}{\left(1-\frac{c^2x^2}{r^2}\right)^{\frac{1}{4}}}\left\{\frac{(2-x^2)\cos\theta_1 + \iota kc(1-x^2)^{\frac{3}{2}}\sin\theta_1}{2-x^2+\iota kc(1-x^2)^{\frac{3}{2}}}\right\}P_{-\frac{1}{2}+kcx}(\mu)$$

$$\times \exp.\iota kc\left\{(1-x^2)^{\frac{1}{2}} - x\cos^{-1}x + \frac{\pi x}{2} - \left(\frac{r^2}{c^2}-x^2\right)^{\frac{1}{2}} + x\cos^{-1}\frac{cx}{r}\right\}, \quad . \quad (19)$$

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* This theorem was given by the author in a paper read before the London Mathematical Society on Nov. 9th. It is proved by the application of contour integration to the function
$$S(x)\left\{\frac{1}{x} + \frac{1}{x-\frac{1}{kc}} + \frac{1}{x-\frac{2}{kc}} + \ldots + \frac{1}{x-\frac{r}{kc}}\right\}.$$

where
$$L = \frac{1}{2\pi\iota}\log x/(x-1+\epsilon).$$

This formula is of most interest in the case of points lying along the axis of x, for the Legendre function then takes the values $(\pm 1)^n$.

For a point at a great distance r on the side from which the incident waves come, expanding the terms containing r,

$$\psi = -2kc^2\, e^{\frac{\iota\pi}{4}}\left(\frac{e^{-\iota kr}}{r}\right)\int_C Lx\, dx\left\{\frac{(2-x^2)\cos\theta_1 + \iota kc(1-x^2)^{\frac{3}{2}}\sin\theta_1}{2-x^2+\iota kc(1-x^2)^{\frac{3}{2}}}\right\}$$
$$\times \exp.\ \iota kc\{(1-x^2)^{\frac{1}{2}} - x\cos^{-1}x + x\pi\}, \quad . \quad . \quad (20)$$

On the opposite side, $\mathrm{P}_{-\frac{1}{2}+kcx}(\mu) = (-)^n$
$$= e^{\iota\pi - (\frac{1}{2}+kcx)},$$

and $$\psi = -2kc^2\cdot e^{-\frac{\iota\pi}{4}}\left(\frac{e^{-\iota kr}}{r}\right)\int_C Lx\, dx\left\{\frac{2-x^2\cdot\cos\theta_1 + \iota kc(1-x^2)^{\frac{3}{2}}\sin\theta_1}{2-x^2+\iota kc(1-x^2)^{\frac{3}{2}}}\right\}$$
$$\times \exp.\ \iota kc\{(1-x^2)^{\frac{1}{2}} - x\cos^{-1}x + 2x\pi\} \quad . \quad . \quad (21)$$

Round the circumference of the sphere the integral takes a much simpler form, but is invalid.

Returning to the formula (19), we note that, using the polar angle ϕ as variable ($x = e^{\iota\phi}$), the contour is traced by making ϕ range from zero to 2π, x takes the value unity at both limits (whilst its logarithm has the values zero and $2\pi\iota$ respectively). But the value $x=1$ corresponds to a large value of the integer $n = -\frac{1}{2} + kcx$.

We therefore, with great accuracy, may write the asymptotic value
$$\mathrm{P}_n(\mu) = \left(\frac{2}{n\pi\sin\theta}\right)^{\frac{1}{2}}\cos\left((n+\tfrac{1}{2})\theta - \frac{\pi}{4}\right) \quad . \quad . \quad (22)$$

throughout the range.

In terms of x,
$$\mathrm{P}_{-\frac{1}{2}+kcx}(\mu) = \left(\frac{2}{kcx\pi\sin\theta}\right)^{\frac{1}{2}}\cos\left(kcx\theta - \frac{\pi}{4}\right). \quad . \quad (23)$$

Therefore at a point (r, θ) not close to the axis of x

$$\psi = -\left(\frac{8kc^3}{\pi\sin\theta}\right)^{\frac{1}{2}}\frac{e^{\frac{\iota\pi}{4}}}{r}\int_C \frac{Lx^{\frac{1}{2}}dx}{\left(1-\frac{c^2x^2}{r^2}\right)^{\frac{1}{4}}}\left\{\frac{(2-x^2)\cos\theta_1 + \iota kc(1-x^2)^{\frac{3}{2}}\sin\theta_1}{2-x^2+\iota kc(1-x^2)^{\frac{3}{2}}}\right\}$$

$$\times \cos\left(kcx\theta - \frac{\pi}{4}\right).\ \exp.\ \iota kc\left\{(1-x^2)^{\frac{1}{2}} - x\cos^{-1}x + \frac{\pi x}{2} - \left(\frac{r^2}{c^2} - x^2\right)^{\frac{1}{2}}\right.$$

$$\left. + x\cos^{-1}\frac{cx}{r}\right\}. \quad . \quad . \quad (24)$$

and if r is great compared with dimensions of obstacle

$$\psi = -\left(\frac{8kc^3}{\pi \sin \theta}\right)^{\frac{1}{2}} . e^{\frac{\iota\pi}{4}} \left(\frac{e^{-\iota kr}}{r}\right)\int_{c} \mathrm{LP}(x) . x^{\frac{1}{2}} \cos \left(kcx\theta - \frac{\pi}{4}\right)$$
$$\times \exp. \iota kc\{(1 - x^2)^{\frac{1}{2}} - x \cos^{-1}x + x\pi\}dx, \quad . \quad . \quad (25)$$

where $\mathrm{P}(x)$ denotes the function

$$\frac{(2 - x^2) \cos \theta_1 + \iota kc(1 - x^2)^{\frac{3}{2}} \sin \theta_1}{2 - x^2 + \iota kc(1 - x^2)^{\frac{3}{2}}}, \quad . \quad . \quad (26)$$

which always lies between positive and negative unity.

To obtain an integral formula, free from the zonal harmonic term, and suitable for the case of points near the axis of x, we note that the asymptotic expansions of $\mathrm{P}_n(\cos \theta)$ and of $\mathrm{J}_0(z)$ are respectively

$$\sqrt{\frac{2}{n\pi \sin \theta}} . \cos\left(\overline{n+\tfrac{1}{2}} . \theta - \frac{\pi}{4}\right), \text{ and } \sqrt{\frac{2}{\pi z}}\cos \left(z - \frac{\pi}{4}\right),$$

and therefore when n is great

$$\mathrm{P}_n(\cos \theta) = \mathrm{J}_0\left(2n \sin \frac{\theta}{2}\right). \quad . \quad . \quad . \quad (27)$$

We therefore, under the integral sign in (19), merely replace

$$\mathrm{P}_{-\frac{1}{2}+kcx}(\mu) \quad \text{by} \quad \mathrm{J}_0\left(2kcx \sin \frac{\theta}{2}\right)$$

to obtain a simpler form.

On the harmonic terms for which n *is greater than* kc.

The quantity ω_c now becomes imaginary, and the exponentials of the previous investigation become those of real quantities. Since ψ cannot be infinite at great distance, the exponential of positive argument cannot be present. The remaining one of order e^{-kc} appears as a multiplier in ψ, and the portion of ψ due to terms for which n is greater than kc, is therefore vanishingly small in comparison with the portion previously found. This supplies the justification for taking the limits of x as zero and unity in the summation. The terms near $n = kc$ are not accounted.

Evaluation of the Integrals.

The form of ψ given in (24) may be written

$$\psi = I_1 + I_2 + I_3 + I_4, \quad \ldots \quad \ldots \quad (28)$$

where

$$(I_1, I_2) = e^{-\frac{\iota\pi}{4}} \int_C U e^{\mp\frac{\iota\pi}{4}} \left\{ 2 - x^2 - kc(1-x^2)^{\frac{3}{2}} \right\} dx$$

$$\times \exp. \, \iota kc \left\{ 2\sqrt{1-x^2} - 2x \cos^{-1} x + \frac{\pi x}{2} \right.$$

$$\left. - \sqrt{\frac{r^2}{c^2} - x^2} + x \cos^{-1}\frac{cx}{r} \pm \theta x \right\}. \quad \ldots \quad (29)$$

$$(I_3, I_4) = e^{\frac{\iota\pi}{4}} \int_C dx \, U e^{\mp\frac{\iota\pi}{4}} \left\{ 2 - x^2 - kc(1-x^2)^{\frac{3}{2}} \right\}$$

$$\exp. \, \iota kc \left(x \cos^{-1}\frac{cx}{r} - \sqrt{\frac{r^2}{c^2} - x^2} \pm \theta x + \frac{\pi x}{2} \right). \quad \ldots \quad (30)$$

The quantity U denoting

$$\left(\frac{2kc^3 x}{\pi r^2 \sin\theta} \right)^{\frac{1}{2}} \cdot \frac{e^{-\frac{\iota\pi}{4}}}{\left(1 - \frac{c^2 x^2}{r^2} \right)^{\frac{1}{4}} \left\{ 2 - x^2 + \iota kc(1-x^2)^{\frac{3}{2}} \right\}} \cdot \frac{1}{2\pi} \log\frac{x}{x-1+\epsilon}. \quad (31)$$

Now consider the integral

$$I = \int_C u e^{\iota kcv} dx \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

where (u, v) are functions of x occurring in any of the four integrals above. If accents denote differentiations with respect to x,

$$I = \frac{1}{\iota kc} \left\{ \frac{u}{v'} e^{\iota kcv} \right\}_C - \frac{1}{\iota kc} \int_C \frac{d}{dx}\left(\frac{u}{v'} \right) e^{\iota kcv} dx.$$

In each of the four cases above, the latter integral may be shown to vanish in comparison with unity [*].

Thus

$$I = \frac{1}{\iota kc} \left\{ \frac{u}{v'} e^{\iota kcv} \right\}_C. \quad \ldots \quad \ldots \quad (33)$$

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[*] Integrals of this class are considered in the paper quoted in the previous footnote.

For the integral I$_1$,

$$v_1 = 2\sqrt{1-x^2} - 2x\cos^{-1}x + \frac{\pi x}{2} - \sqrt{\frac{r^2}{c^2} - x^2} + x\cos^{-1}\frac{cx}{r} + \theta x$$

and

$$v_1' = \theta + \frac{\pi}{2} + \cos^{-1}\frac{cx}{r}.$$

Thus at both limits $x=1$, $x=e^{2\pi \iota}$,

$$v_1' = \pi + \theta - \sin^{-1}\frac{c}{r},$$

similarly,

$$v_2' = \pi - \theta - \sin^{-1}\frac{c}{r},$$

$$v_3' = \pi - \sin^{-1}\frac{c}{r} + \theta,$$

$$v_4' = \pi - \sin^{-1}\frac{c}{r} - \theta, \qquad \cdots \cdots (34)$$

Moreover,

$$\left[U\right]_{x=e^v}^{x=e^{2\pi\iota}} = \left(\frac{2kc^3}{\pi r^2 \sin\theta}\right)^{\frac{1}{2}} \frac{e^{-\frac{\pi}{4}}}{\left(1-\frac{c^2}{r^2}\right)^{\frac{1}{4}}} \cdot \frac{1}{2\pi}\log\left(\frac{e^{2\pi\iota}}{\epsilon}\cdot\frac{\epsilon}{1}\right)$$

$$= \left(\frac{2kc^3}{\pi r^2 \sin\theta}\right)^{\frac{1}{2}} \frac{e^{+\frac{\iota\pi}{4}}}{\left(1-\frac{c^2}{r^2}\right)^{\frac{1}{4}}} \qquad \cdots \cdots (35)$$

Again,

$$\frac{u_1}{U}e^{\iota k c v_1} = e^{-\frac{\iota\pi}{2}} \cdot \exp. \iota k c \left\{\theta + \pi - \sin^{-1}\frac{c}{r} - \sqrt{\frac{r^2}{c^2}-1}\right\}$$

at both limits ;

$$\frac{u_2}{U}e^{\iota k c v_2} = \exp. -\iota k c \left\{\theta - \pi + \sin^{-1}\frac{c}{r} + \sqrt{\frac{r^2}{c^2}-1}\right\},$$

$$\frac{u_3}{U}e^{\iota k c v_3} = \exp. \iota k c \left\{\pi + \theta - \sin^{-1}\frac{c}{r} - \sqrt{\frac{r^2}{c^2}-1}\right\},$$

$$\frac{u_4}{U}e^{\iota k c v_4} = e^{-\frac{\iota\pi}{2}} \cdot \exp. \iota k c \left\{\pi - \theta - \sin^{-1}\frac{c}{r} - \sqrt{\frac{r^2}{c^2}-1}\right\}, \qquad \cdots (36)$$

whence we finally obtain

$$\psi = -\left(\frac{c}{2\pi k r^2 \sin\theta}\right)^{\frac{1}{2}} \cdot \frac{e^{\frac{\iota\pi}{4}}}{\left(1-\frac{c^2}{r^2}\right)^{\frac{1}{4}}} \cdot \left\{ \frac{\exp.\,\iota kc\left(\pi-\theta-\sin^{-1}\frac{c}{r}-\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi-\theta-\sin^{-1}\frac{c}{r}} \right.$$

$$+ \frac{\exp.-\iota kc\left(-\pi-\theta+\sin^{-1}\frac{c}{r}+\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi-\theta-\sin^{-1}\frac{c}{r}}$$

$$+ \frac{\iota\exp.\,\iota kc\left(\pi+\theta-\sin^{-1}\frac{c}{r}-\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi+\theta-\sin^{-1}\frac{c}{r}}$$

$$+ \left. \frac{\iota\exp.-\iota kc\left(\theta-\pi+\sin^{-1}\frac{c}{r}+\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi+\theta-\sin^{-1}\frac{c}{r}} \right\} \quad (37)$$

which holds except close to the axis of x, where the asymptotic expansion of the spherical harmonics is not valid.

At a great distance,

$$\psi = -\left(\frac{c}{\pi k r^2 \sin\theta}\right)^{\frac{1}{2}} e^{\iota\left(\frac{\pi}{4}-kr\right)} \left\{ \frac{\exp.\,\iota kc(\pi-\theta)}{\pi-\theta} + \frac{\exp.\,\iota kc(\pi+\theta)}{\pi+\theta} \right\} \quad (38)$$

The more general formula reduces to

$$\psi = -\left(\frac{c}{\pi k r^2 \sin\theta}\right)^{\frac{1}{2}} \cdot \frac{e^{\frac{\iota\pi}{2}}}{\left(1-\frac{c^2}{r^2}\right)^{\frac{1}{4}}} \cdot \left\{ \frac{\exp.\,\iota kc\left(\pi+\theta-\sin^{-1}\frac{c}{r}-\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi+\theta-\sin^{-1}\frac{c}{r}} \right.$$

$$+ \left. \frac{\exp.\,\iota kc\left(\pi-\theta-\sin^{-1}\frac{c}{r}-\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi-\theta-\sin^{-1}\frac{c}{r}} \right\} \cdot \quad (39)$$

For points on the positive side of the axis of x,

$$P_n(\mu) = 1,$$

whereas the asymptotic expansion gives

$$\left(\frac{2}{kc\pi x \sin\theta}\right)^{\frac{1}{2}} \cos\frac{\pi}{4}.$$

We must therefore multiply the previous result by

$$\sqrt{kc\pi}\,\sin\theta.$$

Thus along this direction,

$$\psi=\left(\frac{c}{r\sqrt{2}}\right)\cdot\frac{e^{-\frac{\iota\pi}{4}}}{\left(1-\frac{c^2}{r^2}\right)^{\frac{1}{4}}}\left\{\frac{\exp.\,\iota kc\left(\pi-\sin^{-1}\frac{c}{r}-\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi-\sin^{-1}\frac{c}{r}}\right.$$

$$+\frac{\exp.\,-\iota kc\left(-\pi+\sin^{-1}\frac{c}{r}+\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi-\sin^{-1}\frac{c}{r}}$$

$$+\frac{\iota\exp.\,-\iota kc\left(-\pi+\sin^{-1}\frac{c}{r}+\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi-\sin^{-1}\frac{c}{r}}$$

$$\left.+\frac{\iota\exp.\,\iota kc\left(\pi-\sin^{-1}\frac{c}{r}-\sqrt{\frac{r^2}{c^2}-1}\right)}{\pi-\sin^{-1}\frac{c}{r}}\right\},\quad.\quad(40)$$

and at a great distance,

$$\psi=-\frac{2\iota c}{\pi}\exp.\,\iota kc\pi\cdot\frac{e^{-\iota kr}}{r}\quad.\quad.\quad.\quad.\quad(41)$$

becoming extremely great in comparison with the effect at the end of the perpendicular axis given by (38) with $\theta=\dfrac{\pi}{2}$.

Along the other side of the axis of x, $P_n(\mu)=(-1)^n$ and $\theta=\pi$. ψ in this case is given by (19), which leads to

$$\psi=I_1+I_2,$$

where,

$$I_1=e^{-\frac{\iota\pi}{4}}\int_{c}U\exp.\,\iota kc\left\{2\sqrt{1-x^2}-2x\cos^{-1}x+\frac{3\pi x}{2}\right.$$

$$\left.-\sqrt{\frac{r^2}{c^2}-x^2}+x\cos^{-1}\frac{cx}{r}\right\}dx.\quad.\quad(42)$$

$$I_2=e^{\frac{\iota\pi}{4}}\int_{c}U\exp.\,\iota kc\left\{x\cos^{-1}\frac{cx}{r}-\sqrt{\frac{r^2}{c^2}-x^2}+\frac{3\pi x}{2}\right\}dx\quad(43)$$

where

$$U=-\frac{kc^2x}{r}\cdot\frac{e^{\frac{\iota\pi}{4}}}{\left(1-\frac{c^2x^2}{r^2}\right)^{\frac{1}{4}}}\cdot\log\frac{x}{x-1+\epsilon}\cdot\frac{1}{2\pi}\cdot\frac{1}{2-a^2+\iota kc(1-x^2)^{\frac{3}{2}}}$$

and

$$[U]_c = -\frac{kc^2 e^{\frac{\iota\pi}{4}}}{r\left(1 - \frac{c^2}{r^2}\right)^{\frac{1}{4}}} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Thus

$$I_1 = -[U]_c \cdot \frac{e^{-\frac{\iota\pi}{4}}}{\iota kc} \cdot \frac{\exp. -\iota kc\left(-2\pi + \sqrt{\frac{r^2}{c^2} - 1} + \sin^{-1}\frac{c}{r}\right)}{-2\pi + \sin^{-1}\frac{c}{r}}$$

$$= -\frac{c}{\iota r} \cdot \frac{1}{\left(1 - \frac{c^2}{r^2}\right)^{\frac{1}{4}}} \cdot \frac{\exp. -\iota kc\left(-2\pi + \sin^{-1}\frac{c}{r} + \sqrt{\frac{r^2}{c^2} - 1}\right)}{2\pi - \sin^{-1}\frac{c}{r}} .$$

Similarly,

$$I_2 = -\frac{c}{r} \cdot \frac{1}{\left(1 - \frac{c^2}{r^2}\right)^{\frac{1}{4}}} \cdot \frac{\exp. \iota kc\left(2\pi - \sin^{-i}\frac{c}{r} - \sqrt{\frac{r^2}{c^2} - 1}\right)}{2\pi - \sin^{-1}\frac{c}{r}} .$$

Thus on the negative portion of the axis of x,

$$\psi = -\frac{c}{r} \cdot \frac{1}{\left(1 - \frac{c^2}{r^2}\right)^{\frac{1}{4}}} \left\{ \frac{\exp. \iota kc\left(2\pi - \sin^{-1}\frac{c}{r} - \sqrt{\frac{r^2}{c^2} - 1}\right)}{2\pi - \sin^{-1}\frac{c}{r}} \right.$$

$$\left. + \frac{\iota \exp. \iota kc\left(2\pi - \sin^{-1}\frac{c}{r} - \sqrt{\frac{r^2}{c^2} - 1}\right)}{2\pi - \sin^{-1}\frac{c}{r}} \right\} \quad . \quad (45)$$

and at a great distance along this axis,

$$\psi = \frac{-c}{2\pi} \left\{ e^{2\pi\iota kc} + \iota e^{2\pi\iota kc} \right\} \frac{e^{-\iota kr}}{r}$$

$$= \frac{-c}{\pi\sqrt{2}} \exp. 2\iota kc\pi \cdot \frac{e^{-\iota\left(kr - \frac{\pi}{4}\right)}}{r}. \quad . \quad . \quad (46)$$

The values of ψ here given, with a time factor $e^{-\iota kVt}$ added, correspond in each case to an incident disturbance

$$\phi = \exp. \iota k(x - Vt).$$

The greatest diffractive effect occurs in the directions, on either side, which lie in the line of the incident disturbance, and the effect near the axis of the disturbance is of a higher order than that at other inclinations. The formulæ are not true close to the sphere, as the asymptotic expansions are not valid there. They give the effect of all harmonic terms not close to $n = kc$.

XVIII. *The Radiation from Ordinary Materials.* By NORMAN R. CAMPBELL, *B.A., Fellow of Trinity College, Cambridge**.

[Plate V.]

Experimental.

§ 1. THE following experiments are an extension of those described in a paper bearing the same title in the Phil. Mag. for April 1905. They have the same object—to determine the nature of the ionizing radiation from ordinary materials, and they follow the same method. But experimental improvements enabled greater accuracy to be attained and the results interpreted with greater certainty.

The observations consisted, as before, in determining the relation between the ionization and the volume of a rectangular vessel with two parallel sliding sides. It was pointed out on page 538 of the former paper that difficulties arose in

Fig. 1.
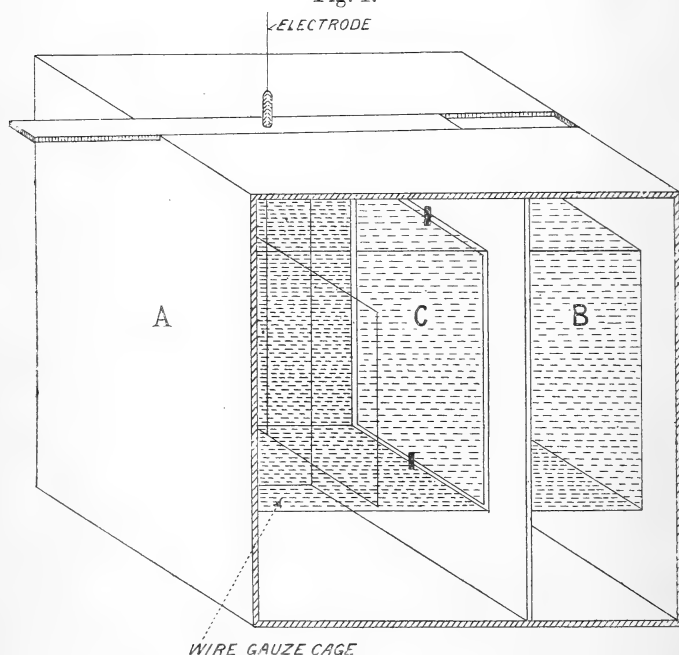
the subsequent calculation of the penetration from the influence of the corners of the box. This disturbance has been obviated by a guard-ring device.

§ 2. A sketch of the vessel employed is given in fig. 1.

* Communicated by the Author.

The dimensions of the box are $32 \times 32 \times 35$ cms.; the front side, which is hinged, is supposed to be removed in order to show the interior arrangements. A rectangular cylinder of coarse wire gauze is fastened to the ends A and B of the box, A being covered with the material under investigation. A plate (C) carrying the same material could slide on the gauze, always parallel to the ends. An electrode of wire netting was supported, through a slit in the gauze, inside the wire-gauze cage by a stiff wire, of which all parts outside the cage were shielded by earthed tubes (not shown). Now when a large difference of potential is established between the wire gauze and the electrode, only the ions inside the cage can carry the current; but rays from outside the cage can pass through the interstices and ionize the air inside as if it were a portion of an infinite volume. The sides of the box parallel to the wire gauze were 8 cms. distant therefrom and covered with aluminium. We shall see that this distance is sufficient to cause the total absorption of the rays from the edges of A or C or from the sides of the box before they reached the cage, and so to prevent them from having any influence on the ionization inside the cage. The dimensions of the cage were $18 \times 18 \times 32$ cms.

Thus, so far as easily absorbable radiation is concerned, the ionization inside the wire-gauze cage will be the same as that in a volume of air of identical dimensions enclosed between two infinite plates of the material covering A and B, for the area of the wire in the gauze is so small that the rays from it produce no considerable effect.

§ 3. For the measurement of the ionization the method described in a recent note* was adopted. The saturation current is measured by balancing it against the known current through a vessel, containing uranium, in which the pressure of the air could be varied at will. The capacity of the compensating vessel was measured by comparing it with that of a small guard-ring condenser, of which the capacity could be calculated, by the method described by McCleiland†.

Throughout the paper the currents are expressed in absolute electrostatic units, unless the contrary is expressly stated.

§ 4. Two series of curves were drawn. In the first the box was surrounded by thick screens of lead in order to cut off the external radiation; in the second the screens were removed. The following metals were investigated:—Lead (1. Commercial sheet), Lead (2. Pure foil for assay); Copper

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* Proc. Camb. Phil. Soc., May 1905.

† Roy. Dublin Soc. Proc. x. 18, p. 167.

(1. Commercial sheet), Copper (2. Purest electrolytic copper specially rolled); Aluminium, Zinc, Iron, Platinum* (all commercial sheet), Tin (Tin-plate), Silver and Gold* ("chemically pure"). The surfaces of all the materials except the gold, platinum and pure copper were cleaned with sandpaper before use, in order to remove any surface radioactivity. In the three cases excepted the metals were newly rolled and might be expected to be clean.

§ 5. A word should be said as to the variations in the ionization which were described on p. 534 of the former paper.

Much less trouble than formerly arose from the sudden discontinuous changes; on a few occasions "jumps" were noticed, but I think it is probable that they are due to experimental defects. However, I am unable to explain their origin or to prevent their occurrence; the observations were simply suspended until the normal state of affairs was re-established. The irregular points, often more than 50 per cent. from the curve, are not plotted.

But the slow continuous variations are still in evidence. They appear to be periodic in some multiple of 24 hours, but my experiments were, of course, not suitable to a detailed investigation of the phenomenon. It is probable that they are due to changes in the intensity of the external radiation. That their magnitude was not sufficient to incommode the work seriously will be seen from the nearness of the experimental points to the curve drawn through them.

It may be mentioned that the observations on Lead (1) were interrupted for a period of eight weeks. On resuming after that time, the observations agreed perfectly with those taken before the interval.

The resulting experimental curves are shown in figs. 2–12 (Pl. V.), where the abscissæ give the distance apart of the sides A and C, and the ordinates the currents $\times 10^5$.

§ 6. The curves can be interpreted on the same principles as those applied in the earlier paper.

Let us consider first the case where the external radiation is cut off. We have then as possible ionizing agents

(1) the easily absorbable radiation from the ends of the box;

(2) the more penetrating radiation from the same;

(3) the radiation from the wire-gauze cage.

Let the dimensions of the cage be $a \times b \times x$, x being perpendicular to the sliding side.

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* I have to acknowledge again the generosity of Messrs. Johnson & Matthey in lending me sufficient quantities of these metals.

(2) will give an ionization proportional to the volume, say $vabx$.

(3) will give an ionization proportional to the area of the wire gauze exposed, say μx.

The ionization due to (1) is more complicated. In the former paper it was calculated on the assumption that the rays were absorbed according to an exponential law; but lately our conception of the absorption of α rays has been completely altered by the important paper of Prof. Bragg*. The calculation must now be based on the theory propounded therein, which is fortunately amenable to mathematical treatment.

§ 7. According to this theory an α particle travels a distance a in air producing approximately the same number of ions per unit length at all points of its path; when the distance a is reached it produces no more ions and may be said to be totally absorbed. If it traverses a thickness t of some other substance than air, its path in air will be curtailed by ρt, where ρ is the density of the substance referred to air $=1$.

Let A be the number of ions produced by the rays from unit mass of the material when those rays are wholly absorbed in air. Then dA will be the number of ions produced by the rays from unit volume, where d is the density of the substance referred to water $= 1$. It is convenient to introduce a quantity I_0 such that $I_0 do$ is the number of ions produced in unit volume of the surrounding air at unit distance by the rays from an element of volume do of the material, so that the ionization in an element of volume dv at a distance r from the radiating element is $\frac{I_0\, do\, dv}{r^2}$. I_0 is connected with A by the following equation :—

$$\int_0^{2\pi} d\phi \int_0^a dr \int_0^\pi d\theta \cdot \frac{I_0 do}{r^2} \cdot r^2 \sin\theta = d\mathrm{A} \cdot do, \quad . \quad (1)$$

$$\therefore \quad 4\pi I_0 a\, do = d\mathrm{A} \cdot do,$$

$$\therefore \quad \mathrm{A} = \frac{4\pi I_0 a}{d}.$$

Now consider an element at O of volume $d\mathrm{S} \cdot dl$ in an infinite slab of the material distant l from the plane surface

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* Phil. Mag. December 1904.

AB of the slab (fig. 13). Take O as pole and ON, the normal, as axis, of polar coordinates. The rays from this

Fig. 13.
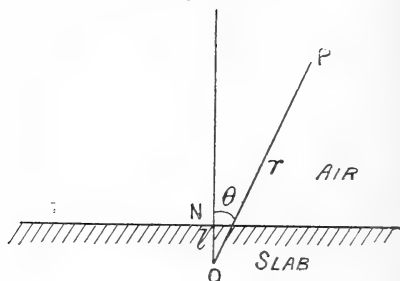
element will cause ionization $\dfrac{I_0}{r^2} dS \, . \, dl \, dv$ in any small volume

of air dv, distant r from O, so long as $r - l \sec \theta$ is not greater than $a - \rho l \sec \theta *$; if r is greater than this there will be no ionization. Now, in an element of volume $r^2 \sin \theta \, dr \, d\theta \, d\phi$ there will be ionization $I_0 \sin \theta \, dr \, d\theta \, d\phi \, dS \, dl$. This has to be integrated within the limits indicated. ϕ can range from 0 to 2π, θ from 0 to the value given by $a - \rho l \sec \theta = 0$, or $\cos^{-1}\dfrac{\rho l}{a}$, r from $d \sec \theta$ to $a - (\rho - 1)l \sec \theta$; hence the ionization is

$$I_0 \int_0^{2\pi} d\phi \int_{l \sec \theta}^{a-(\rho-1)l \sec \theta} dr \int_0^{\cos^{-1}\frac{\rho l}{a}} d\theta \sin \theta \, . \, dS \, dl$$

$$= 2\pi I_0 \int_0^{\cos^{-1}\frac{\rho l}{a}} (a - \rho l \sec \theta) \sin \theta \, d\theta \, dS \, dl$$

$$= 2\pi I_0 \, dS \, dl \left\{ a - \rho l + \rho l \log_e \frac{\rho l}{a} \right\} . \quad . \quad . \quad (2)$$

To get the effect of the whole slab we must integrate this with regard to S, the surface, and l, the depth. We thus find as the whole effect

$$2\pi I_0 S \int_0^{\frac{a}{\rho}} \left(a - \rho l + \rho l \log_e \frac{\rho l}{a} \right) dl = I_0 \, 2\pi S \times \frac{a^2}{4\rho}, \quad . \quad (3)$$

for if ρl is greater than a, no rays emerge from the surface.

* It is assumed here that the rays are emitted with equal intensities in all directions. This seems the most probable distribution, but Bragg inserts an obliquity factor $\cos \theta$. I am unable to find a justification for such a procedure.

If p is the ionization due to a surface S of an infinite slab of the material, then

$$p = I_0 S \frac{\pi a^2}{\rho} = \frac{da}{8\rho} \text{AS.} \quad . \quad . \quad . \quad . \quad (4)$$

But $d = \cdot 0013\,\rho, \quad \therefore \quad p = \cdot 00016\,a\text{AS.}$ We shall write s for $\frac{p}{S}$.

We have now to consider the effect of placing another similar slab parallel to AB and distant x from it. Both slabs will emit rays, but all the rays will not produce their full effect before they strike the opposite slab and are absorbed. By drawing the figure (fig. 14) we see that the

Fig. 14.
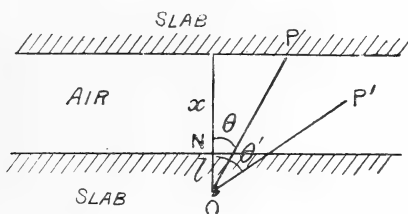
limits within which we have to integrate for an element distant l from the surface of either slab are :—

$$\phi = 0 \text{ to } 2\pi, \ r = l\sec\theta \text{ to } a - (\rho-1)\,l\sec\theta,$$
$$\text{for } \theta = \cos^{-1}\frac{\rho l}{a} \text{ to } \cos^{-1}\frac{x+\rho l}{a},$$

and $\quad \phi = 0 \text{ to } 2\pi, \ r = l\sec\theta \text{ to } (x+l)\sec\theta,$
$$\text{for } \theta = \cos^{-1}\frac{x+\rho l}{a} \text{ to } 0.$$

Hence the effect due to the single element for which $l \not> \dfrac{a-x}{\rho}$,

$$I_0\,dS\,dl \int_0^{2\pi} d\phi \int_{l\sec\theta}^{a-(\rho-1)\,l\sec\theta} dr \int_{\cos^{-1}\frac{x+\rho l}{a}}^{\cos^{-1}\frac{\rho l}{a}} \sin\theta\,d\theta$$

$$+ I_0\,dS\,dl \int_0^{2\pi} d\phi \int_{l\sec\theta}^{(x+l)\sec\theta} dr \int_0^{\cos^{-1}\frac{x+\rho l}{a}} \sin\theta\,d\theta$$

$$= 2\pi I_0 \left\{ x + \rho l \log_e \frac{\rho l}{\rho l + a} - x\log\frac{x+\rho l}{a} \right\} dS\,dl. \quad . \quad (5)$$

P 2

For the whole slabs we have

$$2p_x = 4\pi I_0 S \int_0^{\frac{a-x}{\rho}} \left(x + \rho l \log_e \frac{\rho l}{\rho l + a} - x \log \frac{x + \rho l}{a} \right) dl$$

$$+ 4\pi I_0 S \int_{\frac{a-x}{\rho}}^{\frac{a}{\rho}} \left(a - \rho l + \rho l \log \frac{\rho l}{a} \right) dl \quad . \quad (6)$$

for when the depth l is so great that none of the rays from the element strike the opposite slab, we must return to the previous case.

$$\therefore \ 2p_x = \frac{2\pi I_0 S}{\rho} x \left[\frac{4a - 3x}{2} + x \log_e \frac{x}{a} \right]; \text{ but } p = I_0 \frac{\pi a^2}{2\rho} S$$

$$\therefore \ p_x = p \frac{4x}{a} \left[1 - \frac{x}{a} \left(\frac{3}{4} - \frac{1}{2} \log \frac{x}{a} \right) \right]. \quad . \quad (7)$$

When $x = a$ this becomes p; for $x > a$ the formula breaks down owing to certain assumptions made implicitly above. From the nature of the physical problem it is easy to see that for $x > a \ p_x = p$.

Hence, summing up the three sources of ionization, the curve representing the relation between the ionization (y) and the distance of the sliding side from the end (x) is

$$y = p \frac{4x}{a} \left[1 - \frac{x}{a} \left(\frac{3}{4} - \frac{1}{2} \log \frac{x}{a} \right) \right] + vabx + \mu x. \quad . \quad (8)$$

This curve near the origin is concave to the axis of x, but after the point $x = a$ becomes identical with the straight line $y = p + vabx + \mu x$. Thus p is the intercept of the straight line on the axis of y, and $\tan \theta = vab + \mu$, where θ is the inclination of the straight line to the axis of x.

§ 8. If the screens are removed we shall have to add the effects of (4), the external penetrating radiation (5), the penetrating secondary radiation excited by it, and (6) the easily absorbable radiation excited by it. (4) and (5) will give terms proportional to the volume; let their sum be $v'abx$; (6) will give a relation between ionization and volume analogous to that given by (1). It does not follow, however, that the absorption of the secondary rays will follow Bragg's law, for analogy leads us to expect that the easily absorbable secondary rays will be negatively charged, slow-moving, cathode rays, which are absorbed according to an exponential law. In fact, we may anticipate and say that it was actually

found that p'_x (corresponding to p_x) was given more accurately by the formula

$$p'_x = p'(1 - e^{-\lambda x}) \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7')$$

than by Bragg's formula.

Hence for the unscreened curve we shall have to add to (8) the terms

$$y = p'(1 - e^{-\lambda x}) + v'abx. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9)$$

This curve is of the same form as (8), and is similarly related to p' and v', as regards its straight portion.

§ 9. From what has been said it should be obvious how to deduce, by working backwards, the quantities p, p', v, v', a, λ from the experimental curves. The matter is slightly complicated by the fact that the screens used were only so thick as to cut off two-thirds of the external radiation; the weight of lead required to cut off all the radiation from a box of the size used was almost prohibitive. Accordingly, observations were treated in the following manner.

The values of μ were first ascertained (see § 10), and μx subtracted from the readings recorded before they were plotted in the diagrams from which figs. 2–12 are copied. Smoothed curves were then drawn through the experimental points. In most cases, there was no doubt as to where the curve should lie, but when the points were irregularly distributed it was feared that the drawing of the curve might be influenced by knowledge derived from the former series of experiments. In these cases the help of an unprejudiced friend was invoked; in all cases two independent workers drew substantially the same curve.

The "screened curve" (A) was now subtracted from the "unscreened" (B). The resulting curve (C) gives the effects of (4), (5), and (6), and has the equation

$$y = p'(1 - e^{-\lambda x}) + v'abx. \quad \cdot \quad \cdot \quad \cdot \quad (10)$$

From the intercept and the inclination of the straight portion we can find p' and v' at once. Subtracting $v'x$ from each point of the curve (C) we get the curve (D), which should represent the effects of the easily absorbable secondary radiation, and have the equation

$$y = p_x = p'(1 - e^{-\lambda x}).$$

This is compared with curves drawn from calculation of the formulæ (7) and (7') for different values of a and λ. As stated above, it always agreed best with (7'); the value which gave best agreement between experiment and calcu-

Mr. N. R. Campbell *on the*

lation was assigned to the metal under investigation. We now know p', v', and λ.

Fig. 15 shows the nature of the agreement attained, and also the difference between formulæ (7) and (7′). (7′) is the complete line, (7) the dotted line.

Fig. 15.
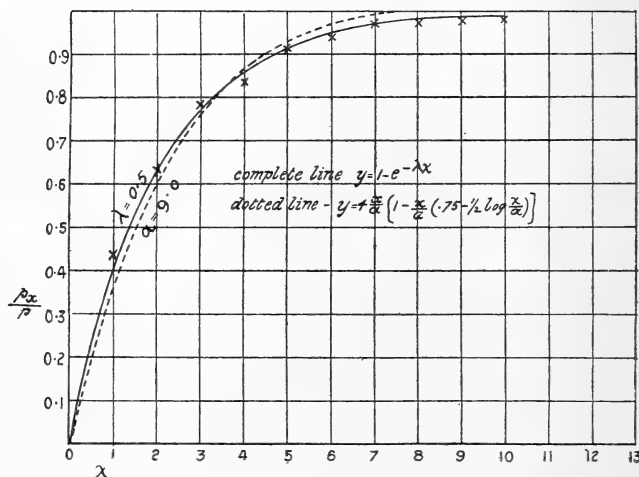
Since only two-thirds of the external radiation was cut out by the screen, the curve (A) must contain half the effect of the agents (4), (5), and (6), as indicated by curve (C). Hence from each ordinate of (A) half the corresponding ordinate of (C) was subtracted; the resulting curve (E) (equation 8) gives the effect of (1) and (2) only, the intrinsic radiations. From this curve p and v can be found immediately, and by subtracting vx all through the curve (F) drawn, whose equation is

$$y = p_x = p \cdot \frac{4x}{a}\left[1 - \frac{x}{a}\left(\frac{3}{4} - \frac{1}{2}\log\frac{x}{a}\right)\right].$$

This was compared graphically with the curves drawn from formula (7) for different values of a; one value was always found to agree fairly well with the experimental results, and was accordingly assigned to the corresponding radiation. We now know p, v and a.

Fig. 16 shows a case of good agreement, and fig. 17 a

case of poor agreement. Table I. gives the calculated values of $\frac{p_x}{p}$ for different values of a and x*.

Fig. 16.
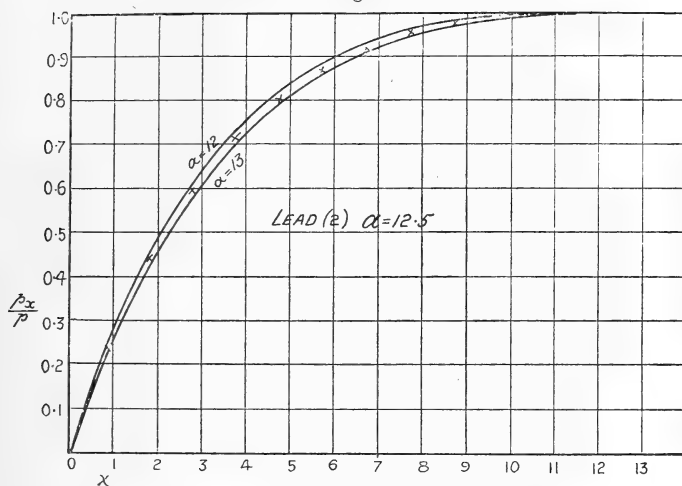
Fig. 17.
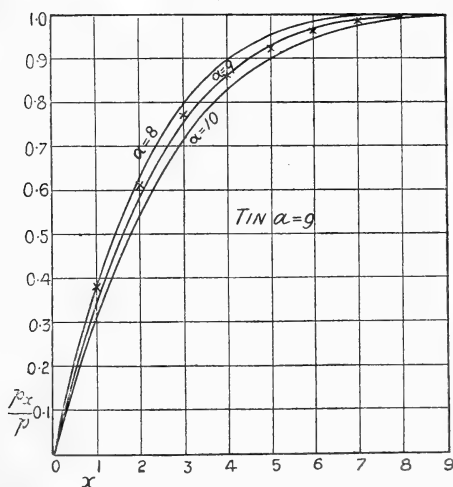
* For the calculation of this table I am indebted to the industry of my brother, Mr. E. R. Campbell. It should be noted that $\frac{p_x}{p}$ is very near 1, when x approaches a; hence the curve (Λ) will appear straight before the ordinate at $x=a$ is reached.

TABLE I.

	$a=6.$	$a=7.$	$a=8.$	$a=9.$	$a=10.$	$a=11.$	$a=12.$	$a=13.$	$a=14.$
$x=1.$	·4839	·4308	·3881	·3532	·3239	·2992	·2779	·2596	·2434
$x=2.$	·7559	·6932	·6394	·5922	·5513	·5154	·4839	·4558	·4308
$x=3.$	·9034	·8520	·8023	·7559	·7133	·6744	·6392	·6071	·5778
$x=4.$	·9728	·9408	·9034	·8648	·8268	·7903	·7559	·7235	·6932
$x=5.$	·9962	·9831	·9609	·9334	·9034	·8725	·8384	·8119	·7836
$x=6.$	1·000	·9980	·9889	·9728	·9522	·9286	·9034	·8776	·8520
$x=7.$	...	1·000	·9987	·9923	·9804	·9645	·9457	·9249	·9034
$x=8.$	...	...	1·000	·9990	·9943	·9854	·9728	·9576	·9408
$x=9.$	...	...	...	1·000	·9993	·9957	·9889	·9788	·9664
$x=10.$	...	...	...	...	1·000	·9995	·9962	·9912	·9831
$x=11.$	...	...	...	...	...	1·000	·9996	·9974	·9930
$x=12.$	...	...	...	...	...	...	1·000	·9996	·9980
$x=13.$	...	...	...	...	...	...	...	1·000	·9997
$x=14.$	...	...	...	...	...	...	...	...	1·000

It should be noted that in all cases except that of platinum, where the secondary radiation is so far larger than the intrinsic, the same value, within half a unit, would have been assigned to a, if it had been calculated from curve (A) instead of (E). Hence an error in the estimation of the proportion in which the radiation was cut down by the screen would only affect the absolute values of the intensities of the radiations, and would leave the penetration constants unchanged.

§ 10. μ was determined by subsidiary experiments. After the form of the screened curve had been determined for any metal, the sliding side was pushed back as far as possible from the opposite end. Two pieces of the same wire gauze as formed the cage were then inserted in the cage parallel to the sliding side, each being about 10 cms. from the nearer end. The ionization in the portion enclosed entirely by wire gauze was then measured; the difference between this ionization and the value of $vabx + \mu x$, as found from the curve for the value of x equal to the distance between the two sheets of wire gauze, must be due to the rays from these two sheets; for the distance from the gauze to the sides of the box is too great to allow the rays from the latter to have any effect at all. Thus we know the ionization given by a known area of the gauze and the quantity μ can be calculated immediately. It was found to be small, the average value being 3×10^{-7}. μ as found in this manner has been subtracted from each ordinate before it was plotted in the diagrams. Similar observations were taken for the unscreened curves, and the value $3·5 \times 10^{-7}$ found for μ.

§ 11. In Table II. the following quantities are given:—

s, the number of ions produced per sec. by the intrinsic absorbable radiation from 1 sq. cm. of the surface of the metal, when totally absorbed in air. (The ionic charge is taken as $3 \cdot 4 \times 10^{-10}$ electrostatic units.)

s', the number of ions produced per sec. by the easily absorbable secondary radiation from 1 sq. cm. of the metal under the conditions of the experiment, when the radiation is totally absorbed in air.

V, the number of ions produced in 1 c.c. by the intrinsic penetrating radiation from the whole box (and the lead screen).

V', the number of ions produced in 1 c.c. by the external radiation and the penetrating secondary radiation excited by it.

a, Bragg's constant for the intrinsic absorbable radiation.

λ, the absorption coefficient of the easily absorbable secondary radiation.

To find s from p of curve (E) we must divide by the total area of the ends of the wire-gauze cage. This area is 2×18^2 cm.^{2}; but since a few ions are probably dragged through the meshes of the gauze, the effective area was taken slightly larger $= 700$ sq. cm.

To find s' from p' of curve (C) we must divide by the same area and multiply by $1\frac{1}{2}$, since p' gives only two-thirds of the secondary radiation.

$V = v$ of curve (E).

$V' = \frac{3}{2}v'$ of curve (C), since that curve only denotes two-thirds of the secondary radiation.

TABLE II.

Material.	s.	s'.	V.	V'.	a.	λ.	§ $\dfrac{v_1 - v_2}{v_1}$.	§. $\dfrac{p_1 - p_2}{p_1}$.
Lead (1)......	270	0	10·2	14·2	12·0* 12·5†	...	·39	0
Lead (2)......	260	0	13·4	26·3	12·5* 12·0†	...	·44	0
Copper (1)...	103	160	2·2	22·0	9·0	0·6	·47	·29
Copper (2)...	110	91	8·1	27·4	9·0	0·5	·51	·29
Aluminium .	117	0	14·8	17·0	6·0* 6·5†	...	·36	0
Tin	144	156	3·1	18·9	9·0	0·5	·57	·35
Silver	146	146	25·5	17·0	8·5	0·9	·27	·33
Platinum ...	74	411	17·3	14·1	12·0	0·4	·30	·56
Gold	78	169	10·4	16·8	10·0	0·6	·43	·46
Zinc	72	51	15·4	16·8	10·0	0·5	·33	·28
Iron	119	124	12·3	10·5	13·0	0·5	·31	·34

* Calculated from " screened " curve.   § See § 14 (8).
†   ,,   ,, " unscreened " curve.

In Table III. the same quantities are given as calculated from the earliest series of experiments. The arbitrary units in which these results were expressed previously have been converted into absolute units by the measurement of the unit of capacity employed there. As this was represented by the capacity of a lead cylinder with a central wire, which had been lying about for a year after it was used and before it was measured, it would not be surprising if a considerable error should attach to the absolute values in the table.

TABLE III.

Material.	s.	s'.	V.	V'.
Lead	247	0	20·7	12·3
Aluminium	64	0	20·5	14·9
Tin (tinfoil)	138	74	5·9	3·1
Platinum	0	59	21·5	7·4
Zinc	30	21	17·7	9·2

§ 12. An attempt was also made to measure more directly the penetration of the rays in a manner similar to that described on p. 539 of the former paper. It was found that no great accuracy could be attained by this method, even with the improved apparatus ; all the quantities examined can be determined with greater certainty from the curves *. A box was constructed $42 \times 42 \times 10$ cms., with a hinged lid $42 \times 42 \times 8$ cms. Covering the top of the box was a grid of thick brass bars leaving nine apertures 12 cms. square ; these were covered with aluminium-foil ·000445 cm. thick. Thus, when the lid of the box was closed, the volume inside was divided by the aluminium-foil into two compartments. The walls of the box and the lid were of wood covered with aluminium ; through the wall of the box passed a rod bearing an electrode insulated from the walls by a guard-ring in the usual manner. By keeping a large difference of potential between the walls and the electrode, the ionization in the volume below the aluminium window could be measured. The lid prevented the deposition of excited radioactivity from the outside air upon the window.

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* The good agreement shown in Table III. (p. 543) of the former paper seems to have been quite accidental: the values given there are *not* in accordance with later results.

The experiments consisted in measuring the ionization through the box, then placing a plate of the metal investigated on the window and noting the increase in the ionization caused by the rays from it, which penetrated the window. The plate was then raised a known distance from the window, and the effect of the layer of air on the intensity of those rays was then noted; comparison of the observations gave the penetration of the rays. The ionization was measured by the compensation method (§ 4).

Only three metals were measured in this manner—lead, copper, and platinum. As an example of the observations recorded some figures are given, which are the mean of a large number of readings.

Normal ionization in the box..	$\cdot821\times10^{-3}$	
		diff. $6\cdot0\times10^{-5}$, ratio $1\cdot00$
With lead plate on window ..	$\cdot881\times10^{-3}$	
		$5\cdot1$ $\qquad$ $\cdot85$
Lead plate $\cdot5$ cm. from window.	$\cdot872\times10^{-3}$	
		$4\cdot4$ $\qquad$ $\cdot74$
Lead plate $1\cdot0$ cm.	$\cdot865\times10^{-3}$	
		$3\cdot1$ $\qquad$ $\cdot52$
Lead plate 2 cm.	$\cdot852\times10^{-3}$	
		$2\cdot3$ $\qquad$ $\cdot38$
Lead plate 3 cm.	$\cdot844\times10^{-3}$	

We can ascertain to what values of a these readings correspond in the following manner. In § 7 the ionization in a layer of thickness x next to the plate has been calculated. The difference between this and the whole ionization is the portion existing beyond this layer; this latter ionization is caused by the rays which have passed through the thickness x. A reference to the analysis will show that a layer of thickness d of a material density ρ is equivalent in all respects to a thickness of air x if $\rho d = x$. Hence if μ_t is the ratio of the ionization caused by rays, which pass through a thickness of material equivalent to a layer of air x, to the whole ionization which would be caused in the absence of that layer, $\mu_t = 1 - \dfrac{p_x}{p}$, where $\dfrac{p_x}{p}$ is found from Table I.

The ratios given in the last column of the figures for lead represent μ_t for the values $\cdot5$, 1, 2, 3 of x. The corresponding numbers, if $a = 12$, are $\cdot86$, $\cdot72$, $\cdot52$, $\cdot36$; if $a = 13$, they are $\cdot87$, $\cdot74$, $\cdot54$, $\cdot39$. We thus see that a for lead lies between 12 and 13, and confirm the results given by the curves.

For platinum the numbers were—

$v=\frac{1}{2}$ $\mu_t=\cdot 84$, $v=1$ $\mu_t=\cdot 73$, $v=2$ $\mu_t=\cdot 54$, $x=3$ $\mu_t=36$.

These give a about 12.

For copper we find

$v=\frac{1}{2}$ $\mu_t=\cdot 82$, $v=1$ $\mu_t=\cdot 66$, $v=2$ $\mu_t=\cdot 44$, $v=3$ $\mu_t=\cdot 29$,

which gives a about 10.

An attempt was made to obtain similar numbers for sulphur, which, being a non-conductor, could not be investigated by the parallel plate method. For some unknown reason it was hard to get consistent results, but the mean of a large number of observations seemed to indicate that a was between 6 and 9.

§ 13. With the same apparatus an attempt was made to determine the sign of the charge carried by the intrinsic absorbable rays from ordinary metals. For this purpose a plate of lead was supported on ebonite blocks $\frac{1}{2}$ cm. above the aluminium window from which it was insulated. A difference of potential of 1000 volts was established between the plate and the window, the field being first in one direction and then in the other. If the rays be charged, the field will tend to accelerate them in one case and to retard them in the other ; it might be expected that the proportion passing through the aluminium window, and therefore the ionization in the box, would be smaller in the latter case than in the former.

An estimation of the order of magnitude of the effect upon the leak which is to be expected can be made easily.

If the rays be negatively charged, they must be of the nature of cathode rays, and their penetration would indicate that their velocity was about 10^9 cms. per second. The velocity v acquired by a particle of mass m and carrying a charge e in falling through a difference of potential V is given by $\dfrac{mv^2}{2}=e\text{V}$. $\dfrac{e}{m}$ is $1\cdot 8 \times 10^7$; $\text{V}=10^{11}$;

$$\therefore \quad v^2=3\cdot 6 \times 10^{18} \quad \text{or} \quad v=1\cdot 9 \times 10^9.$$

Hence the rays would be almost stopped before they reached the window. But if the rays were positively charged, $\dfrac{e}{m}$ would be 10^4, and the velocity acquired in falling through 1000 volts would be only $(2 \times 10^{15})^{\frac{1}{2}}=4\cdot 4 \times 10^7$. The velocity of the α rays of radium is about 2×10^9 ; and hence the

direction of the imposed field would hardly influence the penetration of the rays or the ionization in the box.

When the experiment was made, it was found that the change in the ionization when the direction of the field was reversed could be barely detected by the apparatus, which would have shown a change of 2 per cent. with certainty; but the current seemed slightly greater when the window was charged negatively. This would indicate that the rays were positively charged. They certainly cannot carry a negative charge, but may be without a charge of either sign.

§ 14. One more experiment must be described before we proceed to the discussion of the results. It was directed to ascertaining what proportion, if any, of the activity of lead, the most active metal, could be attributed to the presence of radium diffused throughout its substance as an impurity.

For this purpose 1500 grammes of the metal used in the experiments (lead (1)) were dissolved in nitric acid, and the solution thoroughly boiled to expel any emanation; it was then placed in a bottle of which it filled all but 500 c.c. The air in the bottle could be displaced into a testing-vessel of the usual form, the electrode of which was connected to to the compensating apparatus by which a change of 1 per cent. in the ionization could be detected. When air was drawn through the solution into the testing-vessel, no change could be noted. The solution was then allowed to stand for twenty-four days to allow the emanation to attain radioactive equilibrium. At the end of this time the 500 c.c. of air were displaced into the testing-vessel, but no change in the current could be detected; it did not increase by 1 per cent., nor did it decrease by that amount when fresh air was subsequently introduced, free from emanation.

From the data of § 11 we can calculate easily the increase in the ionization which would have been observed if the whole activity of the lead had been due to the presence of radium. If A is the number of ions produced per sec. by the rays from one gramme of any substance when totally absorbed in air, $s = \cdot00016a$A (§ 7, eqn. 4). For lead, $s = 260$ and $A = 1\cdot35 \times 10^5$; the 1500 grammes would have produced 2×10^8 ions. If this ionization had been caused by radium in radioactive equilibrium, one-fourth of it would have been due to the emanation. Hence the emanation alone would have produced $\cdot5 \times 10^8$ ions per sec. The normal current through the testing-vessel corresponded to something less than 10^5 ions per sec.; the emanation would have increased the leak 500-fold. Since the observed change was

less than 1 per cent., when all allowance is made for error,
it is impossible to attribute any considerable portion of the
activity of lead to the activity of radium ; and radium is the
only impurity to which anyone has ventured to attribute
the activity of ordinary metals.

Discussion of Results.

§ 15. (1) *Comparison of* Tables I. *and* III.—It is clear that
in the main features the new series agrees with the old ; we
may instance the absence of secondary absorbable radiation
from lead and aluminium ; the large ratio of secondary to
intrinsic absorbable radiation for platinum ; and the small
volume ionization for tin.   On the other hand, the absolute
values differ largely : this may be partly due to the un-
certainty regarding the value of the unit of capacity used in
the older series.   It is hard to compare the values of the
penetration, since the first series was not suitable for cal-
culation, but the order in which the metals would be arranged
in respect of this quality would be the same except for lead,
in which case the penetration appears to be larger in the new
than in the old series.   It should be noted, that the inclination
of the straight portion to the horizontal axis of the former
curve depended on the intensity of the absorbable radiation
from the walls : it would be large for lead, and tend to
make the transition from the curved to the straight portion
appear too abrupt.

Since the later experiments are more reliable in principle
and more accurate in execution than the old, in any case of
discrepancy far greater weight should be attached to the
former.

(2) *Value of* s.—The most important feature of this
quantity is its constancy when deduced from wholly different
specimens of the same metal ; the emission of intrinsic
absorbable radiation appears to be an inherent property of
the substance itself.   I am unable to trace any relation
between the value of s for a metal and any other chemical or
physical property.

(3) *Value of* s'.—It might be expected that aluminium
would give little or no secondary radiation, but the absence
of such radiation from lead is very surprising.   Nevertheless
it is in perfect agreement with my earlier experiments and
those of Wood[*].   Perhaps this peculiarity may be connected
with the use of lead as the screen to cut off external radiation,
but the connexion is not clear.   Selective absorption would
give precisely the opposite effect ; moreover, Wood found

[*] Phil. Mag. April 1905.

no difference in the behaviour of lead when shielded with an iron or a lead screen. The large secondary radiation from platinum is remarkable; it may throw some light on the large variations in the activity of that metal which were found by Strutt *.

(4) *Value of* V *and* V'.—Not much reliance can be placed on these values, for a change in the intensity of the external radiation (such as is known to arise from the variation in the quantity of radium emanation present in the air), during the nine or ten days that were required to complete the observations for any one metal, would invalidate the calculation. Moreover, the plates of metal used to cover the parallel sides were not thick enough to cut out all rays from the underlying wood, if these were of a penetration equal to that of β rays. In fact, it is clear that there are inconsistencies in the table : tin-plate is only iron covered with a thin layer of tin, not nearly thick enough to absorb β rays, and yet the value of v is less for tin-plate than for iron. But it does not seem that any information of importance could accrue from a more accurate estimation of these quantities.

(5) *Value of* a.—The variation of Bragg's constant for the different metals is the one point of importance in the whole paper. This variation was suggested in the first paper, and has been proved beyond doubt in the second. It shows that the emission of ionizing rays is a property inherent in the material itself, and cannot be attributed to the presence of a radioactive impurity common to all. This is my main contention.

It is noticeable that in every case except aluminium, the penetration of the rays from ordinary metals is greater than that of the most penetrating rays from radium ($a = 7$, see Bragg, Phil. Mag. Sept. 1905). But in view of the considerations urged by Rutherford †, there seems to be no reason why this should not be the case. An interesting problem arises, whether the rays from ordinary metals are homogeneous. In some cases, *e. g.* lead and platinum, the experimental curve agrees so well with that calculated for the corresponding value of a, that the rays appear to be all of one type. But in others, especially tin and copper, the experimental curve is steeper at first than the calculated ; it is quite possible that we are dealing with two varieties of rays, of which one is more easily absorbed than the other. But at present, the accuracy of experiments on such extremely small ionizations is not sufficient to enable the question to be investigated directly.

* 'Nature,' Feb. 19, 1903.
† Phil. Mag. July 1905.

(6) *Value of* λ.—λ cannot be determined very accurately from the curves, since it will always appear as the difference of two much larger quantities. Its magnitude does not seem to afford an opportunity for a discussion of any relevant interest.

(7) The value of the "window experiments" lies in confirming the conclusions already reached, and in demonstrating in a more direct manner the emission of ionizing rays, similar to α rays, by ordinary metals. The proof of the absence of radium from lead renders impossible one of the suggested explanations of the spontaneous ionization in closed vessels.

(8) It has been suggested to me, that my estimate of the ratio in which the external radiation was cut down by the lead screen might be wholly erroneous, and that what I have called "intrinsic radiation" might only be the residue of secondary radiation due to this error. If this were so, the surface ionization ought to be cut down in the same ratio as the volume ionization when the screen was applied. To show that this was not so, the values of $\frac{v_1 - v_2}{v_1}$ and $\frac{p_1 - p_2}{p_1}$ are tabulated in columns 8 and 9 of Table II. In all but two cases the former is the larger, as would be expected if there were present intrinsic surface radiation.

Conclusion.

§ 16. If the object of this paper has been attained, it has been proved beyond doubt that the emission of ionizing radiation is an inherent property of all the metals investigated; and I see no reason why it should not be extended to all substances. It is not, of course, necessary that this ray-emission should be identified at once with radioactivity—it that word is taken to mean a process of ray-emission accompanied by atomic change. But the constant intensity of the rays, and the probability that the larger portion of them are α rays, which is suggested by the investigation of their charge and their penetration, afford considerable support for that hypothesis; while I know of no other process which affords any analogy. But before the identity can be established irrefutably, further work is required, which I hope to be able to supply in the near future.

Once more it is my privilege to acknowledge my indebtedness to the invaluable advice and kindly interest of Prof. Thomson.

Note.

Perhaps it will not be out of place to remark upon two researches the results of which might seem to be contradictory to mine.

The first of these is described in a paper by McLennan and Burton [*], and consists of the measurement of the variation of the ionization with the pressure in a cylinder of zinced iron, of height 125 cms. and diameter 25 cms. The authors point out that the resulting curve is "nearly a straight line," and hence conclude that the ionization is mainly caused by penetrating radiation.

Now it is clear that the form of such a curve ought to be of the same nature as that of those in figs. 2–12 ; for large pressures, when the radiation from the walls is totally absorbed, it should be a straight line which should cut the axis of ionization at a distance from the origin representing the total ionization due to the walls when the rays are all absorbed. The pressure will have to be considerable before the curve becomes approximately straight, for the density of the air will have to be so great that the rays from any point on the curved surface are absorbed before they strike another point on the same surface. McLennan and Burton's curve is not plotted for pressures higher than 500 cms., at which point it still shows noticeable curvature ; but we may make some estimate of the relation between surface and volume-ionization by assuming that the tangent to the curve at the last point is in the direction of the final straight line. Taking this tangent, we find 10·7 for the total surface ionization and 12·8 for the volume ionization at 500 cms. (equivalent to 1·94 at 760 mms.). The area of the walls is 10780 sq. cms. and the volume of the vessel 61500 c.c. ; hence the ratio

$$\frac{\text{ionization per 1 cm.}^2 \text{ of surface}}{\text{ionization per 1 c.c. of volume}} = \frac{10\cdot7}{10780} \div \frac{1\cdot94}{61500} = 31\cdot5.$$

From Table II. we find the same ratio for zinc to be $\frac{123}{32\cdot2} = 3\cdot8$, for an unscreened vessel. Thus the ratio of the surface to the volume ionization for zinc found by McLennan and Burton is eight times that found by me ; the difference between our experiments is in the opposite direction to that which they affirm. It is very probable from other reasons

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* Phys. Review, vol. xvi. p. 190.

that the value of the external penetrating radiation is exceptionally high in the place where I worked ; this fact may account for some part of the discrepancy.

The other research is that by Eve [*]. He concludes from his experiments that much the larger part of the ionization in a closed vessel is due to the radium emanation in the air enclosed. This explanation seems to me self-inconsistent; for surely if the quantity of emanation in the small volume of air inside the vessel causes an effect so large, the far larger quantity outside the vessel must have an appreciable effect, even though the walls are so thick as to cut out all α rays. In fact, I do not see how Mr. Eve would explain Cooke's result on cutting down the ionization by thick screens —a result which all who have tried it have confirmed.

Mr. Eve confines his remarks to large vessels, and would probably not suggest that my experiments were vitiated by this cause. But I have tried directly what proportion of the ionization can be attributed to the enclosed emanation. This is simply done by storing a volume of air for a time (24 days) sufficient to allow the emanation to die away. I have never found a decrease of more than 5 per cent. between the 2nd and the 24th day, and usually the change was much smaller. Again, the presence of emanation in my boxes would give a term proportional to the volume, and the curve would tend to bend away from the horizontal axis ; the curves drawn show no sign of this. I have therefore omitted all consideration of the emanation in the arguments of § 7.

XIX. *The Chemical and Geological History of the Atmosphere.*
By JOHN STEVENSON, *M.A., F.I.C.*[†]

IV.

Observations on the question as to whether the Amount of Carbonic Acid in the Atmosphere is at present increasing or decreasing ; also Note on the Rate of the Secular Cooling of the Earth.

IN the last article of this series (Philosophical Magazine, Jan. 1905, p. 88) we saw that there is good reason to believe that the amount of atmospheric carbonic acid has varied within wide limits during geological history, limits probably wide enough, when regard is taken of the high absorptive power of carbonic acid for radiant heat, to account

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[*] Phil. Mag. July 1905.
[†] Communicated by the Author.

for the great variations in terrestrial climate or in the temperature of the surface of the earth which are known to have taken place in past geological epochs. The principal reason for holding the above view is the variable or intermittent character of volcanic action, which is one of the principal agencies that produce or evolve carbonic acid into the atmosphere.

A question which naturally suggests itself at this stage is :—What is the state of matters in the above respect at the present time? Is the amount of atmospheric carbonic acid increasing or decreasing, or has it been practically constant in amount for many years (or centuries), and has it still a tendency to remain at the same figure ?

At first sight, it might be supposed that a simple and possibly sufficient answer to this question could be obtained by studying the records of analyses of the atmosphere made at different times, within the last hundred years or so, that is to say since analyses of the atmosphere were made at all. Unfortunately, however, the earlier analyses are quite unreliable on account of the defective analytical methods which were used; and indeed it is very doubtful if any analyses could be accepted for the above purpose except those or some of those which have been made within the last thirty or forty years. Further, it would be very difficult to make a satisfactory comparison of even the most reliable analyses (reliable in so far as analytical methods and careful working are concerned) on account of local variations in the amount of atmospheric carbonic acid and temporary variations due to weather conditions and other causes ; and therefore it is very doubtful if we are in a position as yet to infer anything at all from a comparison of analyses done at different dates.

It is possible, however, that something may be learned in other ways, on the subject before us, especially from a study of the amount of carbonic acid in the sea. The reasons for expecting to learn something in this way are the solubility of carbonic acid in water, the great depth and volume of the sea, the stillness of the water at great depths, and the long time that would therefore be most probably required for carbonic acid entering the ocean at its surface (or at any particular point) to diffuse throughout its whole extent.

It is now a considerable time since it was known or conjectured that the sea and water on the earth's surface generally had a notable influence on the carbonic acid of the atmosphere. In 1855 Peligot (as quoted by Prof. Letts and Mr. Blake in their important work on the carbonic anhydride of the atmosphere referred to in last article) pointed out that water

probably had an important regulating effect in the above respect, by removing from the atmosphere carbonic acid evolved from volcanic and subterranean sources. Later on, in 1880 or so, Schloesing (also as quoted by Letts and Blake) maintained that the oceans were gigantic reservoirs of carbonic acid and functioned as automatic regulators of the amount in the atmosphere. From his analyses and observations, he calculated that the oceans held in reserve a disposable quantity of carbonic anhydride for exchange with the air, ten times greater than the total amount contained in the atmosphere, and *a fortiori* very much larger than the variations in that quantity.

It is very doubtful, however, if Schloesing's estimate of the disposable quantity of carbonic acid in the sea is correct; it may possibly be much too high. No doubt the total quantity of carbonic acid present in the sea in various forms is very great, more than thirty times the amount present in the atmosphere; but as by far the greatest proportion of it is in the combined condition in the form of carbonate and bicarbonate of lime and other bases, it cannot be regarded as being disposable or readily available for interchange with the atmosphere. Prof. Dittmar, as the result of his examination of analyses of sea-water, especially of those made in the course of the 'Challenger' Expedition, concluded that there was rarely any free carbonic acid in sea-water at all, in the ordinary sense of the term "free," for there is usually not enough to form bicarbonate, and the reaction of sea-water is usually alkaline. There was usually, however, more than enough to form the normal carbonate, and anything in excess of this he was inclined to call "loose" carbonic acid.

It is also doubtful if we could as yet make any use for our purpose of the analyses made in connexion with the 'Challenger' Expedition or of any other recorded analyses of sea-water, for the reason that before we made any inference from them we should require to know a good deal about the various agencies that have an influence on the amount of carbonic acid in sea-water. We should require to have some information regarding the amount of carbonic acid produced in the sea by animal respiration, the amount produced by the slow oxidation of animal and vegetable remains in contact with oxide of iron and otherwise, the amount evolved into the sea from volcanic or subterranean sources, the amount decomposed by the growth of vegetation in the sea and the amount removed, or changed from the free to the combined condition, by the action of sea-water on rocks containing silicates of lime and magnesia and other strong bases. It is

quite possible that the normal or ordinary state of matters as regards the balancing of these agencies may be notably different in the cases of the sea and land respectively. Also it is obvious that we ought to know with a high degree of certainty the amount of free carbonic acid in sea-water that corresponds to the ordinary percentage in the atmosphere, and questions regarding analytical methods would have to be very carefully considered. In this connexion it may be noted that Jacobson many years ago observed that the free or loose carbonic acid of sea-water is not all driven off by boiling, even for a considerable time, but that some of it remains until only the dry residue is left.

As it is therefore obvious that the method of enquiry suggested involves the consideration of so many complicated and rather obscure questions, I will not pursue it further, but may remark in passing that it is quite possible that an amount of information on the points mentioned and others pertinent to the subject may yet be obtained sufficient to allow of a satisfactory inference being made.

There is, however, another and a simpler way by which we may arrive at some idea as to how the amount of free or loose carbonic acid in sea-water compares with that in the atmosphere; and that is to compare the average percentage of carbonic acid in sea air with the average percentage in country or continental air. There must always be an interchange going on between the sea and the atmosphere above it; and if we should find that sea air contains more (or less) carbonic acid than country air, then we may conclude that the sea is giving off more (or less) carbonic acid than it takes in from the atmosphere, and that it is therefore richer, or poorer, as the case may be, in free (or loose) carbonic acid than the atmosphere, relatively to other gases present in the atmosphere, and after allowance has been made for solubility or other considerations of a similar nature.

There is a fair amount of evidence available on this point, as there are recorded analyses of the air at a considerable number of places both by sea and land which are fairly comparable with each other; and the general inference which is obtained by comparing them is that sea air is poorer in carbonic acid than country air. There is no doubt a considerable amount of discrepancy between the results obtained by different observers, the difference between the percentage of carbonic acid in sea air and country air being very slight according to some observers and very great according to others. Professor T. E. Thorpe, as the result of a large number of analyses done in 1866, found that the air of the

Irish Channel contained on the average 3·08 parts by volume of carbonic acid in 10,000 parts of air, that the air of the Atlantic Ocean contained 2·95 parts and the air of tropical Brazil 3·28 parts. On the other hand, Beauvais, as quoted by Dr. Phipson in his book on the ' History of the Atmosphere,' states that there is only a trace of carbonic acid in the air over the ocean, and Dr. Phipson also states that Dr. Verhaeghe of Ostend found only $2\frac{1}{2}$ parts of carbonic acid in 100,000 parts of air at the sea-coast, being less than $\frac{1}{10}$ of the amount found in inland districts. These results do not accord very well with those of Professor Thorpe, but still both sets of results indicate that sea air is poorer in carbonic acid than country air; and this conclusion is confirmed by observations taken regarding the influence of the direction of the wind on the amount of carbonic acid in the atmosphere at one and the same place. The observers Reiset at Dieppe, Schulze at Rostock, and Farsky at Tabor in Bohemia, all found (as mentioned by Letts and Blake in their book on the Carbonic Anhydride of the Atmosphere) that when the wind was in a westerly or S.W. direction, that is to say when it was coming from the Atlantic Ocean, there was less carbonic acid in the atmosphere than when it came from the east or the north.

The figures obtained by Müntz and Aubin, referred to in my former paper, may also be regarded as confirming the same supposition. These observers found, on making a summary of their results, that the air of the northern hemisphere contained 2·82 parts by volume of carbonic acid in 10,000, and the air of the southern hemisphere contained 2·72 parts in 10,000. It is obvious that as a much larger area of the southern hemisphere is covered by water than is the case with the northern hemisphere, it is permissible for us to conjecture that the difference observed by Müntz and Aubin may be owing to that circumstance.

It should, however, be noted that the differences in the percentage of carbonic acid in the air observed at different inland places (that is at one inland place as compared with another), and also the differences in the percentage observed at different places on the sea-coast, are or were much greater than the difference between 2·72 and 2·82 ; also that the daily and weekly variations or other variations at longer or shorter intervals in the percentage of carbonic acid in the air at one and the same place are frequently greater than the difference of the above amounts. We cannot therefore put very much stress on the above figures until they are confirmed by fuller and more detailed research ; but still if we simply accept the evidence as it stands, we can say that it is

to the effect that inland or country air is richer in carbonic acid than sea air, and that therefore, on the hypothesis of the regulating influence of the sea and its reminiscent character, the amount of carbonic acid in the atmosphere at present is, or quite recently was, increasing.

It should, however, be added that in making this inference I am assuming that the amount of telluric carbonic acid evolved into the sea is and usually has been less than the amount evolved on land. If this were not so, the inference, if any, that could be drawn from a comparison of the analysis of sea and country air would be different from that just given. However, the assumption seems reasonable in view of our knowledge of the subject. I am therefore still inclined to infer, from the evidence brought forward, that the amount of carbonic acid in the atmosphere is increasing ; and in any case the points referred to in the course of the above discussion are of considerable interest in themselves.

The (probable) increase in the amount of atmospheric carbonic acid is most probably due to the operation of natural forces, though it is just possible that the large quantity of coal which is now raised and consumed may have an appreciable effect on the amount of atmospheric carbonic acid. This in itself is a subject of very considerable interest. It was the subject of Lord Kelvin's address at Toronto in 1897 (referred to at the beginning of my first article), in which he expressed the opinion that possibly there might be enough coal in the world to use up all the oxygen of the atmosphere, and therefore that statisticians, when making estimates regarding the future output of coal or the total available coal-supply of the world, should take into account the effect which the burning of the coal might have on the composition of the atmosphere.

The quantity of coal annually raised at present may be taken in round numbers as 800,000,000 tons. Assuming that it contains on the average 80 per cent. of carbon, the combustion of this amount will yield nearly 2347×10^6 tons of carbonic acid, a quantity which is almost exactly $\frac{1}{20}$ of the total amount of carbonic acid annually removed from the atmosphere by the growth of vegetation, as roughly estimated in my last article ; and quite possibly it is as great a quantity, or even greater, than the amount of carbonic acid which is now annually contributed to the atmosphere from telluric or volcanic sources. At the same time, it is only about the $\frac{1}{900}$ part of the amount of carbonic acid which is present in the atmosphere, and it is obvious that it would take several years' production and consumption of coal at this rate to have an

appreciable effect on the composition of the atmosphere.
Assuming that the exact percentage of carbonic acid in the
atmosphere at present is 0·0277 by volume, it would require
the raising and consumption of coal at the present rate for
ten years to bring the percentage up to 0·0280 even if all
the carbonic acid produced by burning the coal remained in
the atmosphere.   It is doubtful if an increase to such a slight
extent as this could be ascertained with certainty by chemical
analysis ; and if we take into account the regulating influence
of the sea, and of the growth of vegetation and other similar
factors, we shall find the situation such as to make it doubtful
as to whether even the total quantity of coal which has been
raised by human effort and consumed up to the present time,
a quantity which I estimate roughly at 20×10^9 tons (twenty
thousand million tons), has had an appreciable effect on the
composition of the atmosphere.

It is obvious, however, that in making calculations about
the future, we should make allowance for a very considerable
increase in the amount of coal annually raised.   As is well
known, the annual output has increased at a rapid rate for the
last century or more, and the tendency to increase still con-
tinues.   We are probably making a tolerably fair estimate
if we take the average annual rate of increase during the
last 50 years at 4 per cent. ; that is to say, we estimate that
the output for each year has been on the average 4 per cent.
greater than that of the immediately preceding year.   This
is equivalent roughly to an increase in the annual output of
48 per cent. in 10 years and 100 per cent. in 17·7 years; that
is to say, the amount raised is doubled every 17·7 years, or in
integral numbers every 18 years.   At this rate of increase,
the total quantity of coal raised in the next ten years (1906–
1915 inclusive) will be nearly 10×10^9 tons, or 12·48 times
the present annual output.   In 20 years (1906–1925 in-
clusive) the total amount raised will be $24·768 \times 10^9$ tons, or
30·96 times the present annual output ; and this is probably
as large a quantity or even larger than the total amount that
has been raised up to the present time.   In 50 years (1906–
1955) the total would be $126·88 \times 10^9$ tons, a quantity 158
times the present annual output and sufficient to yield (with
the carbon present in the coal reckoned at 80 per cent.)
372×10^9 tons of carbonic acid ; a quantity equal to $\frac{1}{6}$ of the
amount of carbonic acid in the atmosphere at present.   In
100 years (1906–2005) the total amount would be 1028×10^9
tons, or 1285 times the present annual output and sufficient to
yield 3015×10^9 tons of carbonic acid, or 1·3 times the amount
at present in the atmosphere; and therefore, if all the carbonic

acid produced by the burning of the coal remained in the
atmosphere, the amount of atmospheric carbonic acid would
be rather more than doubled (as compared with its present
amount). In 110 years (1906–2015) the total amount of
coal raised would be 1531×10^9 tons, equal to 1914 times the
present annual output, and sufficient to yield 4495×10^9 tons
of carbonic acid, or twice the amount of our present atmo-
spheric carbonic acid. The amount present in the atmosphere
would then (on the supposition mentioned above) be three
times its present quantity, and in volume would amount to
0·084 per cent. of the whole volume of the atmosphere. In
150 years (1906–2055) the total quantity of coal raised would
be 7426×10^9 tons, or about 9283 times the present annual
output, and sufficient to yield 21783×10^9 tons of carbonic acid,
or rather more than 9 times the total amount of atmospheric
carbonic acid at present. The quantity present in the atmo-
sphere then would, on the conditions assumed above, be fully
10 times its present amount, or about 0·28 per cent. of the
whole atmosphere by volume. By this time, if not even at
a much earlier stage, the amount of carbonic acid in the
atmosphere would have a serious effect on human and animal
respiration, and it would therefore be of little use to continue
further the calculations at the above rate of increase (4 per
cent. per annum). Of course, even apart from the effect on
the atmosphere, it is not at all likely that the 4 per cent.
rate of increase or anything very nearly approaching it will
be maintained for 150 years, or perhaps even for a much
shorter period. Considerations regarding the rate of increase
of the population of the world and the probable amount of
coal required for human consumption, and possibly also
considerations regarding practical difficulties connected with
the raising of the coal, obviously give the negative to such a
supposition.

At the same time, it is quite possible that the 4 per cent.
annual rate of increase may be maintained for the next 20 or
the next 50 years, and therefore a very great increase in the
annual output of coal is quite possible. Even if the average
rate of increase should fall to 2 per cent. or so per annum,
the increase in the output in the course of 50 years would
still be very great, and would probably be sufficient to have
an appreciable effect on the composition of the atmosphere.
We must of course also bear in mind that we cannot expect
all the carbonic acid, or perhaps even a large proportion of
the carbonic acid produced by the burning of the coal, to
remain as a permanent addition to the atmosphere. A large
proportion of it will most likely be absorbed by the sea;

and it is also possible that an increase in the amount of atmospheric carbonic acid would be followed by an increase in the luxuriance of vegetation, and therefore by an increase in the rate at which carbonic acid is removed from the atmosphere. Further, a considerable increase in the amount of atmospheric carbonic acid would most probably cause an appreciable decrease in the oxidizing power of the atmosphere. There might thus be a notable decrease in the rate of eremacausis (the slow decay of animal and vegetable remains), which is a very important factor as regards the addition of carbonic acid to the atmosphere ; and it is also of some interest to note that a decrease in the rate of eremacausis would be equivalent to an increase in the rate of the formation of coal or of coal-forming materials.

From the considerations just mentioned, it is obvious that even if the output and consumption of coal should increase to a very great extent, the effect on the composition of the atmosphere might not be at all proportionate to this increase. It is also obvious that it is not possible for us, when there are so many factors in the problem, to estimate in a satisfactory manner either (1) the amount of coal that is likely to be raised in the next 100 years or so, or (2) the approximate effect that this amount (supposing we could form an estimate regarding it) would have on the composition of the atmosphere.

However, I think that, taking a general survey of the various factors involved, we may reasonably conclude that the amount of coal that will be raised in the next twenty years will possibly be sufficient to have an effect on the composition of the atmosphere that will be appreciable by chemical analysis ; and that in the next 100 or 150 years the amount of coal raised and consumed may cause an increase in the amount of atmospheric carbonic acid that will have an effect appreciable in other ways, for example as regards the climate or average temperature of the surface of the earth.

As regards the question of the amount of carbonic acid in the atmosphere in former geological epochs, we may add here that in addition to inferences which may be made from evidence regarding volcanic activity at any particular epoch, it is possible that something may also be inferred from the general character of the animal and vegetable life of the period. If the percentage of atmospheric carbonic acid has varied very considerably in different epochs, it is obvious that the percentage in any particular epoch may have been an important condition or factor in determining the evolution or survival of many forms of animal and vegetable life. In this connexion the observations made by Dr. Phipson in

1893 and 1894 regarding the relative oxygen-producing powers of different classes of plants are of great interest. He gave an account of these observations in the ' Chemical News,' vol. lxx. (1894) p. 223, stating that he " measured the oxygen given off in a certain time by various unicellular algæ and by phanerogamic plants high in the scale. These are then collected, dried at 100° C., and weighed. It was then found that weight for weight the unicellular algæ gave by far the most oxygen. In one experiment last August a mixture consisting chiefly of *Protococcus* and *Chlamidomonas* was thus compared during five days with the common weed *Polygonum aviculare*, and was found to produce, weight for weight, fifty times more oxygen than the latter. In other cases different results were obtained, but in all the excess of oxygen was largely on the side of the unicellular algæ."

From the above experiments by Dr. Phipson it is obvious that unicellular algæ must require a very much greater quantity of carbonic acid in order to gain a definite increase in weight than the higher phanerogamic plants ; and as the latter were late in geological history in making their appearance, it is permissible for us to hold or suggest that their evolution may have been conditioned by the comparative scarcity or the increasing scarcity of carbonic acid in the atmosphere, and that their leaves and general structure have been so modified as to make the most of the carbonic acid available. No doubt another way to look at the question is to regard the unicellular algæ as being in some respects of a less distinctly vegetable character than the higher phanerogamic plants. Probably enough the character of the respiration of the algæ as regards its effect on the weight and bulk of the organism comes nearer to that of animals, in whose case respiration causes a decrease in weight. Still this way of looking at the question may lead to nearly the same conclusion as the one first suggested. The observations are in any case of great interest, and it is quite possible that the appearance or prevalence of special types of organisms at special epochs may have been to a large extent conditioned by the amount of carbonic acid in the atmosphere.

It is also noteworthy that Dr. Phipson found that lilacs, willows, &c. can be killed by an excess of carbonic acid.

A few remarks may also be made here on the bearing that the theory of the variations in the percentage of atmospheric carbonic acid has on the question of the rate of the secular cooling of the earth. As is well known, there is a wide divergence of opinion on this subject, the rate of cooling as calculated by Lord Kelvin from physical data being too high

to admit of the long period of time postulated by geologists as being necessary for the deposition of the observed amount of stratified rocks, and probably also for the evolution or sufficiently gradual appearance of changes observed in the structure of animal and vegetable organisms. Both classes of observers—the physicists and the geologists—make use of data derived from recent observations in order to calculate respectively the rate of cooling of the earth, and the rate of denudation and deposition of rocks ; but it is quite possible that in the course of past ages important changes may have taken place in the rate of cooling and in the rate of denudation. It is quite obvious, from what has been learned regarding the high absorptive powers of carbonic acid for radiant heat, that if the amount of carbonic acid in the atmosphere was much greater in early times than it is now, the rate of radiation of heat from the earth into space would be less than it is at present—assuming the temperature of the surface of the earth to be the same in both cases. There is even a possibility (arising from the circumstance that the earth is always receiving heat from the sun as well as losing heat by radiation into space) that there may have been periods in the past history of the earth when the temperature of the earth as a whole remained constant, or even increased to a slight extent instead of decreasing at all, and that for a considerable number of years at a time. The average rate of the secular cooling of the earth must have been only a very small fraction of a degree per annum, even on Lord Kelvin's estimate of the age of the earth, and the variations in the amount of atmospheric carbonic acid may have been sufficient to cause interruptions of a fairly long period in the process of cooling, just as the succession of day and night and summer and winter may be regarded as causing frequent interruptions for a short period in the same process. On the other hand, we should also observe that the rate of denudation may have varied very much in the course of geological history. If the atmosphere was more extensive in early times than it is now, it would carry more aqueous vapour ; more numerous or more extensive clouds would form, and we should therefore expect the average rainfall to be heavier than it is now, and consequently the rate of denudation to be greater. Further, the greater quantity (by hypothesis) of carbonic acid present in early times and at various epochs, would on account of its chemical activity greatly hasten the solution of limestone and other carbonates, and the disintegration of rocks in general. Also the rate of evolution or the appearance of important changes in the structure of organisms would probably be

much accelerated by notable variations in the amount of atmospheric carbonic acid.

As regards the rate of denudation, it may be added that this may have varied to a very considerable extent in different epochs on account of geological changes of various kinds. Changes may have taken place in the relative area of sea and land, or in the elevation of the land, or in the distribution of sea and land, that would have a very marked influence on the amount of rainfall, and therefore also on the rate of denudation. Reference was made in one of the former articles of this series to the large quantity of water that is contained in stratified and metamorphic rocks ; and the inference that may reasonably be drawn from its presence there, viz., that the ocean in very early times contained a considerably larger quantity of water than it does now, from which again it may be inferred that the area of the ocean was more extensive and that of the land less extensive than at the present day. On the other hand, it is quite possible that the basins of the great oceans were much deeper in very early times than they are now, and therefore also that the relative areas of sea and land may not have changed to a very great extent. In any case, it is obvious that the rate of denudation may have been influenced by quite a variety of circumstances, and it is possible that a careful study of the whole subject, and also a careful study of all the circumstances affecting the secular cooling of the earth, may show the need of revising the estimates of the age of the earth, derived from both lines of enquiry.

XX. *The Dielectric Strength of Air.*
By Alexander Russell, *M.A., M.I.E.E.*[*]

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1. *Introduction.*

PHYSICISTS generally attempt to deduce the dielectric strength of air, at a given barometric pressure, from the results of experiments on the disruptive voltages between equal metal electrodes at given distances apart. They calculate the maximum value of the electric intensity between the metal electrodes on the assumption that the electric field round them is similar to that existing at low voltages. Figures obtained in this way were found, greatly to the disappointment of the early experimenters, to vary widely with the distance apart of the electrodes. Lord Kelvin, however, as far back as 1860 [*], deduced from the results of his experiments with large electrodes that it was " most probable " that the numbers obtained in this way at higher voltages would be " sensibly constant." An examination of the results, which are given below, obtained recently by electricians will show that experiment has amply justified Lord Kelvin's conclusion. The author finds, by considering experimental results obtained both with direct

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    * Proc. Roy. Soc. April 12, 1860 ; ' Reprint,' p. 259.

and alternating pressures, that the limiting value to which the numbers approach is 38 kilovolts per centimetre.

When the electrodes are small, or when the disruptive voltages are only a few kilovolts, the numbers obtained in this way differ largely from 38 kilovolts per centimetre. It is therefore necessary to explain why this is the case. It will be shown in what follows that when the electrodes are small, the air surrounding them may have broken down and become a conductor at voltages which are only a fraction of the disruptive voltage. In this case luminous effects are generally observed at the electrodes. When a high alternating pressure, less than the disruptive voltage, is maintained between small electrodes a few inches apart, each electrode, when the P.D. is sufficiently high, is seen surrounded by a faintly luminous enveloping cloud of a bluish colour, which apparently does not touch the conductor it envelopes. We shall call this cloud the corona. As the pressure is increased, short violet streamers are seen issuing outwards from the corona, the space immediately outside it being the seat of great electrical activity. At higher pressures the streamers are longer, and a hissing noise is heard. When the potential-difference between the electrodes approaches the disruptive value, sparks take place between them, and finally, when all the air is broken down, an arc is suddenly established.

Now, when luminous effects make their appearance, it is obvious that the boundaries of the Faraday-tubes are altered, and, consequently, that the electric field is different from what it is at low voltages. We cannot apply formulæ, therefore, which have been obtained on the assumption that the distribution of the tubes is the same as that for low pressures. We have not attempted to deduce formulæ which will give the dielectric strength of air from the disruptive voltage between two electrodes surrounded with coronæ, as the space occupied by the brush discharges is not clearly defined. There are many cases, however, when there are no luminous effects and where a disruptive discharge ensues the moment that the dielectric stress attains the breaking-down value. We have deduced the dielectric strength of air from the experimental results obtained in these cases.

Many electricians consider that a disruptive discharge always occurs the moment the electric stress at any point of the dielectric between the two electrodes attains a certain maximum value. In what follows, however, we show that in many cases, when some of the air round an electrode breaks down, the new value of the " maximum electric intensity " at the boundary of the broken-down air is less than the old

value at the boundary of the metal, and so there is equi-librium, a corona being formed.

The explanation of the varying numbers obtained when large electrodes are used and the disruptive voltages con-sidered are small, is more difficult. When the minimum distance x between the electrodes is less than 3μ, the sparking potentials are practically independent of the nature of the gas between the electrodes [*]. Since the material of which the electrodes is made exerts an important influence on the sparking potential V, at these small distances, it is highly probable that the carriers of the discharge come from the metal and not from the gas. For a certain distance greater than 3μ, G. M. Hobbs finds in some cases that V remains constant and equal to the minimum spark-potential which in air is about 350 volts [†]. For slightly greater distances V increases uniformly with x.

It is obvious, therefore, that when the electrodes are very close together, we cannot assume that we have a homogeneous medium bounded by rigid equipotential surfaces. Hence, as the equipotential surfaces are unknown, we cannot apply the ordinary electrostatic equations. For these reasons we have in the following paper only considered experimental results obtained for values of x greater than one millimetre. If we had only considered distances greater than half a centimetre (one fifth of an inch), it would have been unnecessary to make any assumptions about the actions that take place at the end of the tube subjected to the maximum electric stress, as the maximum values of the electric intensity, at the instant of discharge, are found to be in satisfactory agree-ment. In order, however, to include in our formulæ the sparking potentials for values of x lying between 0.1 and 0.5 cm., we have found that it is necessary to make the following assumption. At the moment of the disruptive discharge, the pressure on the ends of the Faraday tube subjected to the maximum stress is $V - \epsilon$, where ϵ represents what we shall call the lost volts. When the electrodes are surrounded with coronæ an assumption of this nature must be made [‡], but in this case ϵ will be a function of V and x. In the cases we consider we assume that ϵ is constant and equal to 0.8 of a kilovolt. Making this assumption and,

* G. M. Hobbs, "The Relation between P.D. and Spark-length for Small Values of the latter." Phil. Mag. [6] x. p. 617 (Dec. 1905).

† The Hon. R. J. Strutt, "On the Least Potential-Difference required to produce Discharge through Various Gases." Phil. Trans. vol. 193. A. p. 377 (1899–1900).

‡ H. J. Ryan, "The Conductivity of the Atmosphere at High Voltages." Trans. Am. Inst. El. Eng. vol. xxiii. p. 101 (1904).

for reasons given above, only considering experiments with large electrodes, at appreciable distances apart, we find that the maximum values of the electric intensity at the moment of the disruptive voltage is practically constant for distances varying from a millimetre up to 15 centimetres, and for voltages varying between 4 and 160 kilovolts.

2. *Historical.*

Nearly all experimenters have used equal spherical electrodes. It is therefore necessary to be able to write down at once the value of the electric intensity between two spheres whatever may be their potentials. Kirchhoff * in a very valuable paper has shown how to obtain from Poisson's † equations an expression for the maximum value of the electric intensity in the form of an infinite series. Unfortunately this paper can only be understood by those who are thoroughly familiar with Jacobi's theorems in Elliptic Functions, and so the important results contained in it are known to few physicists. In 1890, Professor A. Schuster ‡ published a table giving the value of the maximum electric intensity between two spheres when one was at potential V and the other at potential zero. He gives, however, no proof of the formula, merely referring to Kirchhoff's work. He reduces the infinite series formula for the case of two spheres close together, given by Kirchhoff, into a remarkably simple algebraical form, and shows that, when the spheres are at potentials V and 0, it applies with sufficient accuracy for practical purposes up to a distance between them equal to one-fifth of their radius. In what follows it will be shown that this Kirchhoff-Schuster formula applies with very considerable accuracy, when the potentials are $+V/2$ and $-V/2$, up to a distance apart equal to their radius. This is proved by actually calculating the values of the series, as it is difficult to see from Kirchhoff's method of proof what are the limitations of his formula. By considering the equipotential surfaces round two particles having equal and opposite charges, the author shows how the first two terms of the Kirchhoff-Schuster formula can be found very simply.

* Crelle's *Journal*, 1860, "Ueber die Vertheilung der Elektricität auf zwei leitenden Kugeln," p. 89; *Gesammelte Abhandlungen*, p. 78.

† *Mémoires de l'Institut Impérial de France*, "Sur la Distribution de l'Electricité à la Surface des Corps Conducteurs." Read 9th May and 3rd Aug. 1812.

‡ A. Schuster, "The Disruptive Discharge of Electricity through Gases," Phil. Mag. vol. xxix. p. 192 (Feb. 1890).

Professor A. Heydweiller[*] carried out a valuable set of experiments on sparking distances in 1892. He also uses Kirchhoff's formula without, however, proving it. Tables of the numerical values of the electric intensity when the spheres are various distances apart and when they are at equal and opposite potentials are calculated. The formulæ are applied to his own experimental results, but as he does not discriminate between the cases when they are and when they are not applicable, and neglects the 'lost volts,' the results vary widely. The experimental results analysed in Table VI. below are taken from this paper. In Mascart and Joubert's 'Leçons sur l'Electricité et le Magnetisme,' vol. ii. p. 610 (1897), a neat proof of a series formula for the maximum electric intensity between two unequal spheres is indicated. Kirchhoff's results are also quoted, and the formulæ are applied with, however, indifferent success.

In this paper the author gives a simple proof by Kelvin's method of images of Kirchhoff's series formula. He shows by elementary algebra that it can be expressed quite approximately enough for all practical purposes by a simple formula. He has also calculated complete tables which enable any one to write down at once the maximum value of the electric intensity between two equal spheres whatever may be their potentials.

In some of the experiments analysed below cylindrical electrodes are used; it is therefore necessary to get the formula for this case also. It will, however, be more instructive to consider the very simplest mathematical cases first, and thus we shall be able to form a clearer picture of the phenomena that happen in the more difficult practical cases.

3. *The Electric Intensity between Two Concentric Spheres.*

In the case of a spherical condenser we have a metallic sphere concentric with a metallic spherical envelope. If the radius of the inner sphere be a and the inner radius of the outer sphere be b, we have

$$-\frac{dv}{dr} = \frac{q}{r^2},$$

where v is the potential at a distance r from their common centre, and q is the charge on the inner sphere. Hence we

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[*] Wiedemann's *Annalen,* vol. xlviii. p. 785 (1893).

easily find that

$$-\frac{dv}{dr} = \frac{\mathrm{V}ab}{r^2(b-a)},$$

where V is the P.D. between the spheres. Now dv/dr has obviously its maximum value R_m when $r=a$, and thus we have

$$\mathrm{R}_m = \frac{\mathrm{V}b}{a(b-a)}.$$

If we suppose that b and V are fixed and a is a variable, we see that R_m increases from $a=0$ to $a=b/2$ and diminishes for greater values of a. Hence, if a be greater than $b/2$ and we gradually increase V, the moment the electric intensity attains a certain value the air immediately in contact with the inner sphere breaks down and becomes conducting. The electric intensity at the surface of this stratum of conducting air round the inner sphere will be greater than the old maximum electric intensity, and hence a new stratum will be broken down. It is unlikely that the boundaries of the strata successively broken down will be exactly spherical, but any lack of symmetry will accelerate the discharge and an arc between the two spheres will certainly be established. Thus when a is greater than $b/2$ the sparking voltages between the two spheres may be used to calculate $\mathrm{R}_{\max}$. the dielectric strength of air.

On the other hand, if a be less than $b/2$, an increase in its value will diminish R_m, and thus equilibrium is possible with a conducting stratum of air round the inner sphere. The outside of this stratum is what we call the corona. As the voltage V is increased the corona grows until its radius is nearly equal to $b/2$, when a disruptive discharge will ensue. We see therefore that the size of the inner sphere has no practical effect on the disruptive voltage provided that its radius be less than $b/2$.

When the radius is greater than $b/2$ we should expect no luminous effects until the final discharge took place. This would occur at the instant when

$$\mathrm{R}_{\max.} = \frac{\mathrm{V}-\epsilon}{b-a} \cdot \frac{b}{a},$$

where ϵ represents the lost volts.

4. *The Electric Intensity inside a Concentric Main.*

Let us now consider the important practical case of a concentric main. A hollow conducting cylinder of inner radius b contains a coaxial conducting cylinder of radius a. If the

cylinders be separated by air, the electric intensity R at a point P in the air at a distance r from the axis of the cylinders is given by*

$$R = -\frac{dV}{dr} = \frac{V}{r \log (b/a)},$$

where V is the P.D. between the cylinders. R has obviously its maximum value R_m when $r = a$. We also have

$$\frac{dR_m}{da} = \frac{V}{\{a \log (b/a)\}^2} \{1 - \log (b/a)\},$$

if V and b remain constant. Hence, if a be less than b/ϵ, where ϵ is the base of Neperian logarithms, R_m will diminish as a increases. In this case a corona will be formed. When the radius of the corona is nearly equal to b/ϵ a disruptive discharge will ensue.

If the radius of the inner cylinder be greater than b/ϵ, a disruptive discharge ensues whenever the intensity at the surface of the inner cylinder equals $R_{max.}$. This was verified roughly by Gaugain †.

The same formulæ apply when the dielectric coefficient of the insulating material between the conductors is not unity. We see, therefore, that the "factor of safety" of concentric mains is not necessarily increased by diminishing the radius of the inner conductor. This result is of considerable practical importance.

5. *The Corona round a Cylinder.*

The luminous effects produced when a cylinder is maintained at a very high alternating potential from earth have been investigated experimentally by E. Jona‡. He found that a cylindrical wire supported by high-tension insulators becomes luminous when the voltage between it and surrounding objects attains a definite value, which depends mainly on the diameter of the cylinder. The corona in this case is practically a concentric cylinder, the diameter d of which varies with the voltage V. In the following table V is in kilovolts and d is in millimetres.

TABLE I.—E. Jona's experiments on the diameter of the corona round a thin wire at various voltages.

V...	12·3	18·2	28	42	57	81	106	127	152	185	196
d ...	0·12	0·25	0·50	1·00	2·00	4·00	6·00	7·60	10·0	12·5	15·0

* Russell, 'Alternating Currents,' vol. i. p. 95.
† *Annales de Chimie et de Physique*, viii. p. 75 (1866).
‡ E. Jona, *Elettricistà*, Rome, xiii. pp. 113–115, April 15, 1904 Science Abstracts, vol. vii. B, p. 605.

From 1 to 15 millimetres V is given roughly by the equation

$$V = 30 + 12d.$$

It will be seen that the diameter of the corona for a pressure of 18 kilovolts is 0·025 cm. Thus if the diameter of the wire is less than 0·025 cm., in a dark room it will be seen surrounded with a corona when the pressure between it and the earth is greater than 18 kilovolts. Approaching an earthed conductor to the wire will increase the luminous effects. E. Jona found that the diameter of the corona was 1·5 cm. whether the wire were 0·01 or 1·4 cm. in diameter, when the pressure was 196 kilovolts.

6. *The Stress in the Dielectric round Two Particles having equal and opposite charges of Electricity.*

Let there be a charge $+q$ of electricity at P (fig. 1) and a charge $-q$ at N. The potential v at a point distant r_1 and r_2 from P and N respectively is given by

$$v = q/r_1 - q/r_2. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The locus of the points, the bipolar coordinates of which satisfy the equation (a) for a given value of v, will give the surface on which the potential is v. Hence these surfaces (fig. 1) can be easily constructed. We see that the equipotential surfaces near P and N are practically spheres.

Fig. 1.
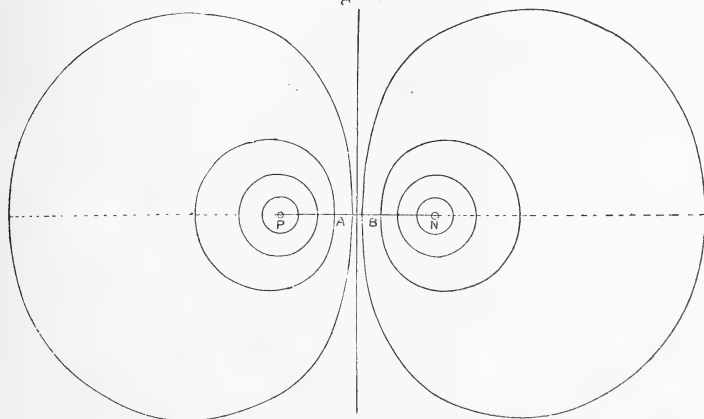
Equipotential surfaces round two points having equal and opposite charges.

Hence the equipotential lines shown in the figure are approximately the same as those of two spheres at a considerable distance apart.

Let $V/2$ and $-V/2$ be the potentials on any two equal equipotential surfaces surrounding P and N respectively. We shall find a formula to determine the maximum value of the electric intensity between these two surfaces. From symmetry the maximum value of the electric intensity R_m in the space between the two will be at the points A and B, where the line joining P and N cuts the surfaces. If $PN = d$ and $PA = a$, we have

$$\frac{V}{2} = \frac{q}{a} - \frac{q}{d-a},$$

and therefore

$$\frac{q}{V} = \frac{a(d-a)}{2(d-2a)}.$$

We also have

$$R_m = q\left\{\frac{1}{a^2} + \frac{1}{(d-a)^2}\right\},$$

and hence

$$R_m = (V/x)f, \quad \ldots \ldots \ldots \quad (b)$$

where x, which equals $d - 2a$, is the minimum distance between the two surfaces and f is given by

$$f = \tfrac{1}{2}(1 + x/a) + \frac{1}{2(1 + x/a)}. \quad \ldots \ldots \quad (c)$$

Now V/x is the average value of the electric intensity along the line joining the nearest points of the surfaces, and is the number which electricians ordinarily give as a measure of the dielectric stress on the insulating medium. We see that f is the factor required to convert this number into the maximum electric intensity.

When x/a is large the surfaces are very approximately spheres of radius a, and (c) can therefore be used to calculate the value of f for two spheres when their distance apart is large compared with the radius of either.

When x/a is small we can show that

$$f = 1 + x/3\rho, \quad \ldots \ldots \ldots \quad (d)$$

approximately, where ρ is the radius of curvature of the equipotential surfaces at the points where the intensity is a maximum. We should expect therefore that, if we had two spheres the radius of each of which was ρ, (d) would give the value of f approximately when x/ρ was small. We shall show later on that (d) gives the value of f in this case, to an accuracy of one in a thousand when x/ρ is $0{\cdot}1$ or less. Even when x/ρ is unity the error is only about 2 per cent.

The two terms given on the right-hand side of (d) are the first two terms in the important Kirchhoff-Schuster formula quoted below.

7. *Proof of the Series-Formula for the Maximum Electric Intensity between Two Equal Spheres.*

Let us suppose that the radii of the conducting spheres X and Y (fig. 2) are each equal to a, that the distance

Fig. 2.
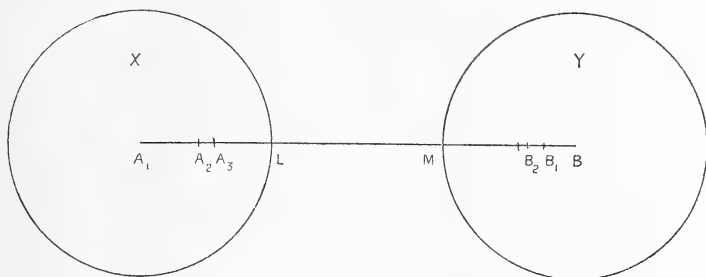
$A_1B = d$; $A_1L = BM = a$; $LM = x = d - 2a$.

between their centres A_1 and B is d, and that the minimum distance LM between them is x, so that $d = x + 2a$. Let us suppose also that these spheres are at potentials V_1 and V_2. We picture Faraday-tubes starting from their surfaces. If their potentials are of opposite sign some of these tubes connect the two spheres and others connect them with neighbouring conductors. We suppose that these other conductors are so far away that they do not appreciably affect the distribution of the tubes in the field between the two spheres. Now, if the spheres be removed we shall show that this field can be exactly reproduced by a series of point charges placed at definite points on the lines AL and BM (fig. 2). The point charges will have the spherical surfaces X and Y for the equipotential surfaces V_1 and V_2 respectively. We can therefore write down at once the potentials and the electric intensities at all external points.

We shall first consider the series of points A_1, A_2 ... B_1, B_2 ... (fig. 2) which are connected by the following relations,

$$BA_1 \cdot BB_1 = a^2 = A_1A_2 \cdot A_1B_1$$

$$BA_2 \cdot BB_2 = a^2 = A_1A_3 \cdot A_1B_2.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We see that the points A_2, B_1; ... A_{n+1}, B_n, are conjugate with respect to the sphere X and the points B_1, A_1; ... B_n, A_n, are conjugate with respect to the

sphere Y.   Let

$$A_1 A_{n+1} = u_{n+1} \text{ and } B\, B_n = u'_n,$$

then the above equations may be written

$$(d-0)u_1' = a^2 = u_2(d-u_1')$$
$$(d-u_2)u_2' = a^2 = u_3(d-u_2')$$

. . . .

In general, we have

$$(d-u_{n-1})u'_{n-1} = a^2 = u_n(d-u'_{n-1}),$$

and thus

$$u_n\{d-a^2/(d-u_{n-1})\} = a^2,$$

or

$$u_n u_{n-1} - \{(d^2-a^2)/d\}u_n - (a^2/d)\,u_{n-1} = -a^2.$$

This form of difference equation is well known * and is readily solved by assuming that $u_n = v_{n+1}/v_n + (d^2-a^2)d$. Making this assumption we find that

$$v_{n+1} + \{(d^2 - 2a^2)/d\}v_n + (a^4/d^2)v_{n-1} = 0,$$

a linear difference equation with constant coefficients. Hence solving in the ordinary way † we get

$$v_n = A a^n(a/d - q)^n + B a^n(a/d - 1/q)^n,$$

where A and B are constants and

$$2q = d/a - \sqrt{d^2-4a^2}/a, \quad \ldots \ldots (1)\ddagger$$

and

$$2/q = d/a + \sqrt{d^2-4a^2}/a, \quad \ldots \ldots (2)$$

so that

$$1/q + q = d/a, \quad \ldots \ldots \ldots \ldots (3)$$

and

$$1/q - q = \sqrt{d^2-4a^2}/a. \quad \ldots \ldots \ldots (4)$$

Now when n is unity $u_1 = 0$, and thus

$$v_2/v_1 = -(d^2-a^2)/d = -a(1+q^2+q^4)/\{q(1+q^2)\}.$$

Substituting for v_2 and v_1 their values, in terms of A and B, in this equation we find that $B = -Aq^2$.   We thus find on

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* Boole's 'Finite Differences,' 3rd ed. p. 233.
† Boole's 'Finite Differences,' chap. xi.
‡ I have called this expression q so as to introduce elliptic function notation.   It is a pure number and has nothing to do with an electric charge.

substituting for v_{n+1} and v_n their values and simplifying, that

$$u_n = aq \frac{1 - q^{4n-4}}{1 - q^{4n-2}}, \quad \cdots \quad (5)$$

and

$$u_n' = \frac{a_n}{d - u_n}$$

$$= aq \frac{1 - q^{4n-2}}{1 - q^{4n}} . \quad \cdots \quad (6)$$

Let us now suppose that charges $Q_1, \ldots Q_n$, are placed at the points $A_1, \ldots A_n$ (fig. 2), and that charges $Q_1', \ldots Q_n'$, are placed at the points $B_1, \ldots B_n$. We shall find the values of these charges so that the potential of the spherical surface X is V_1 and that of the spherical surface Y is zero.

Consider the potential at a point P' at a distance a from B. The potential at this point will obviously be

$$\Sigma \left(\frac{Q_n}{P'A_n} + \frac{Q_n'}{P'B_n} \right).$$

If therefore we choose the ratio of Q_n to Q_n' so that $Q_n / Q_n' = -P'A_n/P'B_n =$ a constant for every point on Y, the potential at P' will be zero, and therefore also the potential of the spherical surface Y will be zero. Since A_n and B_n are conjugate points with respect to the sphere Y we have [*]

$$\frac{Q_n'}{Q_n} = -\frac{P'A_n}{P'B_n} = -\frac{BB_n}{a} = -\frac{u_n'}{a}. \quad \cdots \quad (7)$$

Again, the potential at a point P distant a from A is given by

$$\frac{Q_1}{a} + \Sigma \left(\frac{Q_{n+1}}{PA_{n+1}} + \frac{Q_n'}{PB_n} \right).$$

This will be V_1 if we make $Q_1 = V_1 a$ and

$$\frac{Q_{n+1}}{Q_n'} = -\frac{PA_{n+1}}{PB_n} = -\frac{A_1A_{n+1}}{a} = -\frac{u_{n+1}}{a}. \quad \cdots \quad (8)$$

Hence if we determine the charges Q_n and Q_n' by means of (7) and (8) the potential of the spherical surface X will be V_1 and that of the spherical surface Y will be zero.

From (7) and (8), we have

$$\frac{Q_{n+1}}{Q_n} = \frac{u_{n+1}u_n'}{a^2}$$

$$= q^2 \frac{1 - q^{4n-2}}{1 - q^{4n+2}},$$

[*] Russell, 'Alternating Currents,' vol. i. p. 101.

and thus, since $\qquad Q_1 = V_1 a_1$, we have

$$Q_n = a V_1 \frac{(1/q - q)q^{2n-1}}{1 - q^{4n-2}} . \qquad \cdots \qquad (9)$$

Hence also

$$Q_n' = -a V_1 \frac{(1/q - q)q^{2n}}{1 - q^{4n}} . \qquad \cdots \qquad (10)$$

The electric intensity between the two spheres will obviously have its maximum values R_m at L and M, and thus,

$$R_m = \frac{Q_1}{a^2} + \frac{Q_2}{(a - u_2)^2} + \cdots + \frac{Q_{n+1}}{(a - u_{n+1})^2} + \cdots$$

$$- \frac{Q_1'}{(d - a - u_1')} - \cdots - \frac{Q_n'}{(d - a - u_n')^2} - \cdots$$

Now by (8)

$$\frac{Q_n'}{(d - a - u_n')^2} = - \frac{Q_{n+1}(u_{n+1}/a)}{(a - u_{n+1})^2},$$

and hence

$$R_m = \frac{Q_1}{a^2} + \Sigma \frac{Q_{n+1}}{a} \cdot \frac{a + u_{n+1}}{(a - u_{n+1})^2}.$$

Substituting for u_{n+1} and Q_{n+1} their values from (5) and (9) we get

$$R_m = \frac{V_1}{a} \frac{(1 + q)^2}{1 - q} \sum_1^\infty \frac{1 - q^{4n-3}}{(1 + q^{4n-3})^2} q^{2n-2} . \quad \cdots \quad (11)$$

The value of the electric intensity R_m' at M is given by

$$R_m' = \frac{Q_1}{(d - a)^2} + \cdots + \frac{Q_n}{(d - a - u_n)^2} + \cdots$$

$$- \frac{Q_1'}{(a - u_1')^2} - \cdots - \frac{Q_n'}{(a - u_n')^2} - \cdots$$

Noticing that $d - u_n = a^2/u_n'$ and that $Q_n/Q_n' = -a/u_n'$, we find that

$$R_m' = - \frac{V_1}{a} \frac{(1 + q)^2}{1 - q} \sum_1^\infty \frac{1 - q^{4n-1}}{(1 + q^{4n-1})^2} q^{2n-1} . \quad \cdots \quad (12)$$

We can write down the values of R_m and R_m' when the spheres X and Y are at potentials 0 and V_2 in a similar manner. Hence by the principle of superposition we find that the electric intensity at L when the spheres are at

potentials V_1 and V_2 is given by

$$R_m = \frac{V_1}{a}\frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{4n-3}}{(1+q^{4n-3})^2}q^{2n-2}$$
$$- \frac{V_2}{a}\frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{4n-1}}{(1+q^{4n-1})^2}q^{2n-1} \quad . \quad . \quad (13)$$

The most important case is when

$$V_1 = -V_2 = V/2, \text{ and in this case}$$

$$R_m = \frac{V}{2a} \cdot \frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{2n-1}}{(1+q^{2n-1})^2}q^{n-1} \quad . \quad . \quad (14)$$

$$= \frac{V}{x} \cdot f,$$

where

$$f = \frac{x}{2a}\frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{2n-1}}{(1+q^{2n-1})^2}q^{n-1}, \quad . \quad . \quad (15)$$

and x is the minimum distance between the two spheres. It is convenient to tabulate f for various values of x/a. We see that f is the factor which converts the average electric intensity in the line joining the centres of the two spheres into the maximum electric intensity. In measuring dielectric strengths electricians as a rule merely give the average electric intensity, assuming that f is unity whatever may be the shape of the electrodes.

Another practical case is when one of the spheres is maintained at zero potential. In this case let us suppose that $V_1 = V$, and $V_2 = 0$. Hence by (13)

$$R_m = \frac{V}{x}f_1,$$

where

$$f_1 = \frac{x}{a}\frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{4n-3}}{(1+q^{4n-3})^2}q^{2n-2}, \quad . \quad . \quad (16)$$

In practice it is very difficult to make certain that one sphere is at zero potential, and so this method of testing dielectric strengths is not advisable.

We may write (13) in the form

$$R_m = \frac{V_1}{x}f_1 - \frac{V_2}{x}(2f - f_1)$$

$$= \frac{V_1 - V_2}{x}f_1 + 2\frac{V_2}{x}(f_1 - f). \quad . \quad . \quad (17)$$

Thus if we can calculate the values of f and f_1 for any given value of x/a we have completely solved the problem.

When V_2 is zero or negative we see, since $f_1 - f$ is always positive, that for a given value of $V_1 - V_2$, R_m has its greatest value when V_2 is zero and has its least value when $V_2 = -V_1$.

We shall now give methods and formulæ for calculating f and f_1, and we shall also give tables of their values.

8. *Approximate Formulæ for the Maximum Electric Intensity between Two Equal Spheres.*

Formula (15) may be written in the form

$$f = \frac{x}{2a} \left\{ 1 + \frac{(1+q)^2}{1-q} \cdot \frac{1-q^3}{(1+q^3)^2} q + \cdots \right.$$
$$\left. + \frac{(1+q)^2}{1-q} \cdot \frac{1-q^{2n+1}}{(1+q^{2n+1})^2} q^n + \cdots \right\}.$$

It is easy to see that the $(n+1)$th term of the series in the brackets equals

$$\frac{1+q+q^2+\ldots+q^{2n}}{(1-q+q^2-\ldots+q^{2n})^2} q^n = \frac{q^n + 1/q^n + q^{n-1} + 1/q^{n-1} + \cdots}{\{q^n + 1/q^n - (q^{n-1} + 1/q^{n-1}) + \cdots\}^2}.$$

By formula (3)

$$q + 1/q = d/a = 2 + x/a = y \text{ (say)},$$

and thus [*]

$$q^n + 1/q^n = y^n - n y^{n-2} + \frac{n(n-3)}{\lfloor 2} y^{n-4} - \cdots$$
$$+ (-1)^r \frac{n(n-r-1)\ldots(n-2r+1)}{\lfloor r} y^{n-2r} \cdots$$

We can thus easily express f in terms of y. Substituting and simplifying we find that

$$f = \frac{y-2}{2} \left\{ 1 + \frac{y+1}{(y-1)^2} + \frac{y^2+y-1}{(y^2-y-1)^2} + \frac{y^3+y^2-2y-1}{(y^3-y^2-2y+1)^2} \right.$$
$$+ \frac{y^4+y^3-3y^2-2y+1}{(y^4-y^3-3y^2+2y+1)^2}$$
$$+ \frac{y^5+y^4-4y^3-3y^2+3y+1}{(y^5-y^4-4y^3+3y^2+3y-1)^2}$$
$$\left. + \quad \cdots \quad \cdots \quad \right\}.$$

Now y cannot be less than 2. Hence expanding by the binomial theorem and neglecting $1/y^9$ and higher powers of

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[*] Todhunter's ' Theory of Equations,' 3rd ed. p. 183.

$1/y$, we find that

$$f = \frac{y-2}{2} \left\{ 1 + \frac{1}{y} + \frac{4}{y^2} + \frac{9}{y^3} + \frac{17}{y^4} + \frac{33}{y^5} \right.$$
$$\left. + \frac{64}{y^6} + \frac{126}{y^7} + \frac{252}{y^8} + \ldots \right\}. \quad \ldots \quad (18)$$

It will be seen that the coefficients of $1/y$ are rapidly getting larger, but it has to be remembered that f must equal unity when $y-2$ is zero. We therefore alter the above formula so as to make $f = 1$ when y is 2, and yet make the expanded form of the altered formula agree with (18) as far as the coefficient of $1/y^8$. By this means we secure that the formula (19) gives the correct value of f when y is 2, and again when we can neglect the ninth term in the series formula (15). Expanding $y/(y-2)$ in powers of $1/y$ as far as the term containing the eighth power, and substituting in (18) we get

$$f = \frac{y-2}{2} \left\{ -\frac{1}{y} + \frac{y}{y-2} + \frac{1}{y^3} + \frac{1}{y^4} + \frac{1}{y^5} - \frac{2}{y^7} - \frac{4}{y^8} \right\}$$

approximately, or

$$f = \tfrac{1}{2}(y-1) + \frac{1}{y} + \frac{(y-2)}{2} \left\{ \frac{1}{y^3} + \frac{1}{y^4} + \frac{1}{y^5} - \frac{2}{y^7} - \frac{4}{y^8} \right\}. \quad \ldots (19)$$

Substituting $2 + x/a$ for y, we get

$$f = \tfrac{1}{2}(x/a + 1) + \frac{1}{x/a + 2} + \frac{x/a}{2(x/a + 2)^3} + \frac{x/a}{2(x/a + 2)^4}$$
$$+ \frac{x/a}{2(x/a + 2)^5} - \frac{x/a}{(x/a + 2)^7} - \frac{2x/a}{(x/a + 2)^8}. \quad \ldots (20)$$

The values of f are easily computed by this formula. For values of x/a less than $0\cdot1$ or greater than $0\cdot7$ the error is less than 1 in 1000, whilst for values of x/a between $0\cdot1$ and $0\cdot7$ the error is never as great as 2 in 1000. For practical purposes therefore the formula (20) gives the values of f with sufficient accuracy. We could have made it more accurate by taking more terms into account in the expansion (18), but we have not done so, as we have found by actual computation that the Kirchhoff–Schuster formula

$$f = 1 + \frac{1}{3} \cdot \frac{x}{a} + \frac{1}{45} \cdot \frac{x^2}{a^2} + \frac{73}{53760} \cdot \frac{x^3}{a^3} \quad \ldots \quad (21)$$

sums the series with a most gratifying accuracy until x/a gets greater than $0\cdot7$. It therefore completely covers the part of the scale of our formula which is slightly inaccurate. The formulæ (20) and (21) therefore give the complete practical

solution. It is not easy to give a simple proof of (21), but
we have found above by elementary considerations the first
two terms. If we expand the expression (20) in powers of
x/a we get

$$f = 1 + \frac{11}{32} \cdot \frac{x}{a} - \frac{3}{256} \cdot \frac{x^2}{a^2} + \frac{11}{256} \cdot \frac{x^3}{a^3} \quad \cdot \quad \cdot \quad \cdot \quad (22)$$

The difference between the values of f given by (21) and
(20) when x/a is small is roughly the hundredth part of x/a,
and as f is greater than unity it will be seen that the per-
centage error made by using (22) instead of (21) is small.

In Table III. below the values of the column headed f have
been found directly from the series-formula (15). In calcu-
lating this column I have to acknowledge the help I received
from four of my pupils, Messrs. Hewitt, Hoggett, Ritter, and
Taylor. In the second column the numbers are calculated by
(21), and in the third column by (20).

I am indebted to Mr. Arthur Berry, of King's College
Cambridge, for showing me how the direct calculation can be
greatly simplified. The formula (15) may be written

$$f = \frac{x}{2a} \cdot \frac{(1+q)^2}{(1-q)\sqrt{q}} \left\{ \frac{Kk}{2\pi} - \frac{2}{q^{3/2}} \sum_{1}^{\infty} \frac{q^{3n}}{(1+q^{2n-1})^2} \right\},$$

for [*]

$$\frac{Kk}{2\pi} = \sum_{1}^{\infty} \frac{q^{(2n-1)/2}}{1+q^{2n-1}}.$$

We have used this theorem to check several of our results.
For instance, when x/a is 0·5, q is also 0·5 by formula (1).
Also

$$[\dagger] \quad \sqrt{2kK/\pi} = 2q^{\frac{1}{4}}(1 + \sum_{1}^{\infty} q^{n^2+n})$$

$$= 2q^{\frac{1}{4}}(1 + q^2 + q^6 + q^{12} + q^{20} + \ldots)$$

$$= 2^{\frac{3}{4}}(1 + 0·25 + 0·015625$$

$$+ 0·000244 + \ldots)$$

$$= 2^{\frac{3}{4}}(1·26587).$$

Therefore $\quad \dfrac{kK}{2\pi} = 2^{-\frac{1}{2}}(1·26587)^2$

$$= 1·1331.$$

Also $\quad \sum_{1}^{\infty} \dfrac{q^{3n}}{(1+q^{2n-1})^2} = 0·07001.$

We thus find that $f = 1·1726$, when $x/a = 0·5$.

[*] A. Enneper, *Elliptische Functionen*, p. 179.
[†] A. G. Greenhill, ' Elliptic Functions,' p. 303.

Knowing the values of f we can find the values of f_1 easily by means of an elliptic integral series which is quoted in Kirchhoff's paper. It can be shown that[*]

$$\sum_1^\infty (-1)^{n-1} q^{\frac{2n-1}{2}} \frac{1-q^{2n-1}}{(1+q^{2n-1})^2} = \frac{kk'K^2}{\pi^2}.$$

Hence it follows from (15) and (16) that

$$f_1 = f + \frac{x}{2a} \frac{(1+q)^2}{(1-q)\sqrt{q}} \left(\frac{kk'K^2}{\pi^2} \right.$$

$$= f + \sqrt{\frac{x}{a}} \cdot \frac{x+4a}{2a} \cdot \frac{kk'K^2}{\pi^2}. \quad . \quad . \quad . \quad (23)$$

The values of k, k', and K can easily be found by well-known formulæ. Let us suppose, for instance, that we wish to find the value of f_1 when x/a is $0\cdot5$. We have already found that f is $1\cdot1726$ and q is $0\cdot5$.

Now[†]

$$\sqrt{2K/\pi} = 1 + 2 \sum_1^\infty q^{n^2}$$

$$= 1 + 1 + 0\cdot125 + 0\cdot003906$$

$$+ 0\cdot000031 + \ldots$$

$$= 2\cdot1289.$$

We also have

$$\ddagger \ \sqrt{2Kk'/\pi} = 1 + 2 \sum_1^\infty (-)^n q^{n^2}$$

$$= 1 - 1 + 0\cdot12503 - 0\cdot00391 + \ldots$$

$$= 0\cdot12112.$$

Thus $\qquad k' = (0\cdot12112/2\cdot1289)^2 = 0\cdot0032369$

and $\qquad k = \sqrt{1-k'^2} = 0\cdot99999.$

Hence $\qquad kk'K^2/\pi^2 = 0\cdot01662.$

Thus finally by (16)

$$f_1 = 1\cdot1726 + \{9/(4\sqrt{2})\}(0\cdot01662)$$

$$= 1\cdot1990.$$

This agrees with the value of f_1 found by direct calculation from (16).

[*] A. Enneper, *Elliptische Functionen*, p. 180.
[†] A. G. Greenhill, 'Elliptic Functions,' p. 303.
[‡] Greenhill, p. 303.

For values of x/a greater than unity the values of f_1 can be computed by the remarkably simple formula

$$f_1 = x/a + \frac{1}{x/a+1} + \frac{1}{(x/a+1)(x/a+2)^3}. \quad \cdot \quad \cdot \quad (24)$$

Hence it is unnecessary to tabulate the values of f_1 when x/a is greater than 4. The first row in the following table is taken from Schuster's paper, the second row is calculated by the formula

$$f_1 = x/a + \frac{1}{x/a+1}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (25)$$

and the third row by (24).

TABLE II.—Values of f_1.

x/a.	4.	5.	6.	7.	8.
Schuster's values ...	4·200	5·172	6·144	7·126	8·111
f_1 by (25)	4·200	5·167	6·143	7·125	8·111
f_1 by (24)	4·201	5·167	6·143	7·125	8·111

The values of f_1 given in the last row are the correct values.

TABLE III.—Values of f.

x/a.	q by (1).	f by (15).	f by (21).	f by (20).
0·0	1·0000	1·000	1·0000	1·0000
0·1	0·7298	1·034	1·0336	1·0343
0·2	0·6417	1·068	1·0676	1·0686
0·3	0·5821	1·102	1·1020	1·1032
0·4	0·5367	1·137	1·1370	1·1384
0·5	0·5000	1·173	1·1724	1·1735
0·6	0·4693	1·208	1·2083	1·2095
0·7	0·4431	1·245	1·2447	1·2460
0·8	0·4202	1·283	1·2814	1·2832
0·9	0·4006	1·321	1·3190	1·3210
1·0	0·3820	1·359		1·3594
1·5	0·3139	1·559		1·5594
2·0	0·2680	1·770		1·7704
3·0	0·2087	2·214		2·2149
4·0	0·1716	2·677		2·6777
5·0	0·1459	3·151		3·1513
6·0	0·1270	3·632		3·6317
7·0	0·1125	4·117		4·1165
8·0	0·1010	4·604		4·6044
9·0	0·0917	5·095		5·0946
10·0	0·0838	5·586		5·5865
100·0	0·0098	50·51		50·5098
1000·0	0·0012	500·5		500·5010

For values of x/a greater than 1·5 the values of f given in the last column are correct to four decimal figures. We have shown above by direct calculation that the value of f when x/a is 0·5 is 1·1726. The Kirchhoff-Schuster formula makes it 1·1724. This formula is therefore very accurate for values of x/a less than 0·5, and these are the values which it is so laborious to find by direct computation from (15).

In the following table for the values of f_1 the first column is taken from Table III. The next column is calculated by the equation

$$\Delta = \sqrt{\frac{x}{a} \cdot \frac{x+4a}{2a} \cdot \frac{kk'K^2}{\pi^2}},$$

and the last column for f_1 is got by the equation

$$f_1 = f + \triangle.$$

TABLE IV.—Values of f_1.

x/a.	f from Table III.	Δ.	f_1.
0	1·000	0·00000	1·000
0·1	1·034	0·00001	1·034
0·2	1·0676	0·0008	1·068
0·3	1·102	0·004	1·106
0·4	1·137	0·013	1·150
0·5	1·173	0·026	1·199
0·6	1·208	0·045	1·253
0·7	1·245	0·068	1·313
0·8	1·283	0·095	1·378
0·9	1·321	0·125	1·446
1·0	1·359	0·158	1·517

The values of f_1 given in this table are in exact agreement with the numbers given by Professor Schuster [*].

9. *The Disruptive Discharge between Two Spherical Electrodes.*

The formulæ and tables given above enable us to find the maximum value R_m of the intensity of the electric field round two spherical electrodes provided that the electrodes are not enveloped by coronæ; that is, provided that none of the air surrounding them is broken down. If no coronæ are formed before the disruptive discharge ensues, then we can calculate R_m at this instant, and so find $R_{max.}$ the dielectric strength of the air. As in the case of a concentric main or

* Phil. Mag. vol. xxix. p. 192.

two concentric spheres, it is of importance to know in what cases coronæ can be formed. The problem is now much more difficult as the coronæ are only approximately spherical, the maximum thickness of the stratum of conducting air round each electrode being on the line joining the centre of the two spheres.

If we make the assumption that the surrounding air is broken down to the same depth at every point on the surface of either electrode, we can find whether the value of R_m increases or diminishes with this depth. In the former case a disruptive discharge will certainly ensue, and *a fortiori* it will ensue in the actual case of two spherical electrodes, as the actual breakdown begins at the centre of the spherical face, raising, as it were, a small blister at that point, and so R_m must be greater owing to the greater curvature.

When the distance between the spheres is greater than the radius a, we have, to an accuracy of 1 in a 1000,

$$R_m = \frac{V}{x} \left\{ \tfrac{1}{2}(1 + x/a) + \alpha/(2 + x/a) \right\},$$

where α is $1 \cdot 077$, provided that x/a is less than 7.
Hence

$$R_m = \frac{V}{2} \left\{ \frac{1}{d - 2a} + \frac{1}{a} + \frac{2a\alpha}{d(d - 2a)} \right\}$$

$$= \frac{V}{2} \left\{ \frac{1 + \alpha}{d - 2a} + \frac{1}{a} - \frac{\alpha}{d} \right\},$$

and therefore

$$\frac{dR_m}{da} = \frac{V}{2} \left\{ \frac{2(1 + \alpha)}{(d - 2a)^2} - \frac{1}{a^2} \right\},$$

when V and d are constants. Hence, for values of d less than $a(2 + \sqrt{2(1 + \alpha)})$, that is, for values of d less than $4 \cdot 04a$, R increases as a increases, and thus, on our assumption, a disruptive discharge will ensue.

If the spheres be not further apart than twice their diameter we should therefore expect a disruptive discharge to ensue the moment R_m became R_{max}. For large spheres, experiment shows that this is the case up to a distance apart equal to about three times their diameter. For greater distances apart, the moment R_m attains the value R_{max} the air in the neighbourhood of that point is broken down and a partial corona is formed, the value of R_m at the surface of the corona being less than R_{max}. In these cases, as the equipotential surfaces are no longer spheres, we cannot apply our formulæ.

10. *The Maximum Electric Intensity between a Sphere and a Plane.*

When the plane is at zero potential, we see, by taking the image of the sphere in the plane, that

$$R_m = (V/x) f_p, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where f_p is the value of the factor f given above corresponding to $2x/a$; x being the least distance of a point on the sphere from the plane and a being its radius.

11. *The Maximum Electric Intensity between two infinitely long parallel Cylinders.*

Let us consider the value of the electric potential at points between the two cylinders, the sections of which by the plane of the paper are shown in fig. 3. If q and $-q$ be the

Fig. 3.
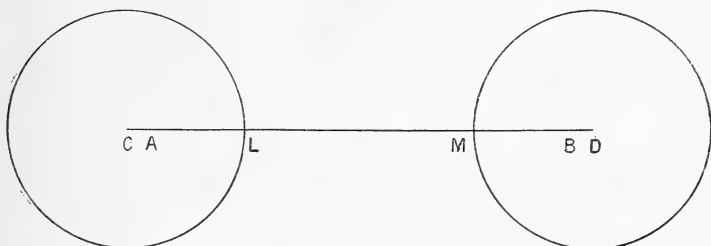
CD $= d =$ the distance between the axes of the two parallel cylinders.
LM $= x =$ the minimum distance between the cylinders.
$a =$ the radius of either cylinder.
A and B are inverse points.   CA . CB $=$ CL$^2 =$ DA . DB.

charges per unit length on the cylinders the axes of which pass through A and D respectively, the potential v at any point P external to them is given by

$$v = -2q \log (AP/BP),$$

where A and B are the inverse points of the circular sections. The maximum values R_m of the electric intensity will be at L and M. The potential at any point p on CD will be

$$v = -2q \log r + 2q \log (c-r),$$

where r is Ap and c is the distance AB.   Hence

$$R = \frac{2q}{r} + \frac{2q}{c-r}.$$

S 2

Now R has its maximum value R_m when r is AL.

Hence $\qquad R_m = \dfrac{2qc}{AL(c-AL)} = \dfrac{2qc}{a(d-2a)} = \dfrac{2qc}{ax},$

where x is the minimum distance between the cylinders. Now, we have *

$$q = \frac{V}{4 \log \{(d+c)/2a\}},$$

where V is the potential-difference between the cylinders and $c^2 = d^2 - 4a^2$.

Hence we have

$$R_m = \frac{V}{x} \frac{c/2a}{\log \{(d+c)/2a\}}$$

$$= \frac{V}{x} f,$$

where

$$f = \frac{y}{\log (1 + x/2a + y)}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (27)$$

and $\qquad y = \{x/a + (x/2a)^2\}^{\frac{1}{2}}.$

Values of f are given in the following table :—

TABLE V.

x/a ...	0·01	0·02	0·03	0·04	0·05	0·06	0 07	0·08	0·09
f	1·00	1·00	1·00	1·01	1·01	1·01	1·01	1·01	1·015

x/a ...	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
f	1·02	1·03	1·05	1·065	1·08	1·10	1·11	1·13	1·14

x/a ...	1	2	3	4	5	6	7	8	9
f	1·16	1·315	1·46	1·61	1·74	1·88	2·01	2·14	2·24

x/a ...	10	20	30	40	50	100	1000	10,000
f	2·39	3·56	4·62	5·62	6·58	11·0	72·5	587

* Russell, ' Alternating Currents,' vol. i. p. 102.

12. *The Disruptive Discharge between Two Parallel Cylinders.*

It is well known in practice that when we have two parallel wires with a high P.D. between them, then in certain cases coronæ envelope the wires. When they are close together, however, this effect is not produced, a disruptive discharge occurring directly the P.D. attains a certain value. It is important therefore to know what distance apart the wires must be in order that coronæ can be formed.

If we assume that $\sec \theta = d/2a$, we find that

$$R_m = \frac{V}{d(1 - \cos \theta)} \cdot \frac{\tan \theta}{\log (\tan \theta + \sec \theta)}.$$

Let d be constant and let a vary, then, solving the equation $dR_m/d\theta = 0$, we find that

$$\log \tan (\pi/4 + \theta/2) = \frac{\sin \theta}{\sin^2 \theta - \cos \theta}.$$

When θ is nearly $70°$ this equation is satisfied, and in this case $d = 5 \cdot 85\, a$ nearly.

Hence making the assumption that the coronæ are cylindrical in shape, we see that R_m diminishes as a increases when d is greater than $5 \cdot 85\, a$. In practice, therefore, we should not expect coronæ to be formed when the wires were at a less distance apart than about three times their diameter.

13. *The Application of the Formulæ to Experimental Results.*

I. With Direct Pressures.

(i.) *Lord Kelvin's tests with large electrodes.*

Lord Kelvin [*] was the first to make accurate tests on the disruptive voltages between electrodes in air. He found that the apparent dielectric strength of a thin stratum of air was much greater than that of a thick one. The apparent dielectric strength in our notation being Vf/x, we have

$$Vf/x = R_{max.} + 0 \cdot 8\, f/x,$$

where V is in kilovolts, and R ., the dielectric strength of air, is a constant. In Kelvin's experiments f was practically equal to unity at all distances, and thus V/x increases rapidly as x diminishes.

From his experimental results[†] Lord Kelvin concludes that a battery of 5510 Daniell cells could produce a spark between two slightly convex electrodes when the minimum

* Proc. Roy. Soc. 1860, or 'Reprint,' p. 24.
† Proc. Roy. Soc. April 12, 1860, p. 259.

distance between them was 1/8th of a centimetre. Taking the E.M.F. of a Daniell cell as 1·07 volts, this makes the dielectric strength $R_{max.}$ of air to be 40·8 kilovolts per centimetre; a result which is only about 6 per cent. higher than the number we give as the average value of R_{max}.

(ii.) A. Heydweiller. 5 cm. spheres (2·5 cm. radius).

In the valuable paper by A. Heydweiller, published in the *Annalen der Physik und Chemie*, vol. xlviii. p. 785 (1893), there are many tables of sparking-distances given both between equal and unequal electrodes. We consider merely the last table he gives, and we choose the 5 centimetre spheres as being likely to give the most accurate results. The height of the barometer was 74·5 cms., and the temperature 18° C. during the test. The columns headed x and V are taken from Heydweiller's paper, f_1 is calculated by the formulæ given above, and $R_{max.}$ is found by

$$R_{max.} = \{(V - 0·8)/x\}f_1.$$

We have assumed that the potentials of the spheres are V and 0 at the instant of the discharge. The results seem to indicate that this was not the case when the electrodes were at their greatest distances apart.

TABLE VI.

Heydweiller's test with 5 cm. spheres ($a = 2·5$).

$x=$distance apart in cms.   V=disruptive pressure in kilovolts.

x.	x/a.	f. (calc.).	V (observed).	$R_{max.}$ (calc.).
0·5	0·2	1·068	18·36	37·5
0·6	0·24	1·081	21·60	37·5
0·7	0·28	1·102	24·54	37·3
0·8	0·32	1·116	27·33	37·0
0·9	0·36	1·132	30·09	36·9
1·0	0·40	1·150	32·85	36·9
1·1	0·44	1·169	35·58	37·0
1·2	0·48	1·188	38·31	37·0
1·3	0·52	1·209	41·01	37·4
1·4	0·56	1·231	43·68	37·7
1·5	0·60	1·253	46·23	37·9
1·6	0·64	1·277	48·66	38·2

The mean value of the numbers in the last column is 37·5, and none of them differ from the mean value by as much as 2 per cent. Hence this experiment gives 37·5 kilovolts per centimetre as the dielectric strength of air.

(iii.) **J. Algermissen.** *5 cm. spheres* $(a = 2 \cdot 5 \ cm.)$.

In the following table the values of x and V are taken from Dr. Zenneck's work ' Elektromagnetische Schwingungen und Drahtlose Telegraphie,' 1905, p. 1011. They are due to J. Algermissen, and are deduced from the average of the values obtained on different days under varying conditions. We have assumed that the potentials of the electrodes were $+ V/2$ and $- V/2$ respectively at the instant of the discharge. As the results in the last column are very approximately constant our assumption is justified.

TABLE VII.

J. Algermissen. 5 cm. spheres $(a = 2 \cdot 5)$.

x is measured in cms. and V in kilovolts.

x.	x/a.	f(calc.).	V (obs.).	R_{max}. (calc.).
1·5	0·6	1·208	46·2	36·6
1·6	0·64	1·223	48·6	36·5
1·7	0·68	1·238	51·0	36·6
1·8	0·72	1·253	53·4	36·6
1·9	0·76	1·268	55·8	36·7
2·0	0·80	1·283	58·2	36·8
2·1	0·84	1·298	60·6	37·0
2·2	0·88	1·312	62·8	36·9
2·3	0·92	1·326	65·0	37·0
2·4	0·96	1·342	67·0	37·0
2·5	1·00	1·360	69·0	37·1
2·6	1·04	1·374	70·8	37·0
2·7	1·08	1·390	72·6	37·0
2·8	1·12	1·406	74·4	37·0
2·9	1·16	1·421	76·2	37·0
3·0	1·20	1·437	78·0	37·0
3·1	1·24	1·452	79·7	37·0
3·2	1·28	1·469	81·3	37·0
3·3	1·32	1·484	83·0	37·0
3·4	1·36	1·500	84·7	37·0
3·5	1·40	1·515	86·4	37·1
3·6	1·44	1·533	88·0	37·1
3·7	1·48	1·549	89·6	37·2
3·8	1·52	1·566	91·2	37·3
3·9	1·56	1·583	92·7	37·3
4·0	1·60	1·599	94·2	37·4
4·1	1·64	1·616	95·7	37·4
4·2	1·68	1·632	97·2	37·4

The mean of the values of R_{max}. in the last column gives the dielectric strength of air as 37·0 kilovolts per centimetre, and the greatest difference between any of the calculated numbers and this value is only about one per cent. It will be seen therefore, that the agreement between theory and

experiment is quite satisfactory. The three final values for $R_{max.}$ obtained in this table closely agree with the mean of the values we deduced from Heydweiller's test. Considerable weight, therefore, must be attached to the results of these experiments in determining the value of $R_{max.}$.

(iv.) J. Joubert and G. Carey Foster. 1 *cm. and* 2 *cm. spheres* ($a = 0\cdot 5$ & 1).

In Foster and Porter's (Joubert's) 'Electricity and Magnetism,' p. 135, tables of the sparking-distances between 1 centimetre and 2 centimetre spheres are given. The results for the 1 cm. spheres are taken from Joubert's *Traité élémentaire d'électricité* (2nd edit.), and those for the 2 cm. spheres were obtained by G. Carey Foster. An analysis of the table for the 1 centimetre spheres shows that if we calculate $R_{max.}$ for sparking-distances of 5, 10, and 15 cms., on the assumption that the air round the electrodes is not broken down to any appreciable depth before the discharge occurs, the values are much too large. This is in accord with the conclusion of § 9. The mean of the values up to a distance of 2 cms. apart makes $R_{max.}$ 42·8. The mean of the values for the 2 cm. spheres makes $R_{max.}$ 42·9.

(v.) É. Hospitalier. 1 *cm. spheres.*

An analysis of the experimental results given by É. Hospitalier in the *Formulaire de l'Électricien*, 21st year, 1904, p. 289, for the sparking-distances between two electrodes, each one cm. in diameter, shows that the potentials of the spheres are not $+ V/2$ and $- V/2$ at the instant of the discharge. The values of $R_{max.}$ calculated on this assumption diminish steadily from the maximum value 44·1 when the spheres are 0·6 of a cm. apart to 40·0 when they are 2 cms. apart. The values of f, however, are little affected by the absolute values of the potentials of the electrodes, provided that x/a is not greater than 0·3. Taking, therefore, the mean of the first three results given, we find that $R_{max.}$ is 42·2.

(vi.) Compagnie de l'Industrie Électrique. *Plate and sphere.*

The Compagnie de l'Industrie Électrique et Mécanique have published tests * on the disruptive voltages between a plate and a ball.

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* Turner and Hobart, 'Insulation of Electric Machines,' p. 33 (1905).

Table VIII.

Compagnie de l'Industrie Électrique.   Plate and 2 cm. ball.

x.	x/a.	f_p (calc. by § 10).	V (obs.).	R_{max}.
0·5	0·5	1·36	18	46·8
1·0	1·0	1·77	26	44·6
1·5	1·5	2·21	31	44·5
2·0	2·0	2·68	35·5	46·5
2·5	2·5	3·15	39	48·1
3·0	3·0	3·63	42·5	50·4
4·0	4·0	4·60	48·0	54·2
5·0	5·0	5·59	54·0	59·5
6·0	6·0	6·57	58·0	62·5

It will be seen that R_{max}. is beginning to increase rapidly (see § 9). The mean of the first four values gives 45·6 kilovolts per centimetre as the dielectric strength of air. In practice the plates used are not large, and so we are only justified in using our formula for f_p when the plate and the ball are close together. We do not attach much importance to this test.

II. With Alternating Pressures.

(i.) C. P. Steinmetz.   *2 inch spheres.*

In a paper on the "Dielectric Strength of Air," published in the Transactions of the American Institute of Electrical Engineers, vol. xv. p. 281, Professor C. P. Steinmetz gives the results of an elaborate and careful research on the disruptive voltages between pointed, spherical, and cylindrical electrodes. Alternating voltage was used of frequency 125, and the shape of the wave was practically identical with a sine curve when a particular smooth-core alternator was used. The ratio of the maximum to the effective voltage in all his experiments with this machine was practically 1·42. The spherical and cylindrical electrodes were put in nitrate of mercury and then rubbed with a clean cloth. When this was done it was found that the disruptive discharge, for a given distance apart of the electrodes, always took place at the same voltage. If the electrodes were merely polished, then at small distances apart the results were very erratic. The accuracy of the results obtained probably lies well within 4 per cent. in most cases. In the experiments the barometer varied from 75·2 to 76·2 cms. This variation introduces an uncertainty of about one per cent. The voltmeter readings may be one per cent. out, and there may be a one per cent.

error in determining the ratio of the maximum to the effective potential-difference. An error is also due to the moisture in the air. This, however, was found to be small. When the electrodes were immersed in "live" steam at atmospheric pressure, the effect of the steam was to *increase* the apparent dielectric strength of the air, a greater voltage being required to produce the disruptive discharge. As pressures up to 160 effective kilovolts were employed, the sparking-distances were large and could be measured with great accuracy. We should expect that, with these high voltages, our formulæ would apply with considerable accuracy, as the disturbing effect of the cathode glow would be small and the field would be approximately symmetrical.

In the following table the results of tests when the electrodes were spheres 2 inches in diameter are analysed. The column headed $R_{max.}$ gives the values of the dielectric strength in kilovolts per centimetre, calculated by the formula

$$R_{max.} = \{(1 \cdot 42V - 0 \cdot 8)/x\}f.$$

TABLE IX.

C. P. Steinmetz. 2 inch spheres ($a = 2 \cdot 54$ cms.).
$\smile = 125$. $E/V = 1 \cdot 42$, where E is the maximum and V the effective value of the alternating voltage.

No. of Experiment.	x.	x/a.	f(calc.).	V (obs.).	$R_{max.}$(calc.).
1	0·318	0·125	1·04	8·95	39·0
2	0·635	0·25	1·08	15·9	37·1
3	1·25	0·49	1·17	26·7	34·7
4	2·74	·08	1·39	51·0	36·2
5	3·69	1·45	1·54	65·2	38·3
6	4·29	1·69	1·63	70·8	37·9
7	5·72	2·25	1·88	83·8	38·9
8	7·62	3·00	2·21	94·0	36·7
9	8·74	3·44	2·42	102·0	39·9
10	10·0	3·95	2·66	101·5	38·0
11	12·9	5·08	3·19	108·0	37·7
12	14·2	5·60	3·44	114·5	39·1

The mean value of $R_{max.}$ obtained from the figures in the last column is 37·8. Considerable importance is attached to this test as the numbers actually observed are given.

The curve in fig. 4 gives the relation between V and x on the supposition that $R_{max.}$ is 38. Steinmetz's experimental results are plotted in this figure for purposes of comparison.

Fig. 4.—Sparking Voltages between 2 inch spherical electrodes. Points marked ⊙ are Steinmetz's experimental results.
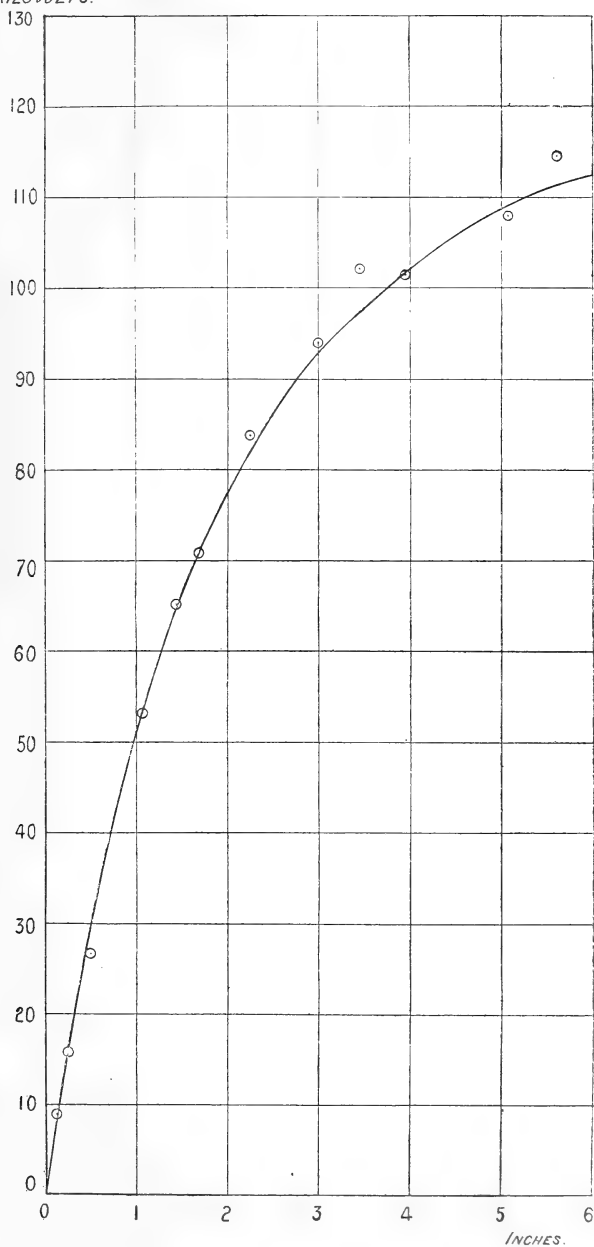
(ii.) Compagnie de l'Industrie Électrique. *2 cm. spheres.*

The Compagnie de l'Industrie Électrique et Mécanique of Geneva have published * a curve giving the sparking-distances between two spherical electrodes, each one centimetre in radius. The frequency of the alternating pressure employed was 50, and the ratio of the maximum to the effective voltage was 1·26. Calculating $R_{max.}$ by the formula

$$R_{max.} = \{(1 \cdot 26V - 0 \cdot 8)/x\}f$$

for values of x from 0·5 cm. to 5 cms., we find that the mean value of $R_{max.}$ is 37·9, which practically agrees with Steinmetz's result for 2 inch spheres.

(iii.) C. P. Steinmetz.   *1, 0·5, and 0·25 inch spheres.*

The analysis of Steinmetz's experiments with 1, 0·5, and 0·25 inch spheres are instructive, but for reasons explained in § 9 they do not give much assistance in obtaining $R_{max.}$ With the 1 inch spheres the mean of the values of $R_{max.}$ obtained up to pressures of 63·7 effective kilovolts is 41·3 kilovolts per centimetre. With the half-inch spheres the mean of the values for pressures up to 31·3 effective kilovolts is 43·1.

When the quarter-inch spheres were 28 cms. apart, the disruptive pressure was 112 effective kilovolts. If we calculate $R_{max.}$ for this pressure as if the spheres were in a vacuum, we find that it is more than six times the dielectric strength of air. In the experiment there must have been coronæ round each of the electrodes after the pressure was about 17 effective kilovolts.

In these experiments the frequency was 125 and E/V was equal to 1·42.

(iv.) E. Jona.   *Point and Plate.   Two spheres.*

E. Jona has published † a table giving the sparking-distances between a point and a plate and also between two equal spherical electrodes each of 2 cms. diameter, for pressures varying from 15 to 240 kilovolts. When the electrodes are far apart it is obvious that the distribution of the Faraday-tubes is considerably affected by the supporting rods connecting the electrodes with the transformer terminals. An important result proved by these experiments is that for all distances greater than 23 cms. the sparking-voltages,

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* Turner and Hobart, ' The Insulation of Electric Machines,' p. 35.

† E. Jona, *Atti dell' Associazione Elettrotechnica Italiana,* vol. vi. p. 3, ' Distanze Esplosive nell' Aria, negli olii ed altri Liquidi Isolanti.''

with the electrodes used by Jona, were the same in the
two cases. For instance, when the maximum value of the
applied P.D. was 240 kilovolts ($\smile = 42$), the sparking-
distance was 47 cms., whether the point and the plate or the
two spherical electrodes were used. Faraday anticipated this
result in his 'Experimental Researches,' § 1499 :—" But
as has long been recognized, the small body is only a
blunt end, and, electrically speaking, a point only a small
ball; so that when a point or blunt end is throwing out its
brushes into the air, it is acting exactly as the small balls
have acted in the experiments already described, and by
virtue of the same properties and relations."

(v.) C. P. Steinmetz. *0·795 cm. cylindrical electrodes* ($a = 0·3975$).

Professor Steinmetz in the paper referred to above also
describes tests on the sparking-distances between cylindrical
electrodes. In one case the electrodes were two copper rods
0·795 cm. in diameter and 71 cms. long. The rods were
slightly curved, so that the sparks ensued across the minimum
distance between them. The radius of curvature at this part
was 198 cms., and so, provided the rods are not further
apart than about 2 cms., we can neglect the curvature and
assume that the field is very similar to that between two
infinitely long parallel straight rods. We can therefore use
the formula (27) for f given in § 11.

TABLE X.

C. P. Steinmetz. Two cylindrical rods slightly curved.
Diameter of rods 0·795 cm. $\smile = 125$, E/V = 1·42.

No. of Experiment.	x.	x/a.	f (calc.).	V (obs.).	R_{max}.
1	0·165	0·42	1·07	4·78	38·9
2	0·203	0·51	1·08	5·70	38·8
3	0·433	1·15	1·18	10·85	38·9
4	0·881	2·21	1·35	18·55	39·1
5	1·115	2·81	1·43	23·7	42·2
6	1·194	3·01	1·46	22·9	38·8
7	1·435	3·61	1·55	23·0	34·3
8	1·753	4·40	1·66	30·5	40·3
9	2·134	5·38	1·80	34·3	40·4
10	2·362	5·96	1·88	33·1	36·8

The mean of the values of R_{max}. given in the last column
makes the dielectric strength of air 38·8 kilovolts per
centimetre.

(vi.) C. P. Steinmetz. 1·11 *inch cylindrical electrodes.*

Experiments were also made with large cylinders 1·11 inch in diameter and 20 inches long. Up to a distance apart of about one-third of an inch we may assume that our formula applies approximately.

We have neglected therefore the experimental results for greater distances. The mean of the values of R_{max}. deduced from the first five experiments is 34 kilovolts per centimetre. We do not attach so much importance to this result as to the preceding as our formula does not apply so accurately.

14. *Table of the Numbers obtained for the Dielectric Strength of Air from the Direct Pressure Experiments.*

TABLE XI.—Direct Pressures.

	Nature of Electrodes.	Authority.	R_{max}.
Table VII...	5 cm. spheres.	J. Algermissen.	37·0
Table VI. ...	5 cm. spheres.	A. Heydweiller.	37·5
§ 13, I., i. ...	Slightly convex surfaces.	Lord Kelvin.	40·8
§ 13, I., v....	1 cm. spheres.	É. Hospitalier.	42·2
§ 13, I., iv...	1 ,, ,,	J. Joubert.	42·8
,, ,,	2 ,, ,,	G. Carey Foster.	42·9

Of the above tests the first three seem to be the most accurate. The mean of the results obtained from these three tests makes the dielectric strength of air 38·4 kilovolts per centimetre.

15. *Table of the Numbers obtained for the Dielectric Strength of Air from the Alternating Pressure Experiments.*

TABLE XII.—Alternating Pressures.

	Nature of Electrodes.	Authority.	R_{max}.
§ 13, II., vi.	1·11 inch cylinders.	C. P. Steinmetz.	34
§ 13, II., ii.	2 cm. spheres.	Comp. de l'Ind. Élect.	37·9
Table IX.	2 inch spheres.	C. P. Steinmetz.	37·8
Table X.	0·313 inch cylinder.	,,	38·8
§ 13, II., iii.	1 inch spheres.	,,	41·3

An examination of the experimental results obtained and the methods of calculating R_{max} from them shows that the second, third, and fourth of the above tests are the only really satisfactory ones. The mean of the results obtained in these three tests is 38·2 kilovolts per centimetre, and this agrees closely with the number we obtained from the direct pressure experiments.

16. *Conclusion.*

We conclude therefore that the dielectric strength of the air at ordinary atmospheric pressures lies between 38 and 39 kilovolts per centimetre, which is about 30 per cent. greater than the value ordinarily given. **J. J. Thomson** * gives the value as approximately 30 kilovolts per centimetre and M. O'Gorman † as 27 kilovolts per centimetre.

The confidence of electricians who are responsible for the working of high-pressure net-works for the distribution of electric power ‡ on the working of their spark-gap safety-valves at the moment the pressure attains a definite value, and the extensive use they make of micrometer spark-gaps § for measuring high voltages, prove that under ordinary working conditions they find that the dielectric strength of air is approximately constant. In ordinary work we may take its value as 38 kilovolts per centimetre.

APPENDIX I.

The Disruptive Voltages for Large Spherical Electrodes.

Table XIII. gives the disruptive pressures in kilovolts between equal spherical electrodes when their radii are 1, 10, 100, and 1000 cms. respectively. The dielectric strength of air has been taken as 38 kilovolts per centimetre, and V and V′ are calculated by the formulæ

$$V = 0.8 + R_{max}.(x/f) \text{ and } V' = V/\sqrt{2},$$

respectively. V′ therefore gives the effective value of the disruptive voltage when the pressure is alternating and sine-shaped.

* J. J. Thomson, ' Electricity and Magnetism,' p. 59 (1904).

† M. O'Gorman, "Insulation on Cables," Journal of the Inst. of Elect. Eng. vol. xxx. p. 666 (1901).

‡ Dusaugey, *Soc. Int. Elect.*, Bull. 5, pp. 109–132. "Méthode de Protection contre les Surtensions actuellement employée dans les Réseaux de Transport d'Energie," Feb. 1905.

§ P. H. Thomas, Amer. Inst. Electr. Engin. Proc. xxiv. pp. 705–742. "An Experimental Study of the Rise of Potential on Commercial Transmission Lines due to Static Disturbances caused by Switching, Grounding, etc.", July, 1905.

TABLE XIII.

Calculated Values of the Disruptive Voltages between large Spherical Electrodes.

V = kilovolts (direct pressure). V' = effective kilovolts (alternating pressure) when $V' = V/\sqrt{2}$.

x in cms.	2 cm. spheres.		20 cm. spheres.		200 cm. spheres.		2000 cm. spheres.	
	V.	V'.	V.	V'.	V.	V'.	V.	V'.
0·1	4·5	3·2	4·6	3·25	4·6	3·25	4·6	3·25
0·5	17·0	12·0	19·4	13·9	19·8	14·0	19·8	14·0
1·0	28·8	20·3	37·7	26·7	38·8	27·4	38·8	27·4
5·0	61·1	43·3	163	115	187	132	191	135
10·0			280	198	370	262	381	269
50·0			604	427	1625	1150	1860	1320
100·0					2795	1980	3690	2610
500·0					6030	4270	16200	11500
1000·0							28000	20000
5000·0							60000	42500

If the electrodes were infinite planes, the direct pressure required to produce the disruptive discharge when they were 50 metres apart would be 190 million volts.

With spherical electrodes of 10 metre or less radius, about 60 million volts would be sufficient to spark over the same distance. We suppose of course that the P.D.'s are established sufficiently slowly to allow the Faraday-tubes to attain their positions of statical equilibrium approximately before the discharge occurs.

APPENDIX II.

The Capacity Currents to the Electrodes.

The formulæ (9) and (10) enable us to find at once the analytical expressions for the electrostatic coefficients of two equal spheres. If Q, Q' denote the charges and V_1 and V_2 the potentials of the electrodes, we have [*]

$$Q = K_{1\cdot1} V_1 + K_{1\cdot2} V_2$$

and

$$Q' = K_{2\cdot2} V_2 + K_{2\cdot1} V_1.$$

In our case $K_{1\cdot1} = K_{2\cdot2}$. By making $V_2 = 0$, we get by (9)

$$K_{1\cdot1} = a\frac{1-q^2}{q} \sum_1^\infty \frac{q^{2n-1}}{1-q^{4n-2}},$$

and by (10)

$$K_{1\cdot2} = -a\frac{1-q^2}{q} \sum_1^\infty \frac{q^{2n}}{1-q^{4n}}.$$

[*] Russell, 'Alternating Currents,' vol. i. p. 89 *et seq.*

Denoting Lambert's series $\overset{\infty}{\underset{1}{\Sigma}} \dfrac{q^n}{1-q^n}$ by $F(q)$ we get

$$K_{1\cdot 1} = a \frac{1-q^2}{q} \{F(q) - 2F(q^2) + F(q^4)\}$$

and

$$-K_{1\cdot 2} = a \frac{1-q^2}{q} \{F(q^2) - F(q^4)\}.$$

By a known transformation due to Clausen [*] we have

$$F(q) = \overset{\infty}{\underset{1}{\Sigma}} q^{n^2} \frac{1+q^n}{1-q^n} ;$$

and this series can be very readily computed.

For instance, when x is $0\cdot 5$ we find by (1) that q is also $0\cdot 5$, and

$$F(0\cdot 5) = 1\cdot 6067, \quad F(0\cdot 25) = 0\cdot 4210, \quad \text{and} \quad F(0\cdot 0625) = 0\cdot 0709.$$

Hence we find that

$$K_{1\cdot 1} = 1\cdot 2534a \quad \text{and} \quad K_{1\cdot 2} = -0\cdot 5252a.$$

The capacity between the two spheres [†] is

$$(K_{1\cdot 1} - K_{1\cdot 2})/2 = 0\cdot 8893a$$

and the capacity [‡] for equal potentials is

$$2(K_{1\cdot 1} + K_{1\cdot 2}) = 1\cdot 4565a.$$

To reduce these values to microfarads we divide by 900,000, a being measured in centimetres.

When the potentials follow the harmonic law we have

$$A_1 = (K_{1\cdot 1} V_1 + K_{1\cdot 2} V_2)\omega$$

and

$$A_2 = (K_{1\cdot 1} V_2 + K_{1\cdot 2} V_1)\omega,$$

where A_1 and A_2 are the effective values of the capacity currents flowing to the two spheres respectively; V_1, V_2 the effective values of their potentials, and $\omega/2\pi$ the frequency of the alternating pressures.

For instance, suppose that we have two 20 cm. spherical electrodes 5 cms. apart, and suppose that the effective value of the P.D. between them is 90 kilovolts. Then, if ω be 1000,

[*] Crelle's *Journal*, vol. iii. p. 95, quoted in Jacobi's *Fundamenta Nova*, pp. 187–188.

[†] Russell, 'Alternating Currents,' vol. i. p. 92.

[‡] Russell, 'Alternating Currents,' vol. i. p. 393.

so that the frequency is nearly 160, and the potentials of the electrodes be equal and opposite at every instant, we have

$$A = \omega KV = 1000 \times 0{\cdot}889 \times 10 \times 90 \times 10^3/(9 \times 10^5 \times 10^6)$$
$$= 0{\cdot}000889 \text{ ampere.}$$

If the potentials of the electrodes be not equal and opposite at every instant, the difference between the effective values of the capacity currents to the electrodes equals the effective value of the current in the earth connexion. Our formulæ are not applicable when the electrodes are surrounded with coronæ. In this case the capacity between them is considerably increased.

The author is indebted to Mr. Arthur Berry for suggesting the use of Clausen's theorem as an aid in calculating the capacity coefficients of two equal spheres.

POSTSCRIPT.

I have received from Principal G. Carey Foster the results of experiments made in his laboratory in 1876 on the sparking distances between 2·6 cm. brass knobs. As these results are of considerable interest I have obtained his permission to publish them.

G. Carey Foster.—2·6 cm. spheres $(a = 1{\cdot}3)$.

x.	x/a.	f.	V (obs.).	R_{max}.
0·05	0·0385	1·013	3·09	46·4
0·1	0·0769	1·026	5·04	43·5
0·2	0·1538	1·051	8·43	40·0
0 3	0·2307	1·077	11·46	38·3
0·4	0·3076	1·105	14·61	38·1
0·5	0·3846	1·131	17·49	37·7
0·6	0·4614	1·159	20·43	37·9
0·7	0·5383	1·185	23·37	38·2
0·8	0·6152	1·213	26·25	38·5
0·9	0·6921	1·242	29·13	39·0

A 'home-made' absolute electrometer was used to measure the voltage. The attracted disk was hung from an ordinary balance and the attraction weighed directly. Up to 30 kilovolts it gave trustworthy readings.

Neglecting the first value of R_{max} as the formula given in the paper is only roughly applicable when the distances are less than one millimetre, we find that the mean value of R_{max} is 39. If we neglect the first two readings the mean

value of $R_{max.}$ is 38·5. Both of these results agree very closely with the final conclusions at which we arrived. It will be seen that the experimental results obtained during the last thirty years on the sparking distances in air at ordinary pressures, when no coronæ are formed, could have been predicted with considerable accuracy from the above results.

Principal Carey Foster also suggested the formula

$$V = 2·13 + 30·6\, x,$$

for the sparking potentials between 2·6 cm. knobs. It is interesting to notice that Baille and many other experimenters subsequently suggested linear formulæ for the relation between V and x.

In reply to a question by Professor Poynting, I have worked out the values of $R_{max.}$ for the case of Heydweiller and Algermissen's tests on the assumption that the ordinary electrostatic equations hold, without modification, at the instant of breaking down. In this case we have

$$R'_{max.} = (V/x)f,$$

where f can be found by formula (20) given above. The values of $R_{max.}$ are those found in Tables VI. and VII.

Heydweiller's Test.

	Maximum value.	Minimum value.	Mean value.
$R_{max.}$	37·5+0·7	37·5−0·6	37·5
$R'_{max.}$	38·35+0·85	38·35−0·55	38·35

Algermissen's Test.

	Maximum value.	Minimum value.	Mean value.
$R_{max.}$	37+0·4	37−0·5	37·0
$R'_{max.}$	37·4+0·3	37·4−0·3	37·4

The last result is so remarkable that I give the complete table.

Algermissen's Test.

x.	R'_{max}.	x.	R'_{max}.	x.	R'_{max}.	x.	R'_{max}.
1·5	37·2	2·2	37·5	2·9	37·3	3·6	37·5
1·6	37·1	2·3	37·5	3·0	37·4	3·7	37·5
1·7	37·1	2·4	37·5	3·1	37·3	3·8	37·6
1·8	37·2	2·5	37·5	3·2	37·3	3·9	37·6
1·9	37·2	2·6	37·4	3·3	37·3	4·0	37·7
2·0	37·3	2·7	37·4	3·4	37·4	4·1	37·7
2·1	37·5	2·8	37·4	3·5	37·4	4·2	37·5

Hence, Algermissen's experimental results give us the ratios of all the values of f, from x equal to 1·5 a to x equal to 4·2 a, with a maximum inaccuracy of less than 1·6 per cent. To fully appreciate this result it is necessary to try and sum the series (15) for any two values of x lying between the given limits.

For sparking distances greater than half a centimetre (one fifth of an inch), therefore, when no coronæ, and consequently no brush discharges, are formed, the error made in assuming that the boundaries of the electrodes form the equipotential surfaces is negligibly small. The disruptive discharge ensues as soon as the maximum value of the electric intensity attains a definite value which is the measure of the dielectric strength of the air between the electrodes under the given atmospheric conditions.

XXI. *A Preliminary Report on the Effect of Combined Stresses on the Elastic Properties of Steel.* By EDWARD L. HANCOCK, *Instructor in Applied Mechanics, Purdue University, Ind.*[*]

[Plate VI.]

FOR some time engineers and others have been interested in the effect of combined stresses on the materials of construction. Lord Kelvin, in the preparation of his article on Elasticity for the *Encyclopedia Britannica*, had a series of tests made on piano-wire. The wire having sufficient weight to hold it straight while suspended, was subjected to a torque at the bottom; when an additional weight was added, it was found that the elastic limit of the wire in torsion was lowered. From these tests it seemed possible that the same thing would be true in the case of compression-torsion (loaded columns), although no tests of

* Communicated by the Author.

this kind were made. Mr. J. J. Guest, in England (Phil. Mag. [5] vol. l. p. 69; Proc. Physical Soc. of London, Sept. 1900), carried out a series of tests to obtain the effect of combined stresses on ductile materials, the materials used in the tests being wrought iron, mild steel, copper, and brass. The results of these tests also lead to the same conclusion, that the elastic properties are lowered when the materials are subjected to combined stresses.

The slight knowledge on this subject and the entire absence of any information available for engineers, have inspired the writer to carry out a series of investigations that should give some information immediately available for practical work. The series contemplated includes the following :—

(*a*) Tests of steel and iron solid rounds and hollow tubes in tension while under torsion.

(*b*) Tests of steel and iron solid rounds and hollow tubes in torsion while under tension.

(*c*) Tests of steel and iron solid rounds and hollow tubes with increasing tension and torsion.

(*d*) Tests of steel and iron solid rounds and hollow tubes in compression while under torsion.

Thus far only a part of series (*a*), on solid steel rounds, has been carried out, and the results given in this report are taken from these tests.

Apparatus for Testing.—The difficulty of making tests of materials under combined stresses lies principally in the fact that no machines are available for such tests. The apparatus used in the tests under consideration is shown in fig. 1 (Pl. VI.). It consists of two specially constructed heads fitted to an ordinary 100,000 lb. tension testing-machine. Each of these heads consists of a flat cast-iron base fitting into the slot in the head of the testing-machine provided for the insertion of the wedges, as ordinarily used. The outer side of this casting is finished to provide for three concentric rows of hardened steel balls, which rest upon a steel plate and are covered by another steel plate. The outer side of this latter plate is a spherical cup. This receives the large casting, or chuck, carrying the wedges and arms. The spherical bearing allows the specimen to " line up " properly. The construction of the heads is seen in fig. 2. Each head is provided with two arms 5 feet long, the lower arms having steel rollers that bear against stationary knife-edges, while the upper arms are provided with knife-edges and stirrups to which the cords carrying the loads are attached.

In order that the loads in torsion might be applied, two

frames were constructed to support the stationary knife-edges for the lower arms, and also to carry bicycle-wheels over which passed the cords that were attached to the upper arms. These wheels were so located that the cord was perpendicular to the arm and on the same horizontal. To the other end of the cord was suspended a small bucket as a receptacle for the sand used in loading. The buckets were counterbalanced by a weight attached to a cord running back over the wheel.

Method of Holding the Specimen.—The large casting of each head carries three lugs provided with slots for the insertion of the wedges. These lugs are so arranged that when a torsion load is applied, the wedges tend to *grip* the specimen, due to the swinging out of the lugs (see fig. 2). This method of holding allows the use of specimens of different diameter, and admits of easy manipulation.

Application of Load in Torsion.—The desired load in torsion was given by allowing a known quantity of fine dry sand to run uniformly into the buckets attached to the cords. This load was transmitted to the arms and a part of it to the specimen (a part was taken up by the friction of the bearings). The use of sand gave a uniform application of load, and was very satisfactory.

Measurement of Twist and Elongation.—The amount of twist was measured by means of an ordinary troptometer on a 10-in. gage length, and the elongations were measured by a Yale-Riehle extensometer on an 8-in. gage length. The extensometer was placed symmetrically between the arms of the troptometer. This means of measuring deformations was entirely satisfactory for the series of tests made, but will have to be changed somewhat for the series (*c*).

General Method of Testing.—The tests of series (*a*) already made consist of two sub-series, (E) and (C), the former on 3 per cent. nickel-steel and the latter on carbon-steel, both being supplied through the courtesy of the Carnegie Steel Co. The metals used had the following composition :—

	Phos.	Mn.	Silicon.	Nickel.	Carbon.
Nickel-Steel....	0·018	0·65	0·022	3·02	0·25 per cent.
Carbon-Steel ...	0·030	0·55	0·024	0·00	0·24 „

In each series three pieces were used. These were about 3 feet long and 0·85 in. in diameter, and turned down for a length of 11 in. at the centre to a diameter of 0·50 inch. In each case a tension-load of 4000 lbs. per sq. inch was applied to hold the specimen in place and cause the wedges to grip properly. One specimen in each series (E_A) and (C_A)

was tested in torsion to one-third the elastic limit by application of sand, troptometer-readings being taken. The specimen while held with this load in torsion was tested in tension to the elastic limit. A second specimen in each series (E_B) and (C_B) was tested in torsion in the same way to two-thirds the elastic limit, and while under this load in torsion was tested in tension to the elastic limit. The third specimen of each series, (E_C) and (C_C), was tested in torsion to the elastic limit, and while under this load was tested in tension to the elastic limit.

Friction of the Heads and Rollers.—The friction of the rollers attached to the lower arms and bearing on the stationary knife-edges was measured and found to be negligible. That is, the amount of upward pressure due to this friction (constant for these tests) was less than could be read on the beam of the 100,000-lb. Olsen testing-machine.

The effective moment due to the friction of the ball-bearings and bicycle-wheels was determined for the tension-load of 800 lbs. (corresponding to a load of 4000 lbs. per sq. in.), this being the constant load in tension used while the torsion tests were being made. The tests were made by placing the heads on the platform of the machine, back to back, and compressing them with a load of 800 lbs. and determining the weight of sand necessary to move the arms. The average of a number of tests gave 16·42 inch-pounds as the effective twisting-moment due to the friction of the heads and bicycle-wheels. This moment was deduced in making calculations.

It is seen from the above that the effect of friction in the tests already made has been accounted for in a definite way, making the tests satisfactory. The plan of work includes the determination of a *friction curve* for the apparatus. This curve will show the twisting-moment lost due to friction for any safe load in tension.

Results of Tests.—Fig. 3 gives a stress-strain diagram of nickel-steel in tension. The curve M is a curve from the average of two simple tension tests of the material, and shows the elastic limit to be 56,000 lbs. per sq. in. The three curves E represent the behaviour of the material in tension while held in torsion : E_A held in torsion at one-third the elastic limit, E_B held in torsion at two-thirds the elastic limit, and E_C held in torsion at the elastic limit. The curves show a lowering of the elastic limit in tension ; a lowering of about 7 per cent. in the case of E_A, about 21 per cent. in the case of E_B, and about 63 per cent. in the case of E_C. When the series of tests contemplated has been completed, it is

hoped that the law of the lowering of the elastic limit may be determined. The modulus of elasticity in tension has also been lowered.

Fig. 4 shows the stress-strain diagram of carbon-steel in tension. The curve C represents the average of two

Fig. 3.

NICKEL-STEEL.—Tension part only of Torsion-Tension Tests.

Series E.
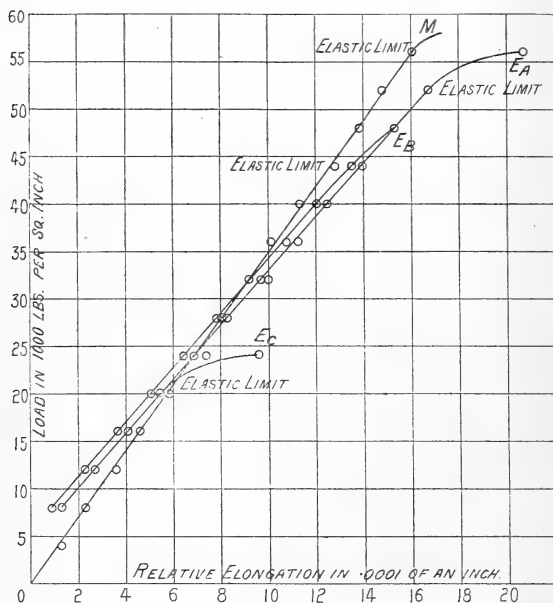
simple tension tests of the material, showing an elastic limit of 34,000 lbs. per sq. in. The curve C_A shows the result of a tension test made while the specimen was held at one-third its elastic limit in torsion; C_B while held in torsion at two-thirds its elastic limit; and C_C while held in torsion at its elastic limit. These curves also show a lowering of the elastic limit: a lowering of about 6 per cent., 30 per cent., and 54 per cent. for C_A, C_B, and C_C respectively. It will be noticed that this is not exactly the same *rate* of lowering as that given by fig. 3. The modulus of elasticity in this case does not seem changed in the case of C_A, but is lowered considerably in the case of C_B and C_C.

In all cases the increase in the troptometer readings while

the loads in tension were being applied—and this notwith-
standing the increased friction of the heads—showed a decided
weakening of the material in torsion due to the combined
stresses. However, sufficient data have not been taken to give
a satisfactory report on this point.

Fig. 4.

CARBON-STEEL.—Tension part only of Torsion-Tension Tests.

Series C.
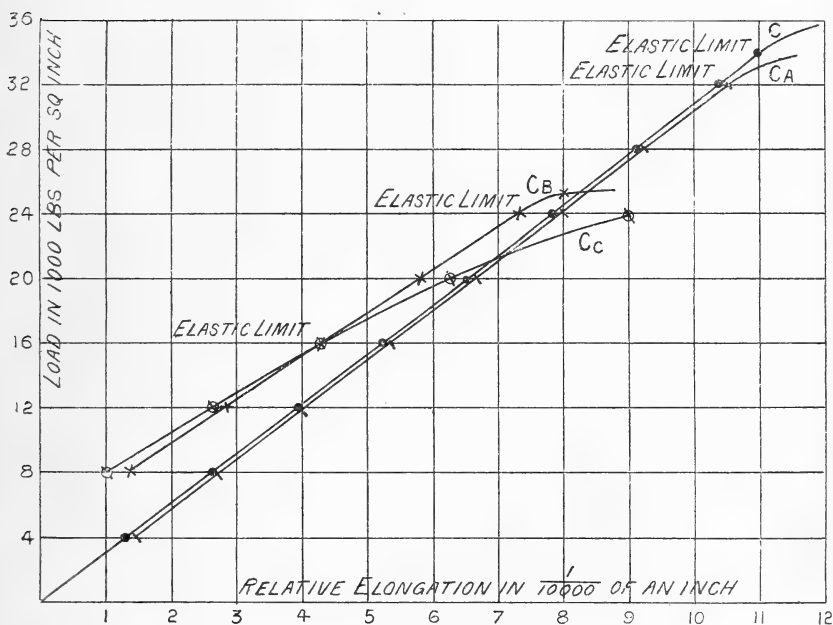
Fig. 5 shows two torsion curves A and B, the former
representing an average of two tests of the material made
while the specimen was under a tensile load of 4000 lbs.
per sq. in., and the latter an average of two simple torsion
tests of the same material. The curves show the expected
lower elastic limit in the case of the specimens subjected to
combined stresses. In the case of this plate the ordinates
represent the shearing stress on the outer fibre, and the
abscissæ the angle of twist measured at the centre of the
specimen.

When the tests have been more completely worked out, it
is planned to compare the values obtained from the tests under

combined stresses with those obtained by computing the formulæ :

$$q_s = \tfrac{1}{2}(p^2 + 4p_s^2)^{\frac{1}{2}},$$

$$q = \tfrac{1}{2}\left[p + (p^2 + 4p_s^2)^{\frac{1}{2}}\right],$$

where p is the load in pounds per square inch in simple tension, p_s the shear on the outer fibre in simple torsion, and q and q_s the normal and shearing stresses on an internal plane of the material.

Fig. 5.

NICKEL-STEEL.—Torsion part only of Torsion-Tension Tests, "A," and Simple Torsion Test, " B."
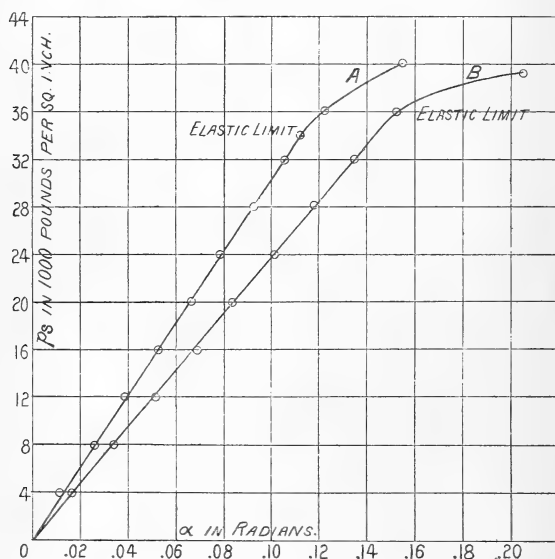
In conclusion, the writer wishes to acknowledge the efficient work of several senior students who made it possible to carry out the tests that have been made : Mr. Carlos Robles Gil, Mr. C. E. Shearer, Mr. F. O. Blair, Mr. W. R. Wheeler, Mr. J. H. Lambert, and Mr. J. W. Krull ; and to acknowledge particularly the helpful suggestions of Professor W. K. Hatt, in charge of the Laboratory for Testing Materials.

June, 1905.

XXII. *On the Production of Vibrations by Forces of Relatively Long Duration, with Application to the Theory of Collisions. By* Lord Rayleigh, *O.M., P.R.S.*[*]

THE problem of the collision of elastic solid bodies has been treated theoretically in two distinct cases. The first is that of the longitudinal impact of elongated bars, which for simplicity may be supposed to be of the same material and thickness. Saint-Venant[†] showed that, except when the lengths are equal, a considerable fraction of the original energy takes the form of vibrations in the longer bar, so that the translational velocities after impact are less than those calculated by Newton for bodies which he called perfectly elastic. It will be understood that in Saint-Venant's theory the *material* is regarded as perfectly elastic, and that the total mechanical energy is conserved. The duration of the impact is equal to the period of the slowest vibration of the longer bar.

The experiments of Voigt[‡], undertaken to test this theory, have led to the conclusion that it is inapplicable when the bars differ markedly in length. The observations agree much more nearly with the Newtonian law, in which all the energy remains translational. Further, Hamburger[§] found that the duration of impact was much greater than according to theory, though it diminished somewhat as the relative velocity increased. I do not think that these discrepancies need cause surprise when we bear in mind that the theory presupposes a condition of affairs impossible to realise in practice. Thus it is assumed that the pressure during collision is uniform over the whole of the contiguous faces. But, however accurately the faces may be prepared, the pressure, at any rate in its earlier and later stages, must certainly be local and be connected with the approach by a law altogether different from that assumed in the calculation. Since the region of first contact would yield with relative ease, we may expect a prolongation of the impact, and in consequence, as we shall see more in detail presently, a diminished development of vibrations. Possibly with higher velocities and longer bars a nearer approach might be attained to the theoretical conditions.

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[*] Communicated by the Author.

[†] Liouville's Journal, xii. (1867). See also Love's 'Treatise on the Theory of Elasticity,' vol. ii. p. 137 (1893).

[‡] Wied. *Ann.* xix. (1883).

[§] Wied. *Ann.* xxviii. (1886).

But it is with Hertz's * solution, under certain conditions, of the problem of impinging curved bodies with which I am now more concerned. He commences with the purely statical problem of contact under pressure. Thus if two equal spheres of similar material be pressed together with a given force P_0, the surfaces of contact are moulded to a plane; and it is required to find the radius of the circle of contact, and more especially the distance (α) through which the centres (or other points remote from the place of contact) approach one another. It appears that the relation between P_0 and α is simply

$$P_0 = k_2 \alpha^{\frac{3}{2}}, \quad \ldots \ldots \ldots \quad (1)$$

where k_2 depends only on the forms and materials of the two bodies. In the particular case above mentioned,

$$k_2 = \frac{\sqrt{2r} \cdot E}{3(1-\sigma^2)}, \quad \ldots \ldots \ldots \quad (2)$$

where r is the radius of the spheres, E Young's modulus, and σ Poisson's ratio.

In applying this result to impacts Hertz proceeds :—" It follows both from existing observations and from the results of the following considerations, that the time of impact, *i. e.* the time during which the impinging bodies remain in contact, is very small in absolute value; yet it is very large compared with the time taken by waves of elastic deformation in the bodies in question to traverse distances of the order of magnitude of that part of their surfaces which is common to the two bodies when in closest contact, and which we shall call the surface of impact. It follows that the elastic state of the two bodies near the point of impact during the whole duration of impact is very nearly the same as the state of equilibrium which would be produced by the total pressure subsisting at any instant between the two bodies, supposing it to act for a long time. If, then, we determine the pressure between the two bodies by means of the relation which we previously found to hold between this pressure and the distance of approach along the common normal of two bodies at rest, and also throughout the volume of each body make use of the equations of motion of elastic solids, we can trace the progress of the phenomenon very exactly. We cannot in this way expect to obtain general laws; but we may obtain a number of such if we make the

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* *Journal für reine und angewandte Mathematik,* xcii. p. 156 (1881); Hertz's Miscellaneous Papers, English edition, p. 146. A good account is given by Love, *loc. cit.*

further assumption that the time of impact is also large compared with the time taken by elastic waves to traverse the impinging bodies from end to end. When this condition is fulfilled, all parts of the impinging bodies, except those infinitely close to the point of impact, will move as parts of rigid bodies; we shall show from our results that the condition in question may be realised in the case of actual bodies." The above-mentioned condition may in fact always be satisfied by taking the relative velocity of impact to be sufficiently small.

The solution of the problem, thus limited, is now easily found. For the case of two spheres the relative acceleration $\ddot{\alpha}$ is connected with P_0 by the equation

$$P_0 = -\ddot{\alpha}/k_1, \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where $$k_1 = (m_1 + m_2)/m_1 m_2,$$

and m_1, m_2 are the masses of the spheres. Eliminating P_0 between (1) and (3), we get

$$\ddot{\alpha} + k_1 k_2 \alpha^{\frac{3}{2}} = 0, \quad \ldots \quad \ldots \quad (4)$$

and on integration as the equation of energy

$$\dot{\alpha}^2 - \dot{\alpha}_0{}^2 + \tfrac{4}{5} k_1 k_2 \alpha^{\frac{5}{2}} = 0, \quad \ldots \quad (5)$$

α_0 being the relative velocity before impact.

"The greatest compression takes place when $\dot{\alpha}$ vanishes, and if α_1 be the value of α at this instant

$$\alpha_1 = \left[\frac{5 \, \dot{\alpha}_0{}^2}{4 \, k_1 \, k_2} \right]^{\frac{2}{5}} \quad \ldots \quad \ldots \quad (6)$$

Before the instant of greatest compression the quantity α increases from zero to a maximum α_1, and $\dot{\alpha}$ diminishes from a maximum $\dot{\alpha}_0$ to zero. After the instant of greatest compression α diminishes from α_1 to zero and $\dot{\alpha}$ increases to $\dot{\alpha}_0$. The bodies then separate, and the velocity with which they rebound is equal to that with which they approach. This result is in accord with Newton's Theory. It might have been predicted from the character of the fundamental assumptions."

"The duration of the impact is

$$2 \int_0^{\alpha_1} \frac{d\alpha}{\sqrt{(\dot{\alpha}_0{}^2 - \tfrac{4}{5} k_1 k_2 \alpha^{\frac{5}{2}})}} = \frac{2\alpha_1}{\dot{\alpha}_0} \int_0^1 \frac{dx}{\sqrt{(1 - x^{\frac{5}{2}})}}$$

$$= \frac{4}{5} \frac{\alpha_1}{\dot{\alpha}_0} \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{9}{10}\right)} = \frac{\alpha_1}{\dot{\alpha}_0}(2 \cdot 9432), \quad \ldots \quad (7)$$

where α_1 is given by (6).

The duration of impact, therefore, varies inversely as the fifth root of the initial relative velocity " [*].

So long as the condition is satisfied that the duration of the impact is very long in comparison with the free periods, vibrations will not be excited in a sensible degree, the energy remains translational, and Newton's laws find application. It would be of great interest if we could enfranchise ourselves from this restriction.   It is hardly to be expected that a complete solution of the problem will prove feasible, but I have thought that it would be worth while to inquire into the circumstances of the *first* appearance of sensible vibrations.   We should then be in a better position to appreciate at least the range over which Newton's laws may be expected to hold.

In the case of spheres the vibrations to be considered are those of the "second class" investigated by Lamb [†].   They involve spherical harmonic functions of the various orders, limited in the present case to the *zonal* kind.   But for each order there are an infinite number of modes corresponding to greater or less degrees of subdivision along the radius. The first appearance of vibrations will be confined to those of longest period, of which the most important is of the second order.   In this mode the sphere vibrates symmetrically with respect both to a polar axis and to the equatorial plane, the greatest compression along the axis synchronizing with the greatest expansion at the equator.   In what follows we shall denote by ϕ_1, ϕ_2, &c. the radial displacements at the pole (point of contact) corresponding to the several modes, the first ϕ_1 being appropriated to that mode in which the sphere moves as a rigid body (spherical harmonic of order 1), the next ϕ_2 to the mode of the second order above described which gives the principal vibration.

Since there is no force of restitution corresponding to ϕ_1, the equation for it takes the simple form

$$a_1 \ddot{\phi_1} = \mathrm{P}_0, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

P_0 being as before the total pressure between the spheres at any time, and a_1 a coefficient of inertia—in this case the simple mass of a sphere.   On the other hand, the equations for ϕ_2 &c. are of the form

$$a_2 \ddot{\phi_2} + c_2 \phi_2 = \mathrm{P}_0 \quad . \quad . \quad . \quad . \quad (9)$$

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* Love, *loc. cit.* p. 154.
† Proc. Lond. Math. Soc. vol. xiii. p. 189 (1882).

c_2 &c. being coefficients of stability to be treated as large. This form applies to all the *lower* modes, for which the force of collision operating at any moment may be treated as a whole. By equation (1) of Hertz's theory $P_0 = k_2 \alpha^{\frac{3}{2}}$, but now that we are admitting the possibility of vibrations α must be reckoned no longer from the centre, but from a point which is at once near the surface and yet distant from it by an amount large in comparison with the diameter of the circle of contact. We may write

$$\tfrac{1}{2}\alpha = \phi_1 + \phi_2 + \dots \quad . \quad . \quad . \quad . \quad (10)$$

inclusion being made of the coordinates of the lower modes only. The sum of *all* the coordinates would be zero, since (in the case of equal spheres) the pole does not move. Thus

$$P_0 = k_2\,(2\phi_1 + 2\phi_2 + \dots)^{\frac{3}{2}}. \quad . \quad . \quad . \quad (11)$$

In the first approximation c_2 &c. are regarded as infinite, so that ϕ_2 &c. vanish. P_0 reduces to $k_2\,(2\phi_1)^{\frac{3}{2}}$, and so from (8)

$$a_1\,\ddot{\phi}_1 = k_2\,(2\phi_1)^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad (12)$$

the solution of which gives ϕ_1 as in Hertz's theory. If P_0 be regarded as a known function of the time, ϕ_2 is determined by (9); but it may be well at this stage to ascertain how far P_0 is modified in a second approximation. Retaining for brevity ϕ_2 only, we have approximately $\phi_2 = P_0/c_2$. Hence

$$a_1\,\ddot{\phi}_1 = k_2\,(2\phi_1)^{\frac{3}{2}} \left\{ 1 + \frac{3}{2}\frac{k_2}{c_2}\Big(2\phi_1\Big)^{\frac{1}{2}} \right\}, \quad . \quad . \quad (13)$$

and we infer that P_0 is changed by a term of the order c_2^{-1}.

We will now pass on to consider the general problem of a vibrator whose natural vibrations are very rapid in comparison with the force which operates. We write (9) in the form

$$\ddot{\phi} + n^2\phi = P_0/a_2 = \Phi, \quad . \quad . \quad . \quad (14)$$

where $n^2 = c_2/a_2$, and is to be treated as very great. If ϕ and $\dot{\phi}$ vanish when $t = 0$, the solution of (14) is *

$$\phi = \frac{1}{n}\int_0^t \sin n\,(t-t')\,\Phi_{t=t'}\,dt'. \quad . \quad . \quad (15)$$

* 'Theory of Sound,' § 66.

If the force operates only between $t = 0$ and $t = \tau$ and we require the value of ϕ at a time t greater than τ, that is is after the operation of the force has ceased, we may write

$$\phi = \frac{1}{n} \int_0^\tau \sin n\,(t - t')\, \Phi_{t=t'}\, dt'. \quad . \quad . \quad . \quad (16)$$

If τ be infinitely small, the force reduces to an impulse, and we get

$$\phi = n^{-1} \sin nt \cdot \int \Phi\, dt; \quad . \quad . \quad . \quad (17)$$

but it is the other extreme which concerns us at present.

In many cases, especially when $\Phi = 0$ at the limits, we may advantageously integrate (16) by parts.   Thus

$$\phi = n^{-2}\,\Phi_{t=\tau} \cos n\,(t - \tau) - n^{-2}\,\Phi_{t=0} \cos nt$$

$$-\frac{1}{n^2} \int_0^\tau \cos n\,(t - t')\, \frac{d\Phi}{dt'}\, dt'. \quad . \quad . \quad (18)$$

Again

$$-\frac{1}{n^2} \int_0^\tau \cos n\,(t - t')\, \frac{d\Phi}{dt'}\, dt' = \frac{1}{n^3}\,\frac{d\Phi_{t=\tau}}{dt} \sin n\,(t - \tau)$$

$$-\frac{1}{n^3}\,\frac{d\Phi_{t=0}}{dt} \sin nt - \frac{1}{n^3} \int_0^\tau \sin n\,(t - t')\, \frac{d^2\Phi}{dt'^2}\, dt', \quad (19)$$

and so on if required.   In this way we obtain a series proceeding by descending powers of n, and thus presumably advantageous when n is great.

As an example, let $\Phi = t$, so that $d^2\Phi / dt^2 = 0$.   The force rises from zero at $t = 0$ to a greatest value at $t = \tau$ and then suddenly drops to zero.   From (18), (19) we find

$$\phi = n^{-2}\tau \cos n\,(t - \tau) + n^{-3} \sin n\,(t - \tau) - n^{-3} \sin nt. \quad (20)$$

Again, take the parabolic law

$$\Phi = t\tau - t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

so that $\Phi = 0$ at both limits,

$$d\Phi / dt = \tau - 2t, \qquad d\Phi / dt^2 = -2.$$

From (18), (19)

$$\phi = -\tau n^{-3} \sin n\,(t - \tau) - \tau\, n^{-3} \sin nt$$

$$+ 2n^{-4} \cos n\,(t - \tau) - 2n^{-4} \cos nt$$

$$= -2n^{-3}\,\tau \cos \tfrac{1}{2}\, n\tau \cdot \sin n\,(t - \tfrac{1}{2}\,\tau)$$

$$+ 4\,n^{-4} \sin \tfrac{1}{2}\, n\tau \cdot \sin n\,(t - \tfrac{1}{2}\,\tau). \quad . \quad (22)$$

If Φ and its differential coefficients up to a high order are continuous within the range of integration and vanish at the limits, the leading term in the development of (16) is of high inverse power in n. An extreme case of this kind is considered by Mr. Jeans[*], who takes

$$\Phi = \frac{c}{\pi} \frac{c}{(c^2 + t^2)}.$$

In this case the solution involves the factor e^{-nc}, smaller when n is great than any inverse power of n. But the force is not here limited to a finite range of time.

The application of these results to the problem of the collision of equal elastic spheres is not quite so straightforward as had been expected. In (9),

$$\Phi = P_0 / a_2 = a_2^{-1} k_2 \alpha^{\frac{3}{2}}, \quad . \quad . \tag{23}$$

α denoting, as in (4), (5), (6), (7), the approach of the spheres. The terminal values of α and of $\alpha^{\frac{3}{2}}$ are zero. Again,

$$\frac{d}{dt}(\alpha^{\frac{3}{2}}) = \frac{3}{2} \alpha^{\frac{1}{2}} \frac{d\alpha}{dt}, \quad . \quad . \quad . \quad . \quad . \tag{24}$$

so that $d\Phi / dt$ vanishes at the limits of the range. But

$$\frac{d^2}{dt^2}(\alpha^{\frac{3}{2}}) = \frac{3}{4} \alpha^{-\frac{1}{2}} \left(\frac{d\alpha}{dt}\right)^2 + \frac{3}{2} \alpha^{\frac{1}{2}} \frac{d^2\alpha}{dt^2}$$

$$= {}_4{}^{\cdot} {}_0{}^2 \alpha^{-\frac{1}{2}} - \frac{21}{10} k_1 k_2 \alpha^2, \quad . \quad . \quad . \tag{25}$$

use being made of (4), (5); and the first part of this becomes *infinite* at the limits where $\alpha = 0$.

Equations (18), (19) give

$$\phi = -\frac{1}{n^3} \int_0^\tau \sin n (t-t') \frac{d^2\Phi}{dt'^2} dt', \quad . \quad . \quad . \tag{26}$$

and in this we have now to consider the two parts

$$\int \sin n (t-t') \alpha^{-\frac{1}{2}} dt' \quad \text{and} \quad \int \sin n (t-t') \alpha^2 dt'.$$

For the second we get on integration by parts, since α^2 vanishes at both limits,

$$-\frac{1}{n} \int \cos n (t-t') \frac{d\alpha^2}{dt'} dt'.$$

which is of order n^{-1}, or less.   In the first part the relation between dt' and $d\alpha$ is, as in (7),

$$dt' = \frac{d\alpha}{\sqrt{(\dot{\alpha}_0^2 - \frac{4}{5} k_1 k_2 \alpha^{\frac{5}{2}})}} \cdot \quad . \quad . \quad . \quad (27)$$

If we exclude the terminal parts of the range, the integral would be of order n^{-1}, or less, so that it is only the terminal parts that contribute to the leading term.   For the beginning we see from (27), or independently, that

$$\alpha = \dot{\alpha}_0 t, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

nearly, so that for this terminal region

$$\int \sin n\,(t-t')\,\alpha^{-\frac{1}{2}}\,dt'$$

$$= \frac{\dot{\alpha}_0^{-\frac{1}{2}}}{n^{\frac{3}{2}}} \left\{ \sin nt \int \frac{\cos nt'\,d\,(nt')}{(nt')^{\frac{1}{2}}} - \cos nt \int \frac{\sin nt'\,d\,(nt')}{(nt')^{\frac{1}{2}}} \right\}.$$

When we suppose n very great, the limits of integration may be identified with zero and infinity; and further by a known theorem

$$\int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \int_0^\infty \frac{\cos x\,dx}{\sqrt{x}} = \sqrt{(\tfrac{1}{2}\pi)}.$$

Thus, so far as it depends upon the early part of the collision,

$$\phi = \frac{3\sqrt{\pi}}{4} \frac{k_2\,\dot{\alpha}_0^{\frac{3}{2}}}{a_2\,n^{\frac{7}{2}}} \cos\,(nt + \tfrac{1}{4}\pi). \quad . \quad . \quad (29)$$

There will be a similar term due to the end of the collision, derivable from (29) by replacing nt with $n\,(t-\tau)$.

If, as I think must be the case, (29) gives the leading term in the expression for a vibration, the next question is as to the order of magnitude of the corresponding energy in comparison with the energy before collision, viz. $\frac{1}{2}$ Mass $\times \dot{\alpha}_0^2$, or $\frac{2}{3}\pi\rho r^3\,\dot{\alpha}_0^2$.

The maximum kinetic energy of the vibration is given by

$$\tfrac{1}{2} a_2\,\dot{\phi}^2{}_{\text{max.}} = \frac{9\pi}{32} \frac{k_2^2\,\dot{\alpha}_0^3}{a_2\,n^5},$$

and the ratio (R) of this to the energy before collision is

$$R = \frac{27}{64} \frac{k_2{}^2 \dot{\alpha}_0}{a_2 n^5 \rho r^3} = \frac{3}{32} \frac{E^2 \dot{\alpha}_0}{(1-\sigma^2)^2 a_2 n^5 \rho r^2}, \quad \cdot \quad (30)$$

if we introduce the value of k_2 from (2).

The precise value of a_2 would have to be calculated from Lamb's theory. It is easy to see that it is decidedly smaller than, but of the same order of magnitude as, the mass of the sphere, viz. $\frac{4}{3}\pi\rho r^3$.

The precise value of R would depend also upon σ, but for our purpose it will suffice to make $\sigma = \frac{1}{2}$. Thus we take

$$R = \frac{E^2}{5\rho^2} \frac{\dot{\alpha}_0}{n^5 r^5}. \quad \bullet \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (31)$$

According to Lamb's calculation [*] for the principal vibration of the second order in spherical harmonics ($\sigma = \frac{1}{2}$)

$$n = \sqrt{\left(\frac{E}{\rho}\right)} \cdot \frac{\cdot 85\,\pi}{\sqrt{3 \cdot r}}, \quad \cdot \quad \cdot \quad \cdot \quad (32)$$

so that approximately

$$R = \frac{1}{50} \frac{\dot{\alpha}_0}{\sqrt{(E/\rho)}}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (33)$$

In (33) $\sqrt{(E/\rho)}$ is the velocity of longitudinal vibrations along a bar of the material in question, and the comparison is between this velocity and the velocity of approach before collision. In steel the velocity of longitudinal vibrations is about 500,000 cms. per second, or about 16 times that of sound in air. It will be seen that in most cases of collision R is an exceedingly small ratio.

The general result of our calculation is to show that Hertz's theory of collisions has a wider application than might have been supposed, and that under ordinary conditions vibrations should not be generated in appreciable degree. So far as this conclusion holds, the energy of colliding spheres remains translational, and the velocities after impact are governed by Newton's laws, as deducible from the principles of energy and momentum.

Terling Place, Witham,
    Dec. 22, 1905.

[*] *Loc. cit.* p. 206.

XXIII. *Note to "Electrical Vibrations and the Constitution of the Atom."* By Lord RAYLEIGH [*].

I FIND that the remark (p. 121) that solutions such as (12), (15), (19), involving a factor e^{+r} "cannot apply to a problem where the disturbance is supposed to originate in the neighbourhood of the sphere" needs qualification. That the solutions *as they stand* cannot be so applied may be admitted; but Profs. Lamb[†] and Love[‡] have shown how by limiting them suitably they may be made to express the advance of waves into a region previously undisturbed. This is a matter of some importance, and I regret that I overlooked the memoirs just cited, in which clear explanations are given.

XXIV. *The Heating Effects produced by Röntgen Rays in different Metals, and their Relation to the Question of Change in the Atom.* By H. A. BUMSTEAD, *Ph.D.*, *Asst. Professor of Physics, Yale University* [§].

THE study of the phenomena of radioactivity during the past five or six years, and, in particular, the brilliant series of experiments and deductions which we owe to Rutherford, have left little room for doubt that a certain proportion of the atoms of radioactive elements are continually breaking up, and that the constant emission of energy by these bodies is a result of this atomic disintegration. This process in any given radioactive body appears to be going on at a fixed and definite rate which is characteristic of the particular substance studied and which is quite un-influenced by any external circumstances whatever. Although radioactive substances have been subjected to the greatest extremes of temperature available in the laboratory, and to great variations in other physical and chemical conditions, no certain results have been obtained (so far as the writer is aware) which point to any corresponding change in the rate of decay of the substance. In fact the process of atomic disintegration has appeared to be quite beyond human control [||].

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[*] Communicated by the Author.
[†] Proc. Lond. Math. Soc. vol. xxxii. p. 208 (1900).
[‡] Proc. Lond. Math. Soc. vol. ii. p. 88 (1904)
[§] Communicated by Prof. J. J. Thomson, F.R.S.
[||] The apparent change (due to high temperature) in the rate of decay of radium-excited activity, discovered by Curie and Danne, has been shown to be due to the fact that the successive products of radium volatilize at different temperatures. *Cf.* Bronson, Am. Jour. Sci. July 1905.

On the other hand, all substances when illuminated with Röntgen rays or with Becquerel rays of the γ type give out a complex secondary radiation, part of which at least is wholly different in character from the primary radiation. For example, the secondary radiation due to Röntgen rays consists in part of negatively charged corpuscles or electrons which are not present in the primary rays. This suggests that there may be some breaking up of the atoms of the secondary radiator; but it is only a suggestion, for it by no means follows that the presence of β rays involves atomic disintegration. The modern theories of electrical conduction imply the existence in conductors (and all bodies are conductors to a greater or less extent) of large numbers of corpuscles not closely bound up in the atomic structure ; and it is quite conceivable that some of these may constitute the secondary β rays, the necessary energy having been, in some way, imparted to them by the primary radiation. The most direct way of discriminating between these two possibilities is to investigate the energy relations when, for example, Röntgen rays are absorbed by matter. If none of the atoms are broken up, then the conservation of energy, in the ordinary sense, will hold : if, on the other hand, some of the atoms are exploded by the Röntgen rays, as dynamite is exploded by a shock, then the total energy after the absorption of the rays may be considerably greater than the energy of the rays themselves. This excess of energy might be expected to manifest itself mainly in the form of heat in the absorbing body; for it is known that a large fraction of the secondary Röntgen rays are very easily absorbed indeed[*]; and Sagnac found that tertiary rays were more easily absorbed than secondary rays[†]. Thus only the secondary rays which are produced very near the surface of the absorber would carry their energy away with them ; those which are set up throughout the mass of the body would be absorbed before reaching the surface, and eventually would warm the absorber.

Assuming for the moment that Röntgen rays are able to cause atoms to break up, it is very improbable that the atoms of different substances are equally susceptible to this effect ; and we should expect to find an inequality in the amount of heat produced when Röntgen rays are equally absorbed in different substances. If, on the contrary, there is no atomic disintegration, the quantities of heat should be equal. It was

[*] J. J. Thomson, 'The Conduction of Electricity through Gases,' p. 263.
[†] Ibid. p. 273.

from this point of view that the problem was proposed to me by Professor J. J. Thomson, during my stay in Cambridge last year ; and the experiments which I am about to describe were carried out in the Cavendish Laboratory under his direction, and owe much to his advice and co-operation.

In considering the various experimental means by which this problem could be attacked, the radiometer seemed to promise certain advantages over other heat-measuring instruments.   For measuring ordinary radiation with this instrument (the development of which is chiefly due to E. F. Nichols), the usual method is to have opaque vanes with a transparent wall near them upon one side ; when one of the vanes is illuminated, it is heated and the molecular reaction causes it to be repelled from the neighbouring wall. But it is quite possible to reverse this procedure and have the wall opaque and the vanes transparent, and, although the attainable sensitiveness is probably less than in the other case, it has obvious advantages when one is dealing with Röntgen rays instead of ordinary light.   The walls can be made of the metals under investigation, and of suitable thickness to absorb a considerable fraction of the rays incident upon them ; while the vanes may be made very transparent to the rays and be thus far less in the way, and be less heated by the rays (independently of the heating of the substances under examination) than would be possible with the thermopile or bolometer.   In the construction of the radiometer and its gradual adaption to the present purpose, I was particularly fortunate in having the advice and assistance of Professor E. F. Nichols who was also in Cambridge, and who most kindly put at my disposal the results of his long experience with the radiometer.   I wish here to express my thanks to him for a much shorter period of apprenticeship to the instrument than would have been possible without his help.

Apparatus.

The radiometer and its adjuncts passed through many preliminary and tentative forms before a satisfactory arrangement was obtained.   The final form which proved fairly sensitive and manageable is here described.

The vanes were made of aluminium-foil which weighed about 1 mg. per square cm.   Each vane measured 8×10 millimetres with its greatest dimension vertical, and the two pieces of foil were stretched between two very thin horizontal rods of glass at top and bottom, which in turn were kept at the proper distance apart by their attachments to the central rod

of the suspended system. The inner edges of the vanes were 4 mm. apart. The various joints were made by very small drops of alcoholic shellac which were baked on with a hot glass rod. The system was put together on a flat brass table and, after a few trials, it was not difficult to get one in which the aluminium vanes were very fairly smooth, plane, and parallel to the central rod. The mirror, for use with telescope and scale, was attached to the central rod 3 cm. above the middle of the vanes ; it was usually about 5 mm. square and was made of specially selected microscope cover-glass. All the mirrors used gave very satisfactory definition. In the final system, the weight of the mirror was 13 milligrams, that of the rest of the suspended system 5 miligrams ; the moment of inertia was approximately 0·002 grm.-cm.[2] The two aluminium vanes were electrically connected by means of a bit of light phosphor-bronze strip. The other details of the suspension are shown in figs. 1 and 2. The working quartz-fibre, F, is 3 cm. long, and is suspended from a glass rod carrying a small magnetic needle, N : as this, in turn, is suspended by a second fibre, F', it may be caused to rotate by means of the bent magnet outside the case ; it thus serves as a sort of torsion head for controlling the zero of the radiometer. This very elegant device is borrowed from the work of Nichols and Hull on the pressure of radiation.

The metals whose heating effects were to be compared were mounted upon an ebonite disk (W, figs. 1 and 2) 6 mm. thick and 7·8 cm. in diameter. Three rectangular holes, 3·6 × 1·4 cm., were cut in this disk, at an angular distance from each other of 120°. Across these holes strips of the metals under investigation were fastened with a little soft wax ; the strips were 2 cm. long and 1 cm. wide and their inner edges were 4 mm. apart. Up to the present, 1 have had time to compare only two metals, lead and zinc. The lead strips were 0·30 mm. thick and the zinc strips 0·82 mm., these thicknesses being chosen because they gave nearly the same absorption of the Röntgen rays used, as will appear later. The thinner lead strips were blocked up from the wheel on small bits of card, so that the surfaces of lead and zinc towards the radiometer vanes were in the same plane. The arrangement of the metal strips on the ebonite disk was as follows : across one of the rectangular openings, two lead strips were placed ; this was for testing the balance of the radiometer when both vanes were influenced by the same metal ; across the second opening a lead and a zinc strip were placed side by side, while the third opening also contained a lead and a zinc strip but in reversed order. The shaft which

Fig. 1.
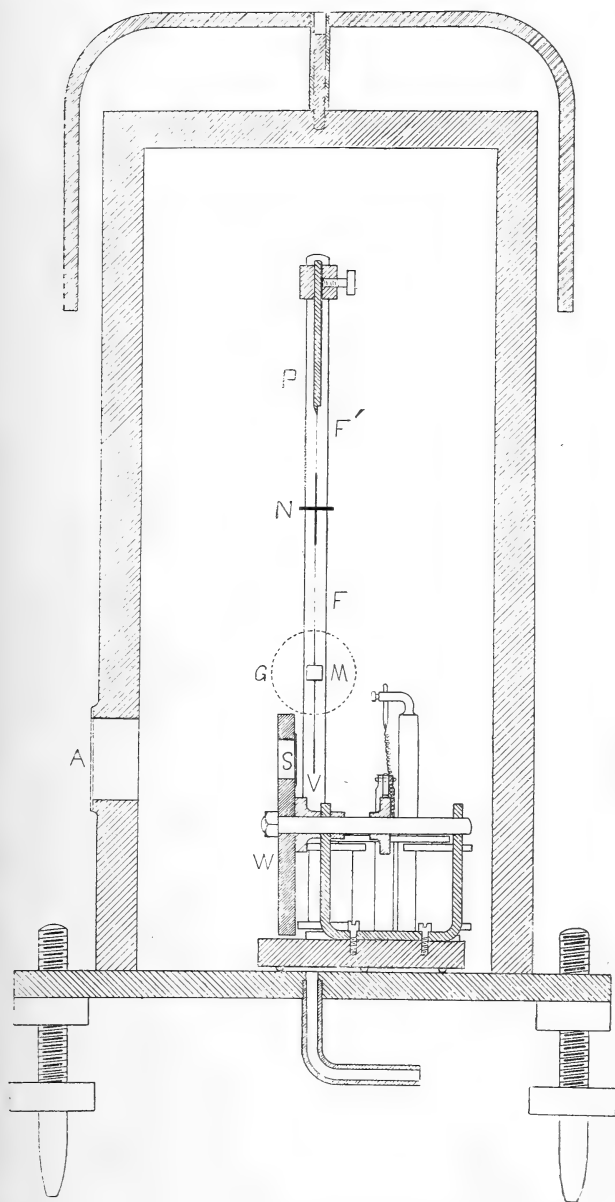
Fig. 2.
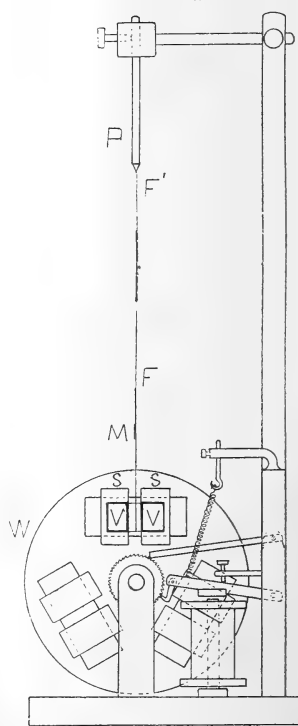
carried the ebonite disk was provided with a ratchet-wheel with sixty teeth; and the disk could be rotated from one position to another by an electromagnet inside the radiometer-case. The current for this purpose was led in by means of wires passing through small glass tubes sealed with sealing-wax, into holes in the base-plate of the instrument. Certain marks on the edge of the ebonite disk which could be seen through the observing window enabled one to be sure that the disk was in proper position in any given case. All the metal strips, before being mounted on the disk, were covered on both sides with leaf aluminium which was held by the thinnest possible layer of soft wax, put on when the metal was hot. This was to give both metals the same surface, so that the loss of heat from the surface for a given rise in temperature might be the same in both cases. The ebonite disk was also covered with aluminium leaf to avoid electrostatic effects; to the same end a small quantity of an impure radium salt was put inside the case in a small open dish. All the metal strips were connected to the shaft of the wheel by thin copper wires; the shaft in turn was connected to the case and to earth.

The portions of the apparatus hitherto described stood upon a brass base-plate turned flat and smooth, which rested on three levelling-screws. Through this plate passed the tube which led to the P_2O_5 bulb, pump, and McLeod gauge. The whole was covered with a heavy, cylindrical, brass casting, 12·5 cm. in internal diameter, 29 cm. high, and with walls 1·4 cm. thick. This heavy metal case was found to be necessary owing to the sensitiveness of the radiometer to thermal disturbances; with a blackened glass bell-jar which was first tried, the zero was so unsteady that nothing could be done with the instrument. As a further protection against thermal disturbance, the cover and base were surrounded with cotton-wool and a felt jacket drawn over the whole. The case was provided with two windows, 2·9 cm. in diameter; one of these (A, fig. 1) was covered with sheet aluminium 1·2 mm. thick, and served to admit the beam of Röntgen rays; the other was of glass for observing the deflexions. Both were put on with sealing-wax and the joints covered with soft wax. The inside of the cover was painted with lampblack in alcohol with a little shellac to make it stick. The bottom of the cover was turned flat, and the joint between it and the brass plate was surrounded by a mixture of equal parts by weight of vaseline, paraffin, and rubber which, after being put on, was glazed over with a small gas-flame. When all the joints

were carefully made, little difficulty was experienced in
maintaining the vacuum for considerable periods; the rise in
pressure due to leakage was usually less than 0·002 mm. per
day.

In the preliminary experiments for testing the working con-
ditions of the radiometer, the ebonite wheel and its metal strips
were replaced by a light ebonite frame which carried two strips
of platinum foil (one opposite each vane), through either of
which a known current could be sent.   By this means one
could readily find the pressure of maximum sensitiveness, and
compare different quartz fibres and different forms of the
suspended system.   The best pressure appeared to be between
0·03 and 0·08 mm., and within this region the variation of
sensitiveness with pressure was slow; in the subsequent work
a pressure between these two limits was usually employed.
From a knowledge of the resistance of the platinum strips and
of the current employed, it appeared that a deflexion of one
millimetre (scale distance, 196 cm.) corresponded to an
emission of about 0·04 erg per second from each square
centimetre of platinum surface.   The deflexions were also
found to be proportional to the energy generated in the strips.
Of course the radiometer action depends primarily on the
temperature of the surface, and with a surface of different
emissivity, the deflexion for a given emission of energy will
be different.

The Röntgen bulb finally employed was a very large one
made by Müller and obtained from Isenthal & Co.   The
diameter of its spherical portion was 17 cm. and the electrodes
were big enough to bear a very heavy discharge.   The anti-
cathode was sealed into the bottom of the water-tube so that
it was in direct contact with the water, and I have frequently
caused the water in the tube to boil without seriously heating
the rest of the tube.   It was provided with an automatic
vacuum adjuster which worked well; the focus was sharp
and the Röntgen rays obtained were very powerful and fairly
steady in intensity and "hardness."   The bulb was driven by
an 18-inch Apps coil and a rotating mercury-jet interrupter.
The coil was in an adjoining room and about 6 metres from the
radiometer; its orientation was adjusted so that it produced
very little effect upon the magnetic "torsion head" of the
instrument—less than 4 mm. with the largest currents used;
this was always in one direction and could be applied as a
correction.   The secondary leads were gutta-percha covered
and supported by silk ribbons; where they passed from one
room to the other, through the wooden frame over a door, they
were enclosed in long glass tubes.   As the rays used were not

very hard there was little difficulty with the insulation, though
there was often a good deal of brush discharge from the leads.
The earthed metallic case of the radiometer protected it com-
pletely from any electrostatic disturbance.   A lead screen,
2 mm. thick, which hung by a quadrifilar suspension from the
ceiling about 2 cm. in front of the aluminium window,
permitted both strips to be exposed to the Röntgen rays, or both
screened, or either exposed while the other was screened.   A
large cardboard screen was kept between the Röntgen bulb
and the radiometer, to diminish the effects upon the instrument
due to the heating of the bulb.

Measurements of the absorption of the lead and zinc were
carried on simultaneously with the radiometer observations.
For this purpose an electroscope was set up 270 cm. from the
bulb ; it had an aluminium window 2 cm. in diameter and
the remainder was covered with sheet lead 2 mm. thick.   As
this did not sufficiently protect it from the rays, a lead wall
was built up of blocks two inches thick between the electro-
scope and bulb with a hole opposite the window ; this gave
satisfactory protection when the window was screened.
Observations were made with a micrometer-microscope and
stop-watch in the usual manner.

Experiments.

The experimental method was based upon the following
considerations :—The Röntgen rays absorbed in the strips will
generate heat throughout the mass of the metals and a steady
state of temperature will be reached when the heat generated
per second is equal to the heat lost per second through the
two surfaces by radiation and convection.   (The possibility of
any appreciable loss by conduction through the ebonite support
of the strips will be considered later.)   The heat lost through
any surface is proportional to its emissivity and, for such
small temperature differences, to its excess of temperature
above its surroundings.   The lead and zinc strips had the
same surface—aluminium leaf—and it is natural to assume
that the emissivity is the same in both cases ; this assumption,
however, will be justified experimentally.   Moreover, it will
appear from theoretical considerations that, in the steady
state, the difference in temperature between the front and back
surfaces of either metal is a small fraction of the excess of its
temperature above its surroundings; and this too will be
confirmed experimentally.   It follows therefore that, very
approximately, half the heat generated in either strip is lost
through each of its two surfaces, and that the total heat
generated per second will be proportional to the temperature

of either surface in the steady state, and hence to the repulsion of the radiometer vane exposed to that surface.

The first step in a series of experiments was to test the balance of the radiometer by exposing the two lead strips simultaneously to the Röntgen rays. The balance was usually found to be fairly good, and could be to some extent adjusted by moving the bulb horizontally before the window; the stand carrying the bulb could be moved by a horizontal screw for this purpose. In two of the series of experiments (quoted below) it was found impossible to get a good balance by moving the bulb; in these cases the lack of balance was determined and applied as a correction. The wheel was then clicked round to the second position, in which one vane of the radiometer was opposite to a lead strip and the other opposite to a zinc strip. This was a somewhat delicate operation as each click caused a violent disturbance of the radiometer (partly magnetic and partly thermal), and it was necessary to get it under control by means of a subsidiary magnet before making another click; otherwise the vanes might have become entangled in the moving wheel and the suspension broken. After the shift from one position to another several hours had to elapse before the radiometer was again fit for use. Several exposures to Röntgen rays were then made in which the zinc and lead strips were exposed separately and both together. The wheel was then moved to the third position, in which the zinc and lead strips were in reversed order, and similar observations taken. It may be said at once that the reversed position of the lead and zinc did not alter the character of the results.

The general nature of the results may be most readily set forth by considering a particular experiment as an example. In fig. 3 the results of this experiment are shown graphically; the abscissæ represent time in minutes, the ordinates deflexions of the radiometer in centimetres on the scale; the position of the wheel was such that a positive deflexion means a repulsion by the zinc of its vane, a negative deflexion a repulsion by the lead of its vane. Between 2^m and 5^m, the zinc strip was exposed to the rays; the rays were then cut off and the zinc allowed to cool for 5 minutes (5^m–10^m), the lead was then exposed for 3 minutes (10^m–13^m); it, in turn, was allowed to cool for 5 minutes (13^m–18^m) and then both strips exposed simultaneously for 3 minutes (18^m–21^m), after which the rays were again cut off. It is plain from the figure that neither metal was exposed long enough to the rays for the steady state to be attained; but it is also plain that the great preponderance of the lead over the zinc is not due to this
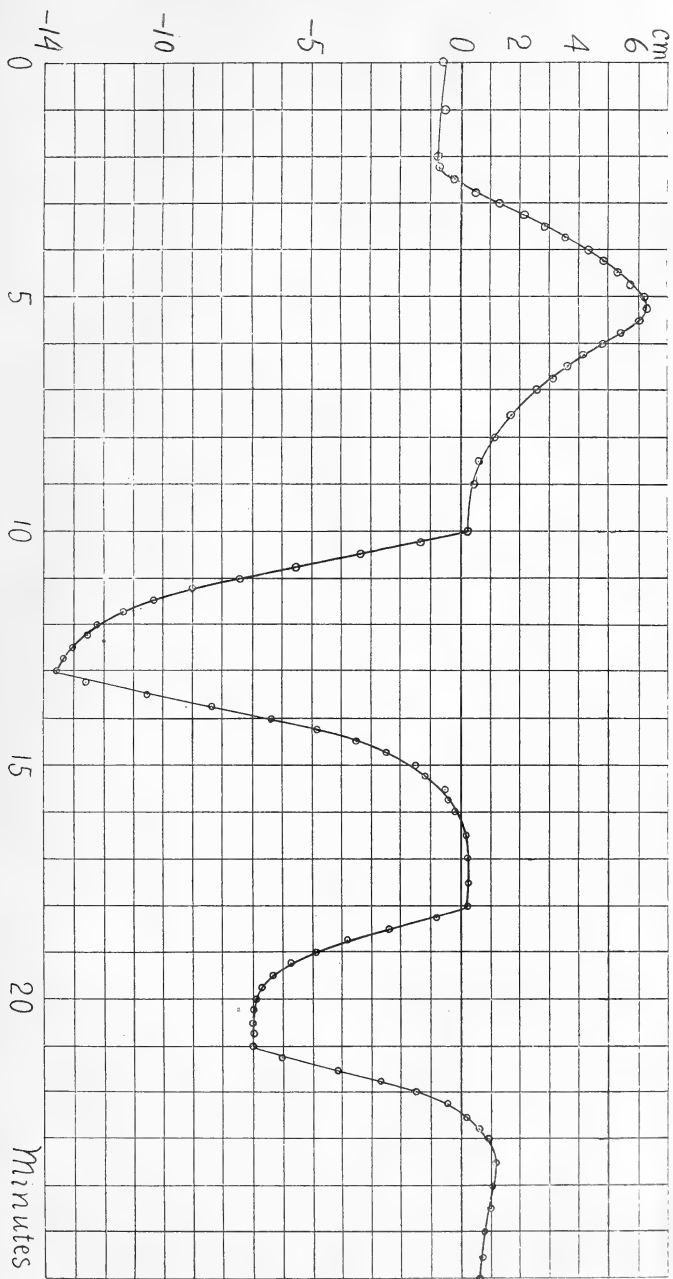
Fig. 3.

cause.   In fact we may from these curves get an approximate idea of what the steady deflexions would have been if it had been safe to continue running the bulb at the rate necessary for such large deflexions.

If λ is the coefficient of absorption of the metal for the rays used, then the energy of the rays at any point in the interior of the strip whose distance is x from the front face is $I_0 e^{-\lambda x}$ and the energy absorbed in an element of thickness dx is $\lambda I_0 e^{-\lambda x} dx$.   Let us assume that the heat generated in the element is proportional to this, say $\alpha \lambda I_0 e^{-\lambda x} dx$.   The equation of the flow of heat under these circumstances is :

$$c \frac{dV}{dt} = k \frac{d^2V}{dx^2} + \alpha \lambda I_0 e^{-\lambda x}$$

with the boundary conditions

$$k \left(\frac{dV}{dx} \right)_{x=o} = hV_0 ; \quad -k \left(\frac{dV}{dx} \right)_{x=l} = hV_l ;$$

where V is the temperature (measured above the surroundings), t the time, c the specific heat of unit volume, k the conductivity, h the emissivity of the surface, and l the thickness of the strip.   The solution of this will have the form

$$V = V_\infty - \Sigma A e^{-\gamma t} ;$$

where V_∞ is the steady value and is the solution of the differential equation with the left-hand member put equal to zero; the successive values of γ in the sum are determined by a certain transcendental equation, and the A's are functions of x into which γ enters.   What is observed is the temperature of the surface where $x = l$.   The variation of this temperature with the time is represented fairly well (except at the very beginning) by a single term of the sum of exponentials, so that we may write as a rough approximation

$$V = V_\infty (1 - e^{-\gamma t}).$$

The variation of the temperature with the time during cooling will have different values of the A's and γ's, but (again excepting the initial part) the curves of heating and cooling will have very nearly the same form.   This is seen in the figure, and it also appears mathematically that, with the values of the conductivity and emissivity involved, there is a nearly uniform temperature throughout the metal within less than a minute after the rays are turned on.   We may therefore, for such accuracy as is needed in this discussion,

take the curve of cooling to be represented by

$$\gamma = V_0 e^{-\gamma t};$$

the time being counted from the moment the rays are cut off. In this equation

$$\gamma = -\frac{1}{V}\frac{dV}{dt};$$

and one may get rough values of γ for lead and for zinc by drawing tangents to the curves of cooling. In the following tables are several values found for different points of the curves in fig. 3 ; the zero for the zinc curve was taken as the mean of the position at 2^m and 10^m and, for lead, the mean of 10^m and 18^m.

<div align="center">

Zinc.

V.	$\dfrac{dV}{dt}$.	γ.
3·0	1·8	0·6
2·0	1·6	0·8
1·5	1·05	0·7
0·8	0·50	0·6
Average		0·68

Lead.

V.	$\dfrac{dV}{dt}$.	γ.
6·0	6·2	1·0
3·0	4·0	1·3
1·5	2·5	1·7
0·8	1·3	1·6
Average		1·4

</div>

In order to determine the steady value from the value at the end of three minutes we have

$$V_\infty = \frac{V}{1 - e^{-\gamma t}}$$

which gives, for lead $V_\infty = 13\cdot95$, and for zinc $V_\infty = 7\cdot4$.

Approximately the same rates of cooling and heating were obtained in all the experiments made upon this point. A second example is given in fig. 4, in which the weaker rays were used for ten minutes instead of three. Curve I is for zinc, curve II for lead; the two experiments were made at different times and with rays of different intensity, so that the *magnitudes* of the deflexion have nothing to do with each other. To show that the inertia and damping of the suspended system have no sensible effect upon the determination of the rates of heating and cooling, curve III is added for comparison.

This was obtained by suddenly moving the controlling magnet at the time $t = 0$; it represents the response of the

Fig. 4.
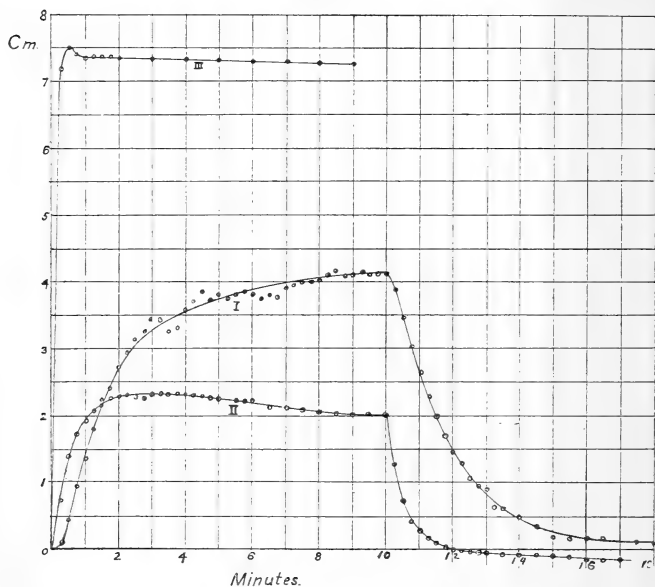
suspension to a sudden impulse. The following values for γ for zinc and lead were obtained from the cooling portions of these curves :

<table>
<tr><td colspan="3" align="center">Zinc.</td><td colspan="3" align="center">Lead.</td></tr>
<tr><td>V</td><td>$\frac{d\mathbf{V}}{dt}$</td><td>γ.</td><td>V.</td><td>$\frac{d\mathbf{V}}{dt}$</td><td>γ.</td></tr>
<tr><td>2·0</td><td>1·2</td><td>0·6</td><td>1·0</td><td>1·5</td><td>1·5</td></tr>
<tr><td>1·0</td><td>0·58</td><td>0·58</td><td>0·5</td><td>0·8</td><td>1·6</td></tr>
<tr><td>0·5</td><td>0·38</td><td>0·76</td><td>0·3</td><td>0·48</td><td>1·6</td></tr>
<tr><td colspan="2" align="center">Average ......</td><td>0·65</td><td colspan="2" align="center">Average ......</td><td>1·6</td></tr>
</table>

Later in the course of the experiments, the aluminium widow was replaced by a glass one and the strips heated by means of a beam of light ; the result of such an experiment

is shown in fig. 5 below. The following values of γ are obtained in the same manner from these curves :—

Zinc.

V.	$\frac{dV}{dt}$.	γ.
4·0	2·15	0·54
2·0	1·22	0·61
1·0	0·57	0·57
Average		0·57

Lead.

V.	$\frac{dV}{dt}$.	γ.
4·0	5·2	1·3
2·0	2·6	1·3
1·0	1·1	1·1
Average		1·23

It will be seen that the agreement is as good as could be expected from the very rough nature of the determinations.

Returning now to the experiments plotted in fig. 3, it is to be observed that the steady deflexion for lead is nearly twice as great as that for zinc. The fractions of the incident rays absorbed by the two strips were determined by means of the electroscope with the pieces of metal from which the strips had been cut ; the pieces of lead and zinc were placed behind a sheet of aluminium of the same thickness as the window of the radiometer. It was found that the lead absorbed 79 per cent. and the zinc 78 per cent. of the rays which got through the aluminium : in this particular case the spark-gap in the automatic adjuster of the Röntgen bulb was 7·5 cm. long.

Thus (unless there has been some error in the experimental method or in the interpretation of the result) it appears that for practically equal absorptions of Röntgen rays in lead and in zinc, about twice as much energy is generated in the lead as in the zinc. It may be said at once that essentially the same result was obtained in all the experiments, with rays of different hardness, with both positions of the strips and with the radiometer vanes sometimes between the window and the strips, as well as in the position indicated in fig. 1. Before giving, however, the numerical results of all the experiments it will, I think, be conducive to clearness to consider the possible sources of error (so far as they have occurred to me) and their influence upon the result.

Possible Sources of Error.

1. After it had been found that the lead strip predominated, and before the rates of heating and cooling of the strips had been worked out, an effect was observed which caused considerable perplexity. The slight drift of the zero-point due to the warming up of the room, after the observer entered,

was found to be always in the direction indicating a repulsion from the lead ; its direction reversed with the reversal of the order of the strips. As the aluminium window was the only part much exposed to changes of temperature, the effect of slightly warming it was investigated. When an incandescent lamp or a warm soldering-iron was held six inches from the aluminium window, the radiometer was deflected away from the lead strip—at first very slowly, then more and more rapidly, until it went off the scale and usually bumped against the zinc strip. As this was in the same direction as the effect when both strips were exposed to the Röntgen rays, it appeared that the whole effect might be spurious. By putting the lamp about two feet away from the window, less violent effects were produced, and it was possible to follow the progress of the deflexion. This was sluggish in comparison with that produced by the rays, and (as might have been expected from varying air-currents in the room) was irregular and some-what erratic in its course. In one respect, however, the behaviour was consistent : if the lamp was left glowing for one or two hours, the radiometer always returned to the neighbourhood of its zero-point and stayed there until the lamp was turned off. It then made another slow and wandering excursion in the opposite direction (repulsion by zinc), and eventually again returned to the middle of the scale. The application of ice to the aluminium window also caused repulsion by the zinc.

The determination of the different rates of heating and cooling of the two strips furnished the explanation of this behaviour. While the window and adjacent parts of the case are slowly warming up, they radiate to the two strips more and more, and cause the temperature of the latter to rise steadily; but the lead, having a smaller time constant than the zinc, has the higher temperature and maintains it until the amount of radiation reaching the strips no longer varies with the time—that is, until the window has reached its stationary temperature. When this takes place the zinc has time to overtake the lead, both temperatures become the same, and the radiometer returns to zero. When the lamp is turned off and the window begins to cool, the lead cools more rapidly than the zinc; and hence there is a similar deflexion in the opposite direction, which again lasts only so long as the temperature of the window is variable.

Although this explanation is reasonable and, in fact, inevitable, the matter here discussed touches so vitally the main conclusion of this investigation that it was desirable to test in a decisive manner the behaviour of the instrument

when the strips were heated by ordinary radiation. For this
purpose, a glass window was substituted for the one of
aluminium and (everything else being left exactly as before)
a beam of light was admitted from an eight-candle power
incandescent lamp, 125 cm. from the window. The result of
such an experiment is shown graphically in fig. 5, where as
before the abscissæ represent time in minutes, the ordinates
deflexions in centimetres ; a positive deflexion means repulsion
by the zinc, a negative deflexion repulsion by lead. From
0^{m} to 7^{m} both strips were exposed to the light, and the
curve clearly shows the temporary deflexion in favour of the

Fig. 5.
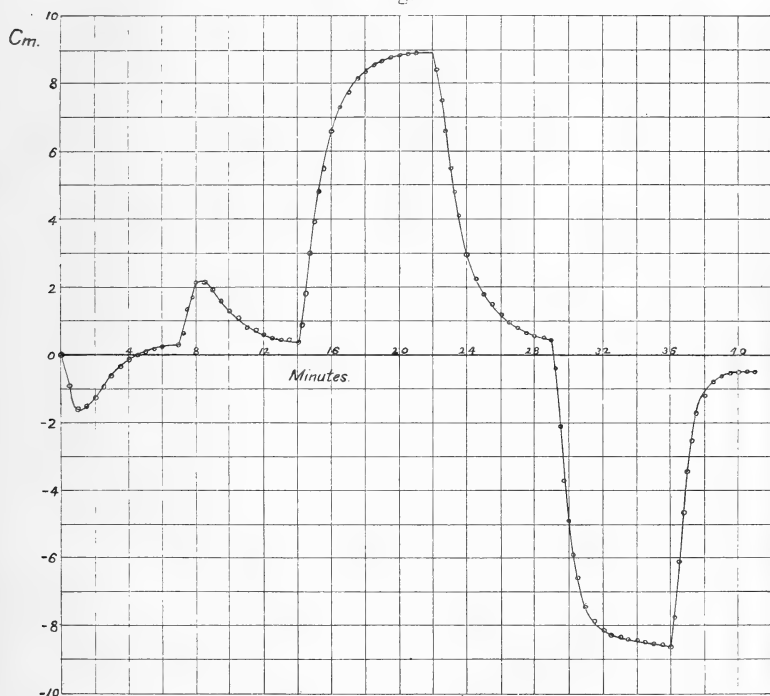
lead, returning to zero as the zinc overtakes the lead ; from
7^{m} to 14^{m} the window was covered by the lead screen so
that both strips cooled, the zinc lagging behind as before,
and so giving a similar temporary deflexion in the opposite
direction. From 14^{m} to 22^{m}, the zinc strip was exposed
while the lead was shielded, and from 29^{m} to 36^{m} the
lead alone was exposed. It will be observed that the
behaviour is perfectly in accord with the above explanation
of the heating effect, and that the deflexions, when the lead

X 2

and zinc are separately illuminated, are equal. The contrast with the effects of the Röntgen rays is sufficiently marked. I have exposed both strips at once to weak Röntgen rays (giving a deflexion of about 2 cm.) for 30 minutes without observing any tendency of the radiometer to return to zero.

2. At first sight there appears to be a possibility that (on account of different heat-conductivities, &c.) one of the metals might lose more heat through the front surface and the other through the rear surface, thus giving rise to different surface temperatures, even though the quantities of heat generated were the same in both cases. A number of considerations show that this cannot account for the effects observed. It is easy to show that, when all the heat enters through the front surface, the ratio of the temperature of the front face to that of the rear face will be (when the steady state is attained)

$$\frac{V_0}{V_l} = 1 + \frac{hl}{k},$$

and the difference will be less than this when the heat is generated throughout the interior of the metals. Now a superior limit to the value of h may be readily obtained, from the rates of cooling of the strips, by assuming infinite conductivity in the metals so that the whole strip has at any instant the same temperature as the surface. Under these circumstances, we have

$$-\tfrac{1}{2}C\rho l \frac{dV}{dt} = hV;$$

where C is the specific heat, ρ the density, and l the thickness of the metal. Also

$$-\frac{dV}{dt} = \gamma V,$$

so that

$$h = \tfrac{1}{2}\gamma C\rho l.$$

With a given curve of cooling of the surface and finite conductivity, the value of h will be less than this; because when the surface has cooled to any temperature the interior will have a higher temperature, and thus not so much heat need be emitted in a given time as when the conductivity is infinite. In this way we find that for the lead strip

$$h < \cdot 007,$$

for the zinc strip

$$h < \cdot 002,$$

and taking $h = \cdot 01$ we find, in the worst case,

$$\frac{V_0}{V} < 1\cdot 004.$$

More direct experimental evidence upon this point was obtained by reversing the position of the ebonite disk with reference to the case, and putting the vanes on the side of the strips next the window instead of on the further side. The effects observed were the same as in the other position.

3. The possibility of unequal values of the emission coefficient for the aluminium-covered surfaces of the two metals is negatived by the experiments with light in which the two metals produced equal effects.

4. If much heat were lost by conduction through the ebonite support, or the copper wire by which the strips were earthed (diameter, 1/4 mm.), the central part of the zinc strip would be cooler than the lead on account of its greater conductivity and thickness. If this were the cause of the higher temperature of the lead, it would act also when the strips are warmed by light, which is not the case. Experiments with light were made with the full aperture of the circular window, so that the ebonite was illuminated as well as the strips ; and also with a cardboard screen in front of the window with a rectangular hole which allowed the light to fall on the strips alone, and not on the ebonite. No difference was observed in the two cases.

5. The observed results cannot be due to electrostatic effects produced by the negative corpuscles of the secondary rays. The strips were earthed, and the two vanes were connected by a conductor ; under these circumstances, no electrostatic effect could be produced except an instability due to *both* vanes having a different potential from the two strips. Such an instability was frequently observed after the vanes had become electrified by bumping against the strips, but no other electrical effect. Moreover, the coincidence of the rates of motion of the radiometer when illuminated by Röntgen rays and by ordinary light, shows that the deflexions produced by the rays were caused by changes of temperature and not by electrical forces.

6. The last consideration also excludes the (somewhat remote) possibility that the effect is produced by direct impact of the secondary rays.

7. It is conceivable that a slight difference in the distance of the lead and zinc strips from the vanes might cause a considerable difference in sensitiveness of the two vanes, and thus account for the preponderance of the lead. If this were the case, however, the effect should have appeared in the experiments with ordinary light. I also tested the effect upon the sensitiveness of varying the vane distance by taking deflexions with the zero near the top and near the bottom of

the scale. For a variation in the vane distance of 0·48 mm. the change in sensitiveness was less than 12 per cent. The positions of the surfaces of the lead and zinc strips were afterwards tested by means of a point gauge, and they certainly did not differ by more than 0·1 mm.

8. It might also be imagined that the measurement of the absorptions by means of the electroscope did not give the relative amounts of energy absorbed by the lead and zinc: that the ionization in the electroscope was not a measure of the energy of the rays. One may express this possibility in slightly different form by saying that, accompanying the rays which ionize the gas, there may be other rays which do not cause ionization but which do carry energy ; and that these latter may be more absorbed in the lead than in the zinc. In order to test this, measurements of the absorption were made with the radiometer itself. One strip was shielded so that as large a deflexion as possible might be obtained with the rays, and an aluminium screen of the same thickness as the window was set up in front of the window ; the lead and zinc of the thickness of the strips were introduced behind this screen. The following deflexions in succession were obtained by a three-minute exposure to the rays in each case :—

$$
\begin{array}{lc}
\text{Al} & 8\cdot4 \\
\text{Al} + \text{Pb} & 0\cdot8 \\
\text{Al} + \text{Zn} & 0\cdot9 \\
\text{Al} + \text{Pb} & 0\cdot6 \\
\text{Al} & 6\cdot6 \\
\end{array}
$$

It is plain that the rays did not remain very constant during this somewhat prolonged use; but it is also plain that, whatever agent it may be which affects the radiometer, it is practically equally absorbed by the lead and the zinc. Taking the means of the readings as they stand, we find that, of the energy which gets through the aluminium, 88 per cent. is absorbed by the zinc and 91 per cent. by the lead.

9. Even if one assumed the substantial correctness of the energy measurements, there still remains the possibility that the difference between the two metals may not be due to atomic disintegration. It is known that secondary rays are generated by the absorption of the primary rays, and it may be supposed that these carry away from the metal a considerable fraction of the energy of the primary rays—a much greater fraction in the case of zinc than in the case of lead, thus giving rise to the observed difference. An examination of this possibility is therefore necessary.

The secondary rays from such metals as lead and zinc consist of two well-defined groups, one of which is completely absorbed in less than 10 mm. of air, the other being much more penetrating. Let us consider first the less penetrating rays. It is stated in the introductory portion of this paper that these rays must form a large fraction of all the secondary rays generated: this statement appears at first sight to be in contradiction with the experimental result obtained by H. S. Allen [*], who found that the ionization produced by these easily absorbed secondary rays from brass is 1/1900 of that which would be produced by complete absorption of the primary rays in the gas. A little consideration, however, shows that, with rays so easily absorbed, a large number must be generated in the metal to permit even so small a fraction to escape.

Let the intensity of the primary rays which have penetrated at a distance x into the metal be

$$I_1 = \bar{I}_1 e^{-\lambda_1 x}.$$

The absorption in an element of thickness dx will be $\lambda_1 I_1 dx$, and we may assume that this is proportional to the intensity of the secondary rays (of the more absorbable type) generated in this element, say

$$I_2\, dx = \alpha \lambda_1 I_1\, dx.$$

The question is as to the value of the fraction α necessary to produce the observed secondary rays outside the metal. Let us assume that half of the secondary rays so generated are propagated straight back to the surface: this will give us an over-estimate of the rays which escape with a given value of α, or an under-estimate of α necessary to account for the observed secondary rays. The effect, at the surface, of the secondary rays generated in the element dx will be then

$$\tfrac{1}{2} I_2\, dx\, e^{-\lambda_2 x},$$

and the total intensity of the secondary rays at the surface of the metal will be

$$\bar{I}_2 = \frac{1}{2} \int_0^\infty I_2\, e^{-\lambda_2 x}\, dx = \tfrac{1}{2} \alpha \lambda_1 \bar{I}_1 \int_0^\infty e^{-(\lambda_1 + \lambda_2) x}\, dx$$

$$= \tfrac{1}{2} \alpha \bar{I}_1 \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

The value of the coefficient of absorption, in air, of the

* Phil. Mag. iii. p. 126 (1902).

secondary rays from brass may be obtained from Townsend's experiments [*], and is 6·9 ; he did not measure the absorption in air of the primary rays, but an estimate of its value may be got from the statement that the number of ions produced by the primary rays when they traverse a layer of air 1 cm. thick is about half as great as the number produced by complete absorption of the secondary rays. Combining this with Allen's result that complete absorption of the primary rays produced 1900 times as many ions as complete absorption of the secondary rays, we get, for the coefficient of absorption in air of the primary rays used by Townsend, approximately 3×10^{-4}. Assuming that the ratio of the coefficients of absorption in the metal is of the same order as that of the coefficients in air, we have roughly

$$\frac{\lambda_1 + \lambda_2}{\lambda_1} = 2 . \times 10^4 ;$$

and as

$$\frac{\bar{I}_2}{\bar{I}_1} = \frac{1}{2000},$$

we get $\alpha = 10$.

Now if the conclusion, toward which the present experiments point, be accepted, it is not surprising that the " fraction " α should be greater than unity ; for if atomic energy is set free it is not unlikely that it is through the mechanism of these easily absorbed secondary rays. Of course, the data from which the above calculation has been made are not sufficiently accurate to give any weight to the actual number obtained for α ; but it is sufficiently clear that a considerable number of these easily absorbed secondary rays must be generated, of which only a very small fraction can escape from the metal.

This conclusion in itself makes it somewhat improbable that a large part of the energy could be transformed into the *more* penetrating secondary rays and so carried away ; but it is desirable to consider the question of this type of rays independently. Sagnac, who has made an extensive series of experiments upon secondary rays [†], finds that heavier metals in general give out more intense, and more easily absorbed, rays (of the type now under consideration) than lighter metals; and, in particular, that this is true of lead as compared with zinc. On the other hand, Townsend (*loc. cit.*) finds that lead is an exception among the heavier metals and, as a matter of fact, gives out less intense and more penetrating rays (of this type) than zinc, and this result has recently

[*] Camb. Phil. Soc. Proc. x. p. 217 (1899).
[†] *Ann. Chim. Phys.* xxii. p. 493 (1901).

been confirmed by Eve [*]. If Sagnac's results were true in general, it would of course dispose of all possibility of explaining the present experiments by means of the energy of the secondary rays; and even from Townsend's observations (when allowance is made for the difference of absorption by air) it is impossible to conclude that the total ionizing power of the secondary rays from zinc is much greater than, or even as great as, that of the rays from lead. But leaving these results out of account, I think it is possible to show that, with the rays which I used and under the experimental conditions, it is not possible to attribute the results obtained to this cause.

The secondary rays from metals are always more easily absorbed than the primary rays which generate them. Barkla [†] has shown that the secondary radiation from gases and light solids is practically identical in character with the primary radiation; and this had previously been found to be the case for such substances as paraffin by Sagnac (*loc. cit.*). But all experimenters, so far as I am aware, are agreed that the secondary radiation from lead, zinc, and other similar metals is less penetrating than the primary; and no evidence has been obtained of any secondary rays more penetrating than the rays which produce them. Thus, if λ_1 and λ_2 are the coefficients of absorption in zinc of the primary and secondary rays respectively, we have

$$\lambda_2 > \lambda_1.$$

Now in the experiment described at length above, the absorption by the zinc strip was about $0 \cdot 8$; the two-tenths observed behind the zinc included not only the primary rays which got through, but also some of the more penetrating secondary rays from the rear surface. Hence $e^{-\lambda_1 l} < 0 \cdot 2$; and as $l = 0 \cdot 08$ cm.,

$$\lambda_2 > \lambda_1 > 20.$$

If we assume as before that one-half of the secondary rays generated in any element are propagated straight back toward the front surface (which will exaggerate the intensity of the secondary rays), we get for the intensity of the secondary rays escaping from the front surface

$$\overline{I}_2 < \tfrac{1}{2}\, \alpha\, \overline{I}_1 \frac{\lambda_1}{\lambda_1 + \lambda_2}\left(1 - e^{-(\lambda_1 + \lambda_2)l}\right).$$

Writing this

$$\tfrac{1}{2}\, \alpha\, \overline{I}_1\, \lambda_1\, l\, \frac{1 - e^{-(\lambda_1 + \lambda_2)l}}{(\lambda_1 + \lambda_2)\, l},$$

[*] Proc. Roy. Soc. lxxiv. p. 474 (1905).
[†] Phil. Mag. viii. p. 674 (1904).

the fraction is seen to be of the form

$$\frac{1-e^{-x}}{x},$$

which (for positive values of x) increases as x diminishes. Hence

$$\dot{I}_2 < \tfrac{1}{4}\, \alpha\, \overline{I}_1\, (1-e^{-40\,l}).$$

The primary rays absorbed in the zinc strip

$$A_1 = \overline{I}_1\,(1-e^{-\lambda_1\, l}) > \overline{I}_1\,(1-e^{-20\, l})\,;$$

hence

$$\frac{I_2}{A_1} < \tfrac{1}{4}\, \alpha\, \frac{1-e^{-40\,l}}{1-e^{-20\,l}} = \tfrac{1}{4}\, \alpha\,(1+e^{-20\,l})\,;\ \text{ or }\ \frac{\overline{I}_2}{A_1} < 0\cdot 3\, \alpha,$$

in which, of course, if there is to be no generation of fresh energy, α must be less than unity.

We may make a similar calculation for the rear surface, but it is sufficient to observe that even if all the effect observed behind the metal is due to secondary rays, the ratio of these to the primary rays absorbed cannot be greater than 0·2. So that by considerably exaggerating all possibilities in favour of the production and escape of secondary rays, we are unable to give to the secondary rays which escape an intensity one-half as great as that of the primary rays absorbed. Now in order to account in this way for the observed difference in the heating of the lead and the zinc, we must assume that at least half of the energy of the primary rays absorbed in the zinc escapes in the form of secondary rays: this is on the supposition that none so escapes from the lead; if the lead loses any energy in this manner, then the fraction for the zinc must be greater than one-half. It appears therefore that the experimental results cannot be accounted for in this way.

Numerical Results.

The measurements of the absorptions of lead and zinc ran so nearly parallel throughout the course of the experiments, with rays differing considerably in penetration, that it was eventually considered unnecessary to make a separate correction to each measurement with the radiometer. The absorptions were measured from time to time during the investigation, and the results are given in the following table; the numbers give the fractions of the primary rays absorbed by aluminium of the same thickness as the window, and by lead and zinc of the same thickness as the strips, the latter being behind the aluminium except in the first three experiments when no

aluminium was interposed. The last experiment in the
table is the one made with the radiometer instead of the
electroscope.

Absorptions.

Experiment.	Al.	Pb.	Zn.	$\frac{Pb}{Zn}$.
I.	......	0·925	0·924	1·001
II.	......	0·926	0·920	1·007
III.	......	0·944	0·947	0·997
IV.	0·57	0·862	0·856	1·007
V.	0·45	0·790	0·780	1·013
VI.	0·47	0·830	0·803	1·034
VII.	0·45	0·809	0·805	1·005
VIII.	0·31	0·684	0·684	1·000
IX.	0·34	0·676	0·648	1·043
X.	0·32	0·675	0·655	1·031
XI.	......	0·907	0·880	1·030
	Average			1·016

The average value of the ratio is applied as a correction in
the energy measurements below.

The energy measurements are given in the following table.
Column I. gives the time of exposure of the strips in minutes.
Columns II. and III. give the deflexions in centimetres,
produced by exposure of the lead and zinc respectively.
These are the observed deflexions, corrected for the effect
of the induction-coil upon the magnetic "torsion-head";
this was always down-scale and varied from 0·1 to 0·3 cm.;
it was determined each time by allowing the coil to run with
the lead screen in front of the window of the radiometer.
The plus and minus signs indicate the direction of the de-
flexion; thus a change of sign in either column represents a
shift of the wheel, so that the positions of the strips are
interchanged. Columns IV. and V. contain the deflexions
corrected for lack of balance of the radiometer. In experi-
ments 1, 2, 3, and 9, the radiometer was sensibly balanced
when tested by the two lead strips; in 4, 5, and 6, there was
a lack of balance of 15 per cent. in the positive direction;
in 7 and 8 the lack of balance was 10 per cent. in the same
direction. Accordingly, in the first case the positive de-
deflexions (zinc) are reduced 15 per cent., in the second case
the positive deflexions (lead) are reduced 10 per cent. The
changes in the conditions are explained by the fact that
between the groups of experiments mentioned the radio-

meter, for one reason or another, had to be taken down and re-adjusted.

Columns **VI.** and **VII.** contain the deflexions reduced to the steady state, by dividing by $(1 - e^{-\gamma t})$. The values of γ used were : for lead 1·5, for zinc 0·67. Column **VIII.** contains the ratios of **VI.** to **VII.**; it is the ratio of the heat generated per second in the lead strip to that generated in the zinc strip.

The first three experiments were made with the vanes between the window and the strips ; the last six with the strips between the window and the vanes.

Exp.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.
1	1·5	−3·45	+1·30	......	......	−3·85	+2·06	1·87
2	1·5	−6·84	+2·26	......	......	−7·65	+3·58	2·13
3	1·5	+5·20	−1·90	......	......	+5·82	−3·01	1·93
4	1·5	−5·51	+2·43	......	+2·16	−6·18	+3·41	1·81
5	1·5	−6·67	+2·64	......	+2·25	−7·46	+3·55	2·10
6	1·5	−7·07	+2·96	......	+2·51	−7·90	+3·96	2·00
7	1·5	+7·09	−2·30	+6·39	......	+7·12	−3·63	1·96
8	1·5	+9·76	−3·21	+8·80	......	+9·85	−5·07	1·94
9	3·0	−13·75	+6·4	......	......	−13·95	+7·4	1·88

The average of the ratios in Column **VIII.** is 1·96 ; reducing to equal absorptions, we get, as the result of these measurements, that when Röntgen rays are equally absorbed in lead and in zinc, the quantity of heat generated in the lead is 1·93 times the quantity generated in the zinc.

The necessity of the writer's returning to America has temporarily interrupted this investigation. Further experiments are, however, now under way in which other metals will be compared, and the experimental conditions varied as much as possible. An attempt will be made to detect the effect with a thermopile as well as with the radiometer ; and to ascertain whether cathode rays produce similar effects, as might be expected. If the present results are due to some hitherto unsuspected source of error, it is hoped that the error may reveal itself as the conditions are varied.

It is in no perfunctory spirit that I wish to express my thanks to Professor J. J. Thomson for permission to work in the Cavendish Laboratory and for the help which I constantly derived from his advice and suggestions, some of which were vital to the success of the experiments. I take pleasure in acknowledging my great indebtedness to him.

Conclusions.

The present experiments indicate that when Röntgen rays are equally absorbed in lead and in zinc, approximately twice as much heat is generated in the lead as in the zinc. It does not appear possible to attribute this result to errors in the measurements or in the theory underlying the experimental method.

To account for this effect the writer has been able to think of only one hypothesis which is not in more or less direct conflict with experimental facts. This hypothesis is that, by means of Röntgen rays, the atoms of certain elements may be artificially broken up and that the energy thus liberated forms a part (and perhaps the greater part) of the energy which appears when the rays are absorbed by matter.

New Haven, Conn., U.S.A.
    Oct. 20, 1905.

XXV. *On the Surface Elasticity of Saponine Solutions. By* S. A. Shorter, *B.Sc., Assistant Demonstrator in the Physical Laboratory of the University of Leeds* *.

THE question of the concentration of dissolved substances in the surface-layers of solutions has been the subject of much research since the time of Plateau's well-known experiments on the superficial viscosity of liquids. The present paper deals with the specially interesting example of this phenomenon afforded by solutions of saponine. The investigation may be divided into three parts :—

(1) Examination of the mechanical properties of the surface-layer by means of the torsion-balance ;
(2) Theoretical investigation of the effect of a viscous-elastic medium on the motion of the disk of the torsion-balance ;
(3) Absolute measurement of the surface-rigidity.

* Communicated by the Author.

The Mechanical Properties of the Surface Layer.

On examining the surface in the usual way by means of the torsion-balance, it was found that the effect of the surface on the motion of the disk depends upon the dimensions of the various parts of the apparatus. If a very powerful wire is used, twisting the torsion-head sets up oscillations which are damped by the surface. If a very weak wire is used, a large rotation is required to set up visible motion, and this motion consists of a simple drift towards the position of equilibrium of the wire. With wires of intermediate strength it is possible to obtain some indication of a temporary position of equilibrium. The disk moves rapidly forward, suddenly comes to rest for an instant, and then moves forward again. With a wire of suitable torsional moment it is possible to obtain a type of motion which clearly demonstrates the elastic nature of the forces called into play by the shearing of the surface. On twisting the torsion-head the disk moves forward, comes to rest, and then moves back *against the torsional couple of the wire*, comes to rest, moves forward again, and so on : in this way executing a few rapid oscillations before settling down to a simple drift. These oscillations do not take place about a fixed centre. The motion is a compound of damped oscillatory motion and translational motion in the direction of the initial twist of the wire.

It is difficult to obtain these oscillations with a fresh surface. They are more readily obtained after the solution has been standing for some time. If, however, the motion is started by giving the vessel containing the solution a small rapid rotation, oscillations are easily obtained provided the moment of inertia of the suspended system is not too great. If the suspending wire has a very small torsional moment, the motion set up is damped oscillatory motion without translation.

It is clear that shearing of the surface gives rise to forces of an elastic nature. These forces are not permanent, so that the surface may be described as *viscous-elastic*. We will next investigate mathematically the effect of a viscous-elastic medium on the motion of the disk.

Mathematical Theory of the Motion.

Suppose a circular disk of radius r_1 and moment of inertia I to be suspended by a wire of torsional moment ϵ. Let the disk touch centrally the surface of a saponine solution contained in a circular vessel of radius r_2. Let θ be the angular displacement of the disk from the position of zero torsion of

the wire, and let L be the moment of the viscous-elastic forces acting on the disk.

Neglecting the inertia and ordinary viscosity of the liquid, the equation of motion of the disk will be

$$L + \epsilon\theta = -I\frac{d^2\theta}{dt^2}.$$

We will assume that the quantity L obeys the equation

$$\frac{dL}{dt} = \mu\frac{d\theta}{dt} - \frac{L}{\tau},$$

where μ is related to the absolute coefficient of surface-rigidity by the equation

$$\mu = \frac{4\pi n\, r_1^2\, r_2^2}{r_2^2 - r_1^2}.$$

The dimensions of n (which we need not formally define) are of course those of an ordinary coefficient of rigidity multiplied by a length.

Eliminating L between the first two equations, we have

$$\frac{d^3\theta}{dt^3} + \frac{1}{\tau}\frac{d^2\theta}{dt^2} + \frac{\epsilon + \mu}{I}\frac{d\theta}{dt} + \frac{\epsilon}{I\tau}\theta = 0,$$

the solution of which takes one of the forms

$$\theta = Pe^{-pt} + Qe^{-qt} + Re^{-rt}$$

or

$$\theta = Ae^{-\alpha t} + Be^{-\beta t}\cos(\gamma t + C),$$

according as

$$4\,(\epsilon + \mu)^3\,\tau^4 - (\mu^2 + 20\,\epsilon\,\mu - 8\,\epsilon^2)\,I\,\tau^2 + 4\,\epsilon\,I^2 \lessgtr 0.$$

There are three possible types of motion : (1) when τ is so *small* that the above expression is positive ; (2) when the expression is negative ; (3) when τ is so *large* that the expression is positive. For convenience we shall refer to these as (1) oscillatory motion of the first type ; (2) aperiodic motion ; (3) oscillatory motion of the second type. The two types of oscillatory motion are quite distinct. When τ is very small the disk oscillates under the action of the wire and the surface exercises a damping effect on the motion. When τ is very large the disk oscillates under the combined action of the elastic forces of the disk and surface.

Motion according to the equation

$$\theta = Ae^{-\alpha t} + Be^{-\beta t}\cos(\gamma t + C)$$

is not necessarily oscillatory in the ordinary sense of the

term. In the accompanying figure are drawn a number of
curves representing various types of motion according to this
equation. Curve I. represents the type naturally suggested
to the mind by a casual examination of the equation. It
may happen, however, that the steady decrease of the term
$Ae^{-\alpha t}$ is so rapid as to disguise the oscillatory nature of the
motion. Curves II., III., and IV. illustrate this point, and

Fig. 1.
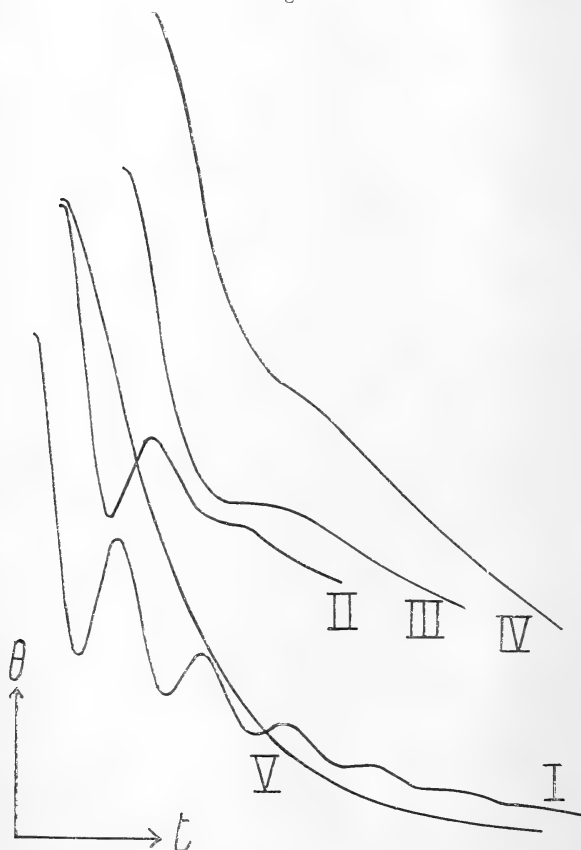
show the continuity of this motion with the aperiodic motion
represented in Curve V.

A very dilute solution of saponine affords on standing an
example of continuous change from oscillatory motion of the
first type to that of the second type. When the surface is
fresh the disk may be set into oscillation by twisting the
torsion-head. The surface viscosity gradually increases till

the motion becomes aperiodic (Curve V.). On further stand-
ing, the motion changes to oscillatory motion of the second
type. The transition is first indicated by a momentary
retardation after the rapid initial motion (Curve IV.). This
becomes more and more marked till it amounts to an actual
stoppage (Curve III.). After this a slight backward motion
can be detected. This increases (Curve II.), and soon it
becomes possible to obtain two oscillations, then three, and so
on; till on twisting the torsion-head we obtain several, as in
Curve I.

A study of the above curves will show that it is quite
useless to attempt to measure the surface-rigidity *statically*
by means of the torsion-balance. The temporary position of
equilibrium indicated by a stoppage or a backward movement
of the disk does not afford any measure of the ratio between
the torsional moments of the wire and the surface. In the
following investigation, the elasticity has been measured
dynamically. If the torsional moment of the wire is very
small, we have

$$\theta = A + B e^{-\frac{t}{2\tau}} \cos\left(\tfrac{1}{2}\sqrt{\frac{4\mu\tau^2 - I}{I\tau^2}} \cdot t + C\right),$$

provided that

$$4\mu\tau^2 > I.$$

Hence

$$\mu = \frac{(4\pi^2 + \lambda^2)\,I}{T^2},$$

where T is the period and λ the logarithmic decrement of the
oscillations set up by giving the vessel an initial rotation.

Measurement of the Surface Rigidity.—Fresh Surfaces.

Apparatus.—The disk was made of brass and had a diameter
of 7·10 cms. The upper end of the suspending wire was
clamped to a bracket firmly fixed to the wall. The lower
end of the wire was clamped centrally to a short piece of
brass rod whose lower end could be screwed into the centre
of the disk. A small mirror was fixed to the rod so as to
enable the motion of the disk to be observed by means of
a distant telescope and scale. The moment of inertia of the
suspended system was 2050 grm.cm.² In all the experiments,
the period of oscillation under the action of the wire was
very large compared with the period under the action of
the surface, so that the torsional moment of the wire was
negligible. The vessel for holding the solution was made of

glass, and had a diameter of 15·50 cms. The coefficient of surface-rigidity was calculated from the formula

$$n = \frac{\pi \left(r_2{}^2 - r_1{}^2\right) \mathrm{I}}{r_1{}^2 \, r_2{}^2 \, \mathrm{T}^2}.$$

The omission of the term involving the logarithmic decrement is admissible even when the damping is excessive.

The vessel was placed on a graduated turntable, and the disk was set into motion by sharply turning the table through about half a degree. The oscillations were timed by observing the end points since, of course, the centre of oscillation was indeterminate. A stop-watch reading to $\frac{1}{5}$ sec. was used.

Method of Experiment.—The disk was cleaned by polishing in the lathe with fine emery and afterwards wiping with a clean cloth. The glass vessel was cleaned by treatment with hot chromic acid, followed by repeated washings with distilled water. The disk was suspended and allowed to attain a state of steady oscillation. The vessel was placed centrally on the turntable and adjusted concentric with the disk. The solution was then run in from a large separating funnel—a little having previously been run to waste in order to ensure a fresh surface—care being taken to avoid the formation of bubbles on the surface. During this operation the motion of the disk was observed through the telescope.

Six solutions were used, containing one part of saponine in 10^2, 10^3, 10^4, 10^5, 10^6, 10^7 parts of water respectively. In the case of the first three solutions, the slow oscillatory motion was stopped immediately the solution came into contact with the disk. Before coming to rest the disk generally executed a few rapid oscillations of small amplitude. These oscillations could be set up by rotating the turntable. In the case of fresh surfaces it was never possible to observe more than four oscillations, so that the time of oscillation could only be measured very roughly. Different experiments carried out with these three solutions gave for the initial period values ranging from 2·2 to 2·5 sec., corresponding to absolute values of the surface-rigidity varying from 83 to 65 dynes per cm. This variability is not surprising when we consider, first, that it is difficult to measure the time of oscillation accurately; and, secondly, that we are dealing not with a perfectly fresh surface, but with one a few seconds old, and during this initial period the elasticity is increasing very rapidly, so that the value obtained will depend upon the exact moment of taking the observation. It is probable, too, that the manner of formation of the surface has some

influence on the initial elasticity. Some of the earlier experiments seemed to indicate that the initial elasticity was slightly greater if the liquid was poured out so as to splash and form bubbles on the surface, than if it was poured out with the minimum of mechanical disturbance, but this result was not satisfactorily verified by subsequent experiments.

No difference in the initial elasticity between the concentration limits 1 in 100 and 1 in 10,000 could be detected. Between the concentrations 1 in 10,000 and 1 in 100,000 a rapid change occurs. Experiments carried out with solutions of intermediate strengths gave the following results :—

TABLE I.

Concentration: 1 part in	T (initial).
15,000	2·5 to 3 sec.
16,000	3 sec to ∞ .
17,000	Motion oscillatory of the first type with excessive damping.
20,000	

It does not follow that any specially marked change occurs between the concentrations 1 in 15,000 and 1 in 17,000. The point at which the effect of the surface changes from elastic to viscous depends upon the dimensions of the apparatus. Thus, by using a vessel with a diameter of 8 cms., all the above solutions show an initial elastic effect. By using a disk with a very small moment of inertia and having the annular space between the disk and vessel narrow, an initial elastic effect can be obtained with solutions much more dilute than the above. There is a limit, however, in practice to the effectiveness of such arrangements, due to the fact that a certain amount of liquid moves with the disk and surface, so that however light the suspended system there is a lower limit to the inertia of the total moving system. In the case of the 1 in 100,000 solution, it is impossible to obtain an initial elastic effect even with the most favourable arrangements. The surface has, however, a marked initial viscosity.

The excessive rapidity of formation of the elastic surface in the case of the stronger solutions is shown by the fact that it is impossible to destroy, or even temporarily impair, the elasticity by mechanical means such as stirring the liquid with

a glass rod. In fact, the only effect of such an operation is
to set up oscillations of the second type.

In the case of the 1 in 10^5 and 1 in 10^6 solutions, there was
a rapid increase of superficial viscosity on standing, and the
surface gradually developed an elastic effect as previously
described. The 1 in 10^7 solution showed no trace of elasticity,
or even of marked viscosity, after 24 hours. The properties
of elastic surfaces formed after prolonged standing will be
next investigated.

The Elasticity of Long-exposed Surfaces.

After prolonged exposure of the surface the elastic forces
attain a high degree of permanence. In many cases the disk
may be made to oscillate for about 20 seconds. At this
stage the surface cannot be described as viscous. For small
motions which do not shear it beyond its elastic limit, the
surface behaves as an elastic rather than a viscous body. It
is only under excessive shear that the surface shows a viscous
effect.

On measuring the surface elasticity at intervals extending
over several days, it was found that after the initial period
of rapid increase there were considerable fluctuations during
the course of the day. The elasticity was generally greatest
in the morning and decreased during the day. These
fluctuations were most pronounced if the temperature of the
room was much higher during the day than during the night.
Further investigation of this point showed that these fluctua-
tions were not primarily due to temperature changes. The
decreases which occurred during the day were found to be
due to the condensation of water-vapour from the air of the
room on the surface. Thus the effect was very marked if
after a cold night the room was warmed and occupied by a
number of persons so that the air became saturated with
moisture at a temperature much above that of the solution.

In order to eliminate this effect a special form of apparatus
was designed. This is shown in the accompanying diagram
(fig. 2). A is a bell-jar 35 cms. high and 18 cms. in diameter.
From a short piece of brass rod passing through the centre of
the stopper B was hung, by means of a thin steel wire, the
rod carrying the disk and mirror. C is a wooden base with a
circular groove into which the mouth of the bell-jar could be
fitted. A steel screw was fixed through the centre of the
base. To the upper end of this was fixed a circular piece of
wood, D, which served as a platform for the vessel. To the
lower end was soldered a horizontal radial arm, E, which
projected slightly from under the base which was supported

on three levelling-screws. The bell-jar was luted to the base
by means of mercury. In the case of long-exposed surfaces,
oscillations could be set up by merely pressing in a horizontal
direction against the end of the arm E, which was slightly

Fig. 2.
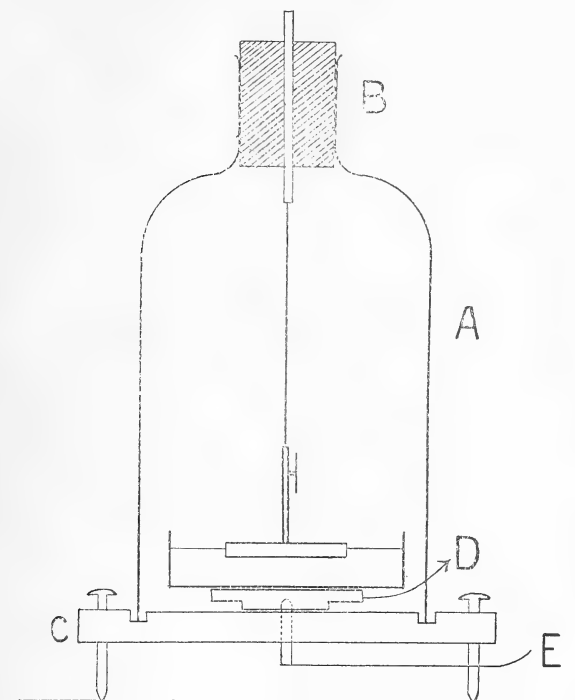
flexible, without visibly moving it. Owing to the fragility of
the surface this method was preferable to actually rotating
the platform through a comparatively large angle. The
motion was observed by focussing a short-focus telescope on
the reflexion of the filament of an incandescent electric lamp.
By this means several well-defined lines were obtained, in
spite of the thickness and curvature of the glass; and by
observing the motion of one of these relative to the dust-
particles in the eyepiece of the telescope, the end-points of
the oscillations could be quite accurately noted.

The condensation of moisture on the surface was prevented
by placing a number of shallow vessels containing calcium
chloride on the base. A deeper vessel containing a little of
the drying agent was suspended by means of a loop of wire
from the stopper.

Experiments carried out in this apparatus showed that the continuous evaporation promoted by the calcium chloride had a marked effect. The fluctuations in the density were totally eliminated, the elasticity steadily increasing to a maximum much greater than any value attained when the solution was exposed to the air of the room.

In the following table are summarized the results of a number of experiments carried out with solutions of different strengths.

TABLE II.

Age of Surface (days).	Concentration : 1 part in							
	100.	500.	10,000.	100,000.	200,000.	10^6.	10^7.	10^8.
1	332	371	535	390	...	364	81	0
2	...	...	...	490	464	427	100	...
3	393	427	581	556	531	460	...	...
4	408		...	...	...	...	...	...
5	467		612	630	...	464	...	0
6	468		626	628	...	465	...	
7	474		629	626	533	471	186	
8	478		648			...	200	
12	523		648			...		
14	543		644			516		
17	574							
21	595							
24	597							
30	593							

The most striking fact illustrated by the above table is that increase of concentration beyond a certain point is unfavourable to the process of surface separation, causing a slight decrease in the maximum value of the surface-rigidity and a very great decrease in the speed with which the maximum is attained. There is a marked difference in this respect between a 1 per cent. and a ·0005 per cent. solution. The latter attains a well-defined maximum in three days, while in the case of the strong solution the increase proceeds very sluggishly and irregularly for a period of about three weeks before a maximum is reached.

The action of continuous evaporation in causing a higher degree of elasticity to be attained may be partially explained in the following way. When the solution is exposed to the air of a room whose temperature fluctuates, there will be certain periods during which water is evaporating from the surface and other periods when water-vapour is being condensed on the surface. When the elasticity has reached a stage when the increase is very slow, continued evaporation

will have very little effect. On the other hand, the conden-
sation of moisture on the surface will have a marked effect
in diminishing the surface elasticity, the magnitude of this
effect depending on the relation between the rates of forma-
tion of the water-film by condensation, and of the separation
of the saponine in the new surface-layer thus produced.
Hence the effect of alternating periods of evaporation and
condensation, even though the amount of evaporation is much
greater than the amount of condensation, will be to prevent
the surface from ever attaining the degree of elasticity
attained when the surface is kept " dry."

This effect was graphically illustrated by an experiment
carried out during the cold weather in the early part of
January. The above apparatus was used, but no drying
agent was employed. The nights were very cold and the
room was strongly heated during the day. In the morning
the inside of the bell-jar was covered with drops of water,
which had distilled over from the solution. During the day
the walls gradually dried, the moisture distilling back to the
now colder interior portions of the bell-jar, *i.e.* to the surface.
Corresponding to this to-and-fro distillation of moisture were
observed fluctuations in the elasticity. A 1 per cent. solution
was used, and the value of the coefficient of rigidity fluctuated
from about 400 dynes per cm. in the morning to about
350 dynes per cm. in the evening. The experiment was
conducted for several days, and there was no progressive
increase such as occurs when continuous evaporation is
promoted (see Table II.).

In the following table are given the values of the coefficient
of rigidity attained by different solutions after standing about
24 hours (*a*) exposed to the air of the room, (*b*) in the
apparatus.

TABLE III.

Concentration : 1 part in	*a.*	*b.*
10,000	404	535
1,000,000	92	390
10,000,000	0	81

It will be seen that the more dilute the solution the greater
the effect of continuous and rapid evaporation. It is probable
that in the case of very dilute solutions the effect is more

direct than that indicated above. The evaporation, if suffi-
ciently rapid, may maintain a permanent difference in con-
centration between the body of the liquid and the portions
near the surface, in spite of the tendency for diffusion, thus
giving the equivalent, from the point of view of surface
separation, of a stronger solution. This effect will only
operate to any large extent when increase of concentration
greatly increases the power of forming an elastic surface,
i. e. in the case of very dilute solutions.

The University, Leeds,
    Sept. 5, 1905.

XXVI. *Notices respecting New Books.*

Lehrbuch der Physik. Von O. D. CHWOLSON, Prof. Ord. an der
    Kaiserl. Universität zu St. Petersburg.   Dritter Band : *Die
    Lehre von der Wärme.*   Uebersetzt von E. BERG. Mit 259
    eingedruckten Abbildungen. Braunschweig : F. Vieweg und
    Sohn.   1905.   Pp. xi + 988.

THE translation of this excellent and thoroughly up-to-date text-
    book is nearing completion, and Vol. III. is now before us.
It is divided into fourteen chapters.   After a general introduction,
in Chap. I., on heat considered as a form of energy and its relation
to other forms, the author deals, in Chap. II., with thermometry,
including modern methods of measuring very high and very low
temperatures.   The next four chapters are devoted to the deter-
mination of expansion coefficients, specific heats, thermo-chemistry,
the laws of cooling, and thermal conductivity.   So far the
treatment follows, more or less, the usual arrangement adopted
in standard treatises on the subject.   The greatest amount of
originality is displayed in the remaining portions of the book,
which form more than one-half of the entire volume.   Here the
author deals (in Chapter VIII.) with the principles of thermo-
dynamics (and a more masterly exposition of the subject could
not be wished for), with the applications of thermodynamics to
various phenomena studied in the earlier chapters, and to changes
of physical state (Chaps. IX. to XI.).   In Chapter XII. we have an
account of the properties of saturated vapours and hygrometry.
Chapter XIII. deals with unsaturated vapours, the critical state,
and the theory of corresponding states.   The concluding
Chapter XIV. is devoted to the Phase Rule and the theory of
solutions.

Little need be added to the praise already bestowed upon this
comprehensive and remarkably lucid treatise.   To the advanced
student of physics it is a veritable treasure, and its detailed and
careful explanations, with its wealth of excellent illustrations,
particularly fit it for purposes of private study; while its numerous
and systematically arranged bibliographical references render it a
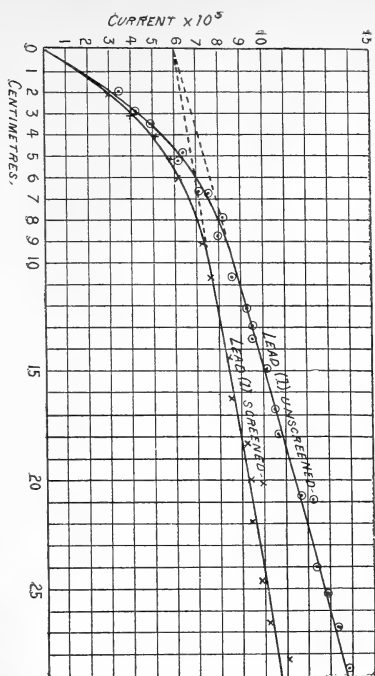most valuable work of reference to all interested in physics.

Fig. 2.

Fig. 3.
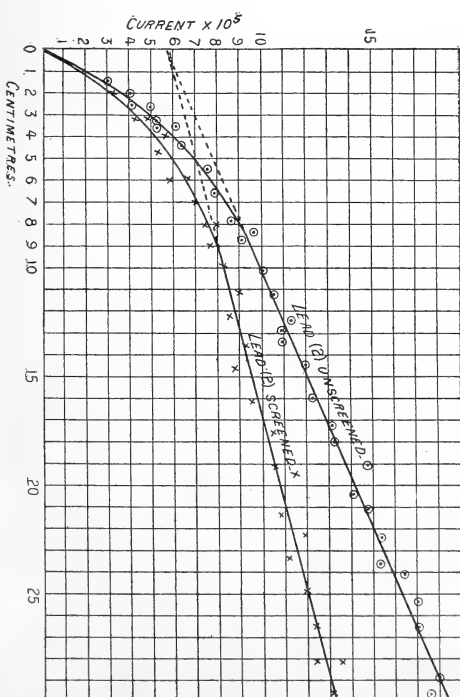
Fig. 2.
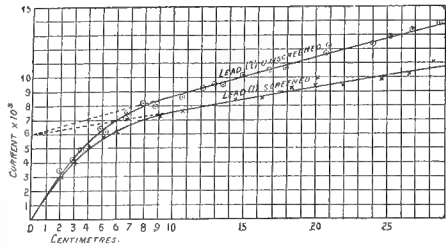
Fig. 3.
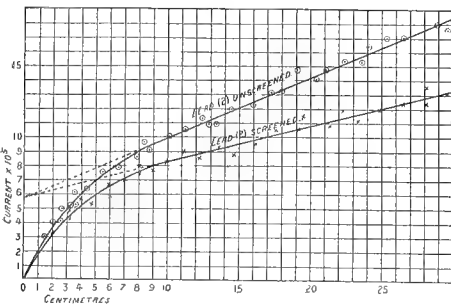
Fig. 4.
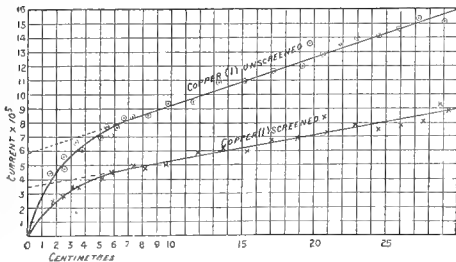
Fig. 5.
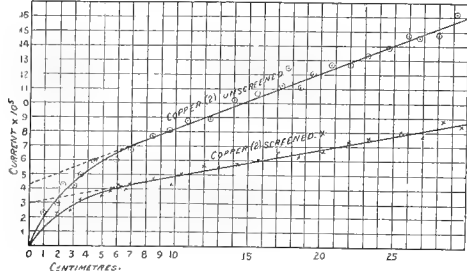
Fig. 6.
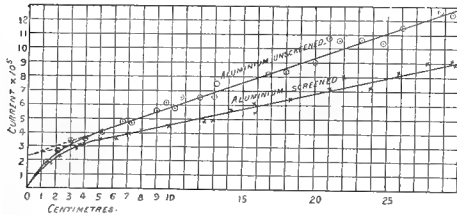
Fig. 7.
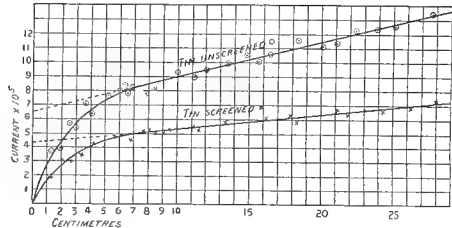
Fig. 8.
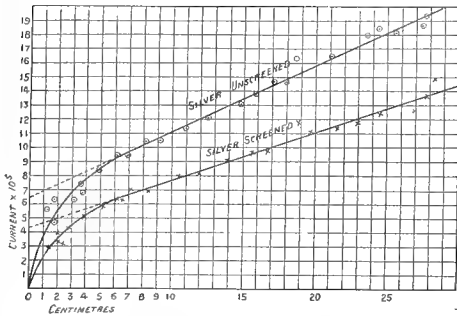
Fig. 9.
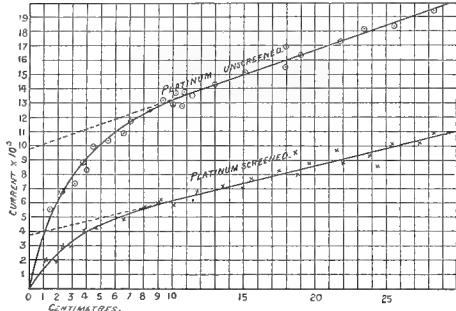
Fig. 10.
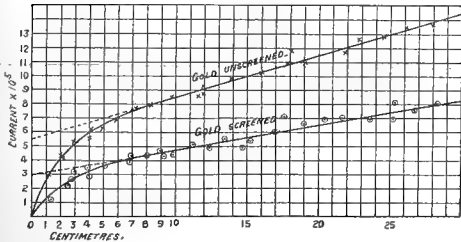
Fig. 11.
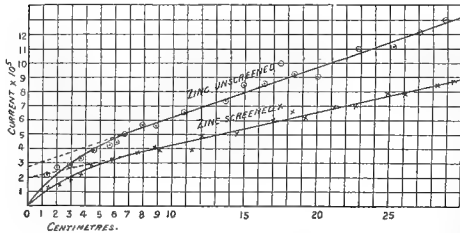
Fig. 12.
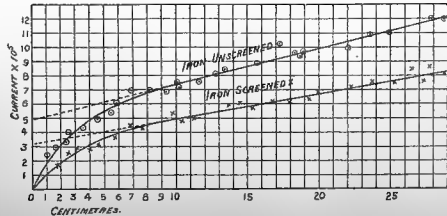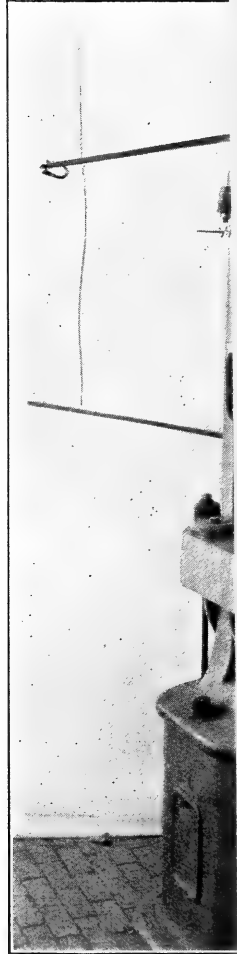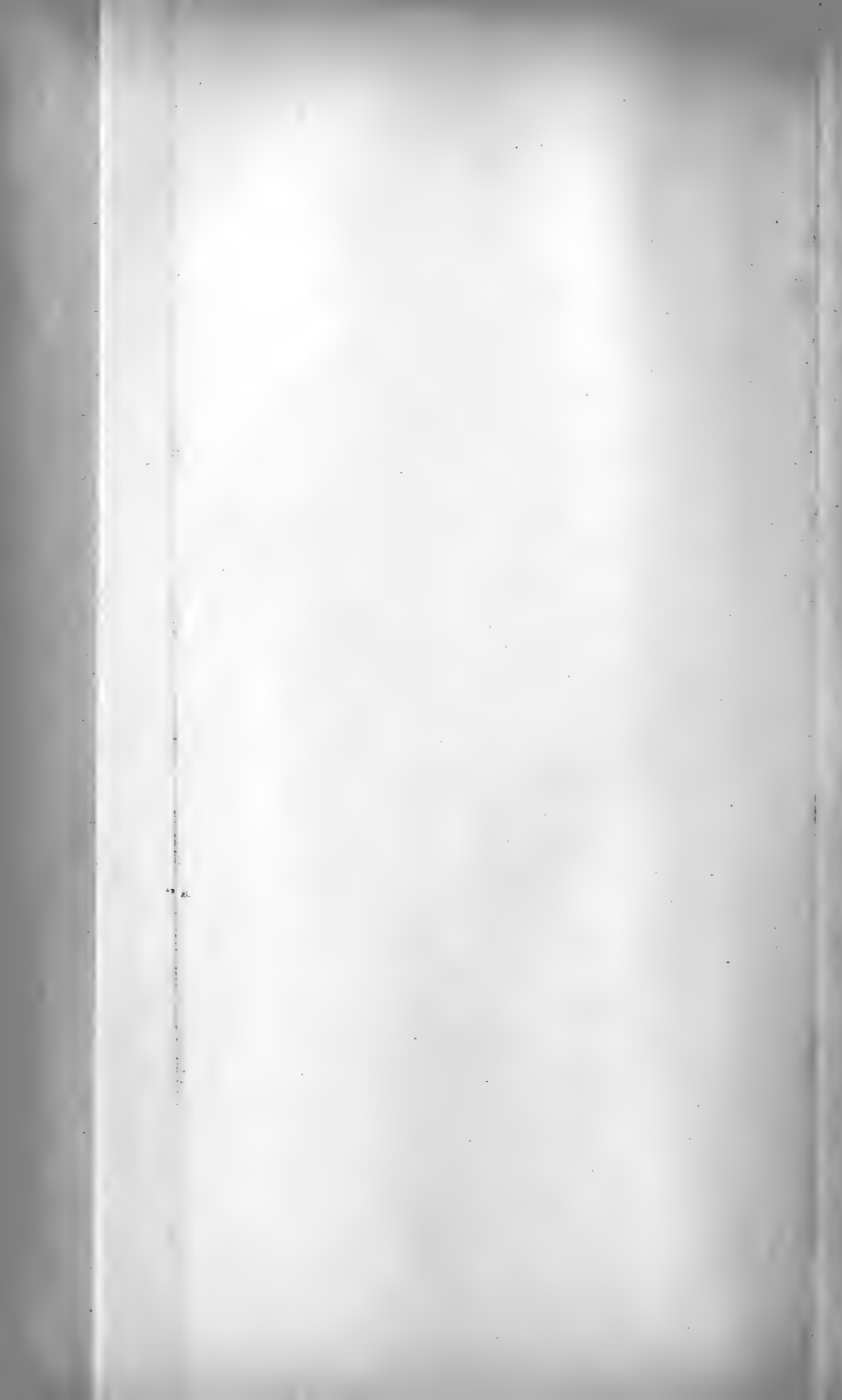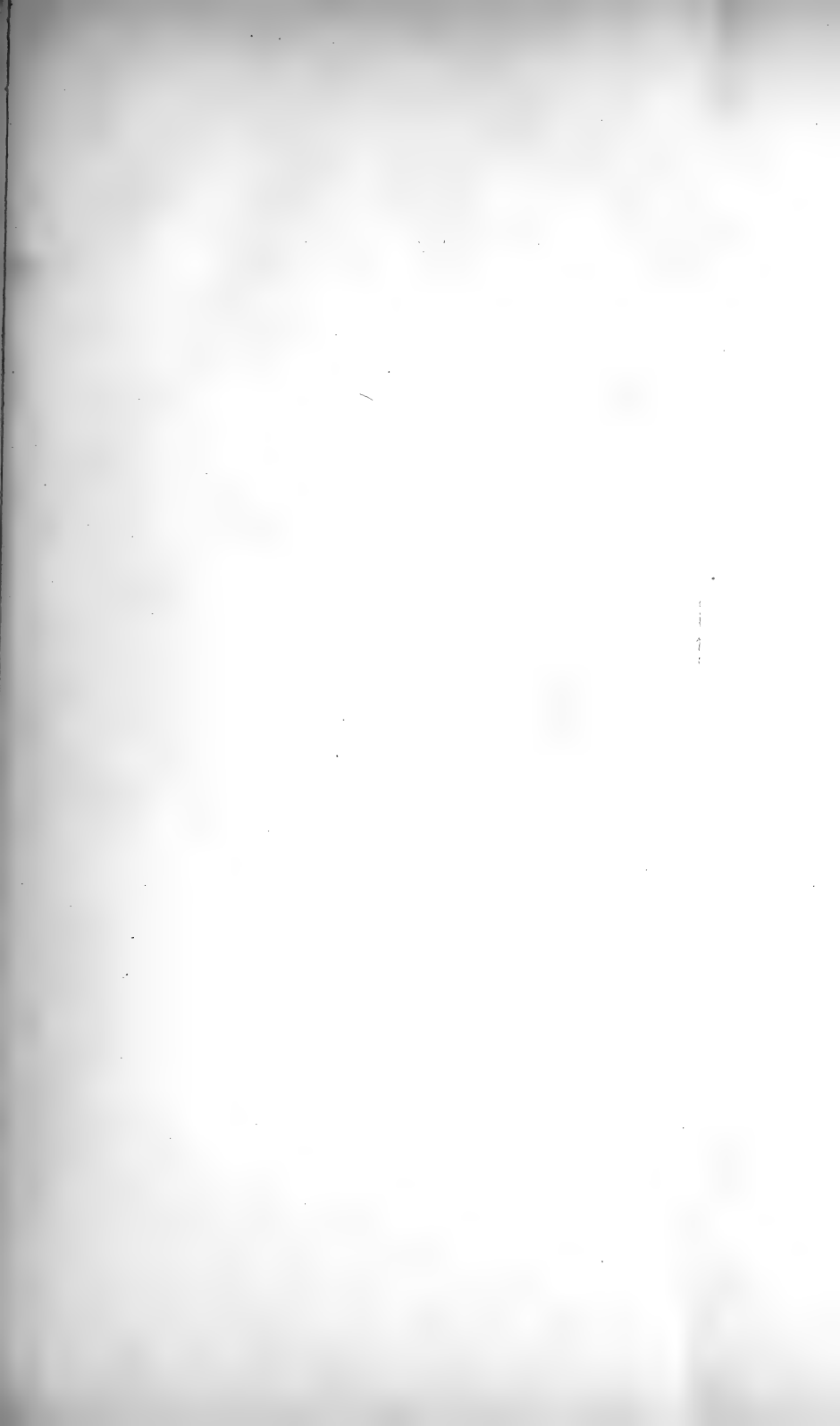
Fig. 1.
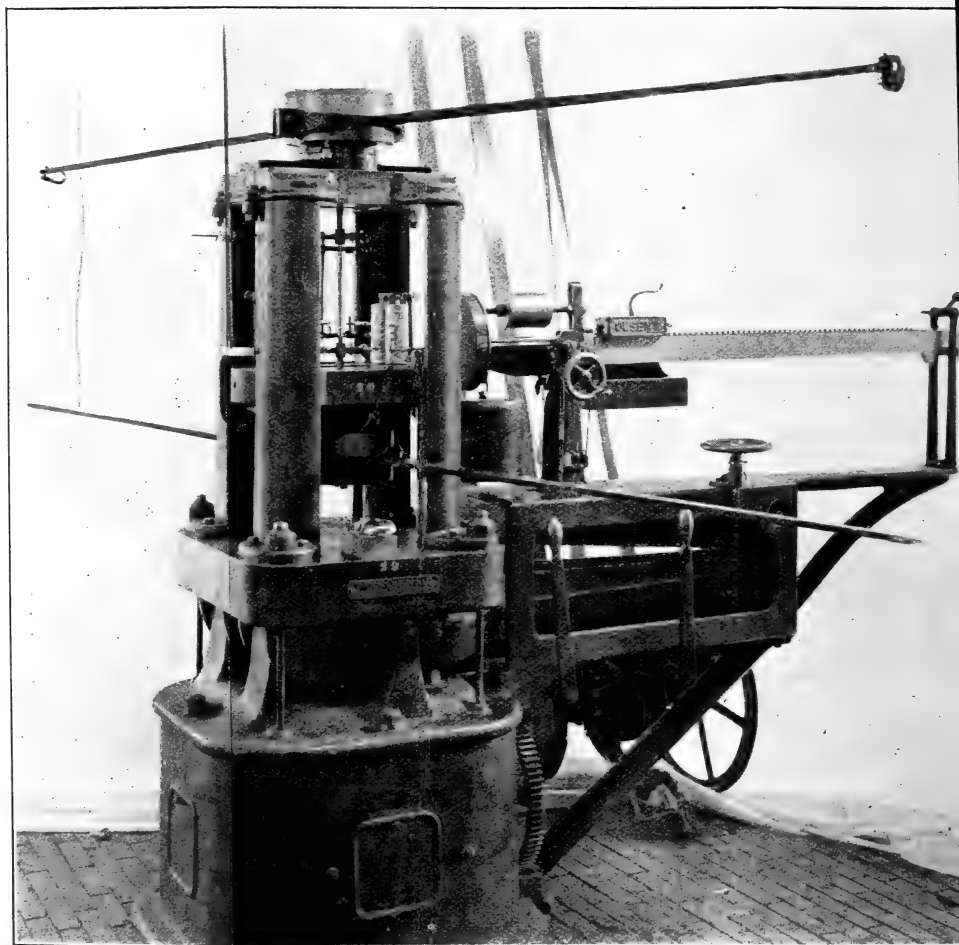
Fig. 2.

Chuck & Ball Bearing
[assembly.]
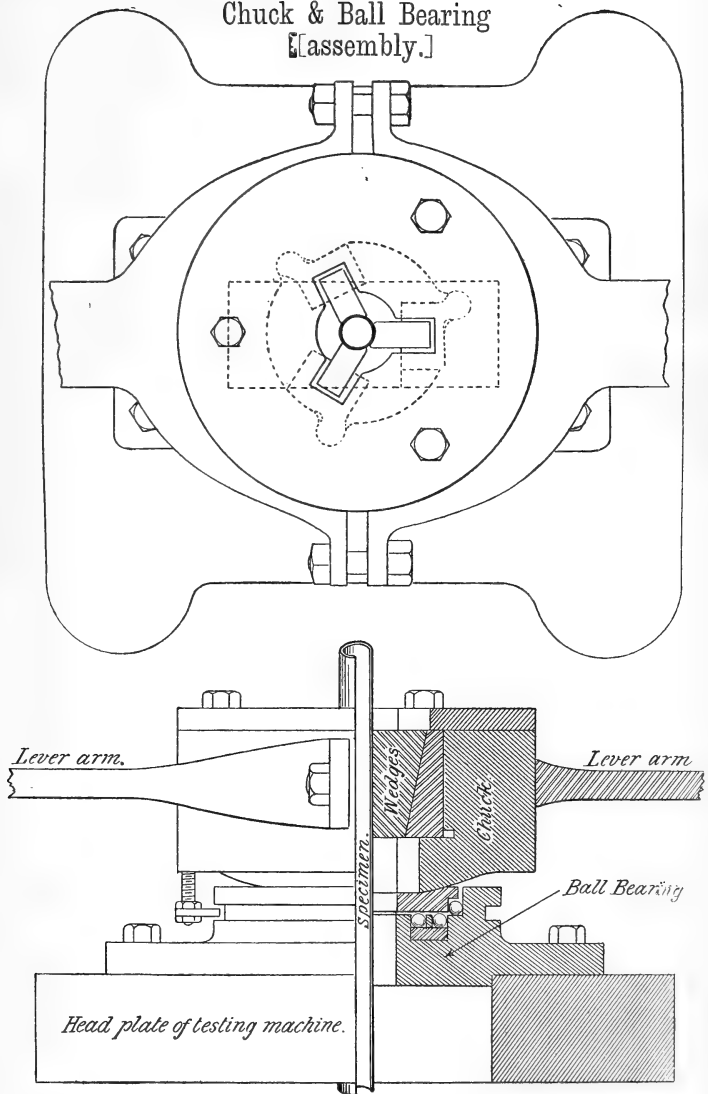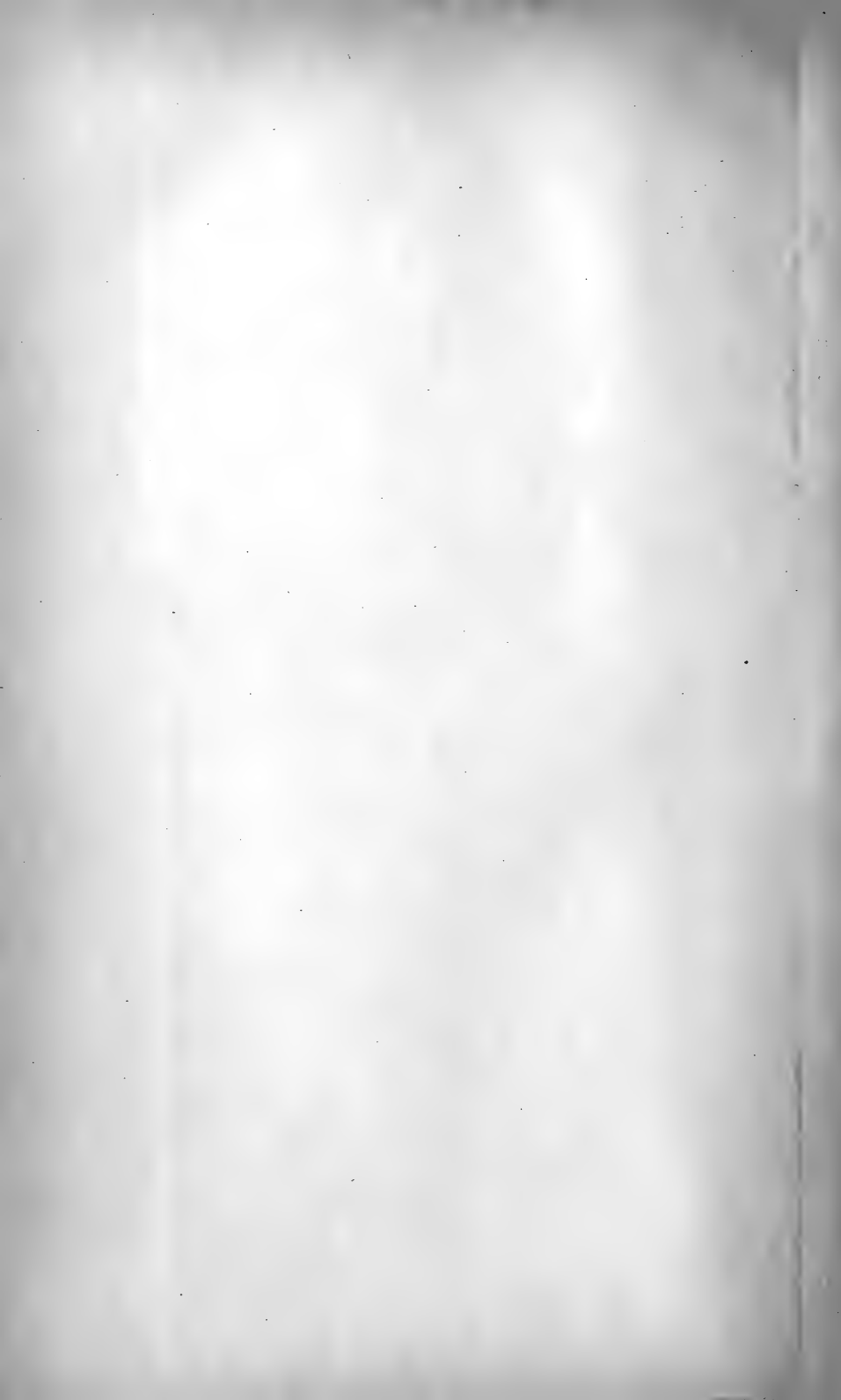
Lever arm.

Lever arm

Wedges

Chuck

Specimen.

Ball Bearing

Head plate of testing machine.

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

◆

[SIXTH SERIES.]

MARCH 1906.

XXVII. *On the Asymmetrical Action of an Alternating Current on a Polarizable Electrode.* By P. G. Gundry, *Ph.D.* (*Gött.*), *B.Sc.* (*Lond.*), *Assoc. R.C.Sc., late Assist. Lecturer and Demonstrator in Physics at University College, Cardiff*[*].

THE great advance made by Kohlrausch[†] in the study of polarization by the use of alternating currents led to an extended series of researches in the subject, of which especially may be mentioned those of Warburg[‡], E. Neumann[§], M. Wien[‖], Krüger[¶], and others. In all these investigations (with the exception of E. Neumann's) the so-called initial polarization capacity was considered, and the strength of the alternating current used was kept sufficiently small for this condition to be satisfied. There are, however, cases in which it is well-nigh impossible with measurable currents to keep within these limits, and E. Neumann has considered the effect of an extension beyond initial polarization capacity in the Wheatstone bridge.

From another side, however, in recent years the consideration of the action of an alternating current on a polarizable electrode has come into prominence, namely, in the

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[*] Communicated by Principal E. H. Griffiths, Sc.D., F.R.S. The greater part of this paper forms part of an Inaugural-Dissertation in the University of Göttingen.

[†] Kohlrausch, Pogg. *Ann.* cxlviii. p. 443 (1872).

[‡] Warburg, Wied. *Ann.* lxvii. (1899), and earlier papers.

[§] E. Neumann, Wied. *Ann.* lxvii. p. 500 (1899).

[‖] M. Wien, Wied. *Ann.* lviii. p. 37 (1896), and others.

[¶] F. Krüger, *Zeitsch. f. phys. Chemie*, xlv. p. 1 (1903).

so-called "Electrolytic Coherer." This instrument was introduced by Fessenden[*] and Schlömilch[†], the latter of whom made the first penetrating investigation in the subject. Schlömilch, as is well known, found that the current through an electrolytic cell, in which a very fine point of platinum formed the anode, and to which an E.M.F. of several volts was laid, was greatly increased by excitation with electric waves. He found a negative result when the point was a cathode.

The explanation of Fessenden, that this resulted from a warming of the electrolyte in the immediate neighbourhood of the small electrode, was proved untenable. Later experiments (of Rothmund and Lessing) show a coherer and not an anticoherer effect, when the electrolyte (*e. g.* phosphoric acid and hypophosphorous acid at higher temperatures) has a negative resistance temperature coefficient. The work of Reich[‡], in which he found a difference when sending a non-oscillating discharge of a Leyden-jar through the cell according to the direction of the discharge, led him to explain the phenomenon as a depolarization effect.

A great addition to the knowledge of the facts was made by Rothmund and Lessing[§]. In the first place, they showed that the coherer action was even relatively more pronounced when the applied E.M.F. was under that required for decomposition, and that under these circumstances the very small polarization current was increased when the point was a cathode as well as when it was an anode. The effect of the waves was, however, much greater for an anodically polarized point than for a cathodically polarized one. The magnitude of the effect was found to depend in very great measure on the sign and magnitude of the direct-current polarization. It was also shown that the E.M.F. of electric batteries, in which a small point, made of different metals, formed one of the electrodes, was altered by the action of electric waves.

On the theoretical side there is, so far as I know, no satisfactory explanation. The idea of the depolarization set forth by Rothmund and Lessing, by which one phase of the alternating current passes easily from electrolyte to electrode owing to oxidation, the other with difficulty, based on a comparison with the aluminium rectifier, appears to help but little to a clear understanding of the question. A satisfactory

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[*] Fessenden, *Elektrotech. Zeitsch.* xxiv. pp. 586 & 1015 (1903).
[†] Schlömilch, *ibid.* xxiv. p. 959 (1903).
[‡] M. Reich, *Phys. Zeitschr.* v. p. 338 (1904).
[§] Rothmund and Lessing, Drude *Ann.* xv. p. 193 (1904).

explanation must account for the very marked dependence on the degree of polarization of the electrode.

We know, however, very little indeed of the conditions which prevail in an oxygen- or hydrogen-charged platinum electrode, and the hope of obtaining a satisfactory explanation of the above phenomenon seems remote until this gap in our knowledge is filled.

The object of the present paper is to show that in a polarizable electrode in general (and the results obtained for the simplest of all cases, here treated, are applicable in principle to all polarizable electrodes), so soon as we pass beyond the stage of initial capacity, the alternating current gives rise to an asymmetry, which would appear as a coherer effect analogous to the above. In order that this asymmetry may be sufficient to account for the coherer effect used in practice, certain conditions must hold (afterwards to be developed), which, though not in antagonism with any known fact, and, indeed, corroborated to some extent by what is known of the conditions, must be proved or disproved by a greater experimental material than I am at present able to bring forward. With this reservation, however, I believe that the application to this practical case is not without interest.

The case here considered is that of a mercury electrode. The conditions are much better known than in the case of platinum, and the changes of concentration, taking place as they do, not in the electrode but in the liquid, are easier to follow. Another advantage that mercury offers, is that the electrode can always be set up in a perfectly definite condition, and with comparative ease in a pure state.

The experiments, to be afterwards described, consisted essentially in the sending of an alternating current of frequency varying between $80\smile$ and $5000\smile$ per second through an electrolyte between two electrodes, of which one was very small, the other more than a thousand times as large, so that the latter may be considered as unpolarizable in comparison with the former. A galvanometer in the circuit showed that the alternating current, before symmetric, was rendered asymmetric by the passage through the electrolytic cell. In other words, there was superposed on the alternating current a direct current, and the production of this direct current involves the existence of an asymmetry in the E.M.F. of the polarization produced by the alternating current.

This asymmetry seems, however, to be a natural consequence of the osmotic theory of Electromotive Force in general and its application to polarization in particular. According

Z 2

to this theory we arrive at the Nernst formula for the E.M.F. of polarization :

$$E = \frac{R\theta}{n} \log_e \frac{c_0}{c},$$

where R is the gas-constant, θ the absolute temperature, and n the valency of the ion concerned.

c_0 is the ionic concentration in the neighbourhood of the large electrode (which remains practically unaltered), and c the concentration of Hg ions in the layer of electrolyte immediately overlying the small electrode.

If now c oscillates symmetrically about a mean value, the logarithm, being an asymmetrical function, changes asymmetrically.

We have

$$E = \frac{R\theta}{n} \log_e \frac{c_0}{c + j \sin mt}$$

in the direction corresponding to a cathodic polarization. Without the alternating current

$$E = \frac{R\theta}{n} \log \frac{c_0}{c}.$$

The **E.M.F.** in this direction will be increased or diminished according as $j \sin mt$ is negative or positive. It follows, however, from the asymmetry of the function, that the increase when $j \sin mt$ is negative has a far greater effect on the polarization (E.M.F.) than the diminution when $j \sin mt$ is positive. This asymmetry falls away naturally when j is so small as compared with c that, in the expansion of the logarithm in series, only the first term need be retained, *i. e.* in the condition of initial capacity. So soon, however, as this condition is passed, the asymmetry comes into play, and this all the sooner and the more markedly the smaller c is. It follows, therefore, that the mean value of the E.M.F. is not zero, but that a direct current is produced, which flows from the small electrode to the large electrode in the solution. The asymmetry brings with it an effect equivalent to an apparent dilution of the salt in the neighbourhood of the small electrode, even when the mean concentration really remains unaltered (or even when the mean concentration is really increased by the alternating current, as will be shown sometimes to be the case).

Physically worded, this effect arises because a diminution from a small initial value of the concentration causes a much

greater E.M.F. alteration, than the same concentration increase.

So soon as the alternating current ceases to flow, the E.M.F. also instantaneously ceases to exist. There is no after-effect.

The quantitative consideration of this effect requires an hypothesis of the mechanism of polarization, and is intimately bound up with the nature of polarization capacity. In antagonism to what he called the " Ladungsstromtheorie " of Helmholtz, which looks on the action of the alternating current as the charging and discharging of the condenser-like double-layer, Warburg introduced what he called the " Leitungsstromtheorie," which disregards the part played by the double-layer, and recognizes, alone, the changes of ionic concentration which take place in the neighbourhood of the electrode and the diffusion which goes on simultaneously. These two points of view lead to different results, which may be tested by experiment: viz., according to the Helmholtz view, the polarization-capacity is independent of the frequency; according to the Warburg view, it varies as $\frac{1}{\sqrt{\text{Frequency}}}$, while at the same time the phase-alteration is different according to these two theories. Krüger, taking into account both these elements of polarization capacity, has shown that the one or the other predominates, according to the nature of the electrode and the concentration of the electrolyte. In dealing, however, with the asymmetry, it is clear that the Warburg form alone comes into consideration. The double-layer can introduce no asymmetry, and, indeed, the experiments of Krüger show that, in the cases considered, this part of the polarization capacity is small compared with the other.

The method I have here used is as follows :—An alternating current of pure sine form passes through the electrolyte. The resulting changes of concentration follow also an harmonic law, and the asymmetry of the resulting polarization, calculated by the Nernst formula, gives an E.M.F. in one direction, which causes the observed direct current. It may be objected, that the symmetry of the current and therefore of the concentration changes is disturbed by this very effect. The direct current observed, however, was always a small fraction of the alternating current, and the alteration of the ion-concentration may relatively without error be considered as due to the undisturbed alternating current. Of course a method of successive approximation may be applied; *i. e.* from the calculation I have made, a nearer approximation of the

actual concentration changes may be calculated, and so the E.M.F. of asymmetry reckoned once more; but this nearer approximation would make no essential alteration and would add seriously to the complication.

Following, now, the method of Warburg[*], we suppose

$i_0 \sin mt$ the alternating current,

c_0 the Hg-ion concentration in the neighbourhood of the large electrode (practically unaltered),

c the same in the neighbourhood of the small electrode.

c changes, partly through diffusion, and partly through the quantity of Hg-ions brought in and out of solution by the alternating current.

Let q = area of the small electrode,

and ϵ = the charge of electricity on a gramme-ion.

For the diffusion, we have :

$$\frac{\partial c}{\partial t} = \kappa \frac{\partial^2 c}{\partial z^2}, \quad \cdots \quad \cdots \quad (1)$$

where z is measured perpendicular to the small electrode (supposed a small plane), and κ is the velocity of diffusion.

The boundary conditions are:—

where $z = 0$,

$$\frac{i_0 \sin mt}{q\epsilon\kappa} = \left(\frac{\partial c}{\partial z}\right)_{z=0} \quad \cdots \quad \cdots \quad (2)$$

and where $z = \infty$, $\qquad c = c_0 \cdot \quad \cdots \quad \cdots \quad (3)$

The solution is :

$$c = c_0 + B e^{-z/\zeta} \cos\left(mt - \frac{z}{\zeta} + \phi\right)$$

where

$$\zeta = \sqrt{\frac{2\kappa}{m}}, \quad \phi = \frac{\pi}{4}, \quad B = \frac{\zeta i_0}{2q\epsilon\kappa \cdot \sin\phi} = \frac{i_0}{q\epsilon\sqrt{\kappa m}}.$$

Where $z = 0$, we have

$$c = c_0 + B \cos\left(mt + \frac{\pi}{4}\right) \quad \cdots \quad \cdots \quad (4)$$

The polarization is

$$p = \frac{R\theta}{n} \log_e \frac{c}{c_0}$$

$$= \lambda \left\{ \frac{c - c_0}{c_0} - \frac{1}{2}\left(\frac{c - c_0}{c_0}\right)^2 + \frac{1}{3}\left(\frac{c - c_0}{c_0}\right)^3 - \cdots \right\},$$

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[*] Warburg, *loc. cit.*

where
$$\lambda = \frac{R\theta}{n} = 0.0244 \text{ volt.}$$

$\dfrac{c - c_0}{c_0}$ is always less than 1.

The observed direct current I corresponds with an E.M.F. which is the mean value of p.

$$I = \frac{1}{W \cdot \dfrac{2\pi}{m}} \int_0^{\frac{2\pi}{m}} p\, dt, \quad \cdots \cdots \quad (5)$$

where W is the total resistance in circuit.
Therefore

$$I = \frac{\lambda}{W \cdot \dfrac{2\pi}{m}} \int_0^{\frac{2\pi}{m}} dt \left\{ \frac{B}{c_0} \cos\left(mt + \frac{\pi}{4}\right) - \frac{1}{2}\left(\frac{B}{c_0}\right)^2 \cos^2\left(mt + \frac{\pi}{4}\right) \right.$$
$$\left. + \ldots \ldots \right\}$$

Now

$$\int_0^{\frac{2\pi}{m}} \cos^{2p+1}\left(mt + \frac{\pi}{4}\right) dt = 0 ;$$

$$\int_0^{\frac{2\pi}{m}} \cos^{2p}\left(mt + \frac{\pi}{4}\right) dt = \frac{1}{2^{2p}} \cdot \frac{2p(2p-1)\ldots(p+1)}{p!} \cdot \frac{2\pi}{m}$$

where p is any integer.
Therefore

$$I = \frac{\lambda}{W} \left\{ -\frac{1}{2}\left(\frac{B}{c_0}\right)^2 \frac{1}{2^2} \cdot \frac{2}{1} - \frac{1}{4}\left(\frac{B}{c_0}\right)^4 \frac{1}{2^4} \cdot \frac{4 \cdot 3}{1 \cdot 2} \right.$$
$$\left. - \frac{1}{6}\left(\frac{B}{c_0}\right)^6 \frac{1}{2^6} \cdot \frac{6.5.4}{1.2.3} - \ldots \ldots \right\}. \quad \cdot \quad (6)$$

This series is convergent if $\dfrac{B}{c_0} < 1$. This condition is always fulfilled, as

$$\frac{B}{c_0} \cos\left(mt + \frac{\pi}{4}\right) = \frac{c - c_0}{c_0} < 1.$$

The series may be summed, but the summation is more easily obtained by direct integration.

As may easily be shown,

$$\int_0^{2\pi} \log(1 + a^2 + 2a\cos\theta) = 0,$$

and therefore

$$\int_0^{2\pi} \log(1 + k\cos\theta)\, d\theta = \int_0^{2\pi} \log(1 + a^2 + 2a\cos\theta)\,d\theta - 2\pi\log(1 + a^2)$$

$$= -2\pi\log(1 + a^2)$$

$$\left(\text{where } k = \frac{2a}{1 + a^2}\right)$$

$$= -2\pi\log\left(\frac{2}{1 + \sqrt{1 - k^2}}\right).$$

$$\mathrm{I} = \frac{\lambda}{2\pi\mathrm{W}} \int_0^{2\pi} \log\left\{1 + \frac{\mathrm{B}}{c_0}\cos\left(\theta + \frac{\pi}{4}\right)\right\}$$

$$= -\frac{\lambda}{\mathrm{W}} \log_e\left(\frac{2}{1 + \sqrt{1 - \frac{\mathrm{B}^2}{c_0^2}}}\right).$$

Taking once more the series (6), and neglecting $\left(\dfrac{\mathrm{B}}{c_0}\right)^4$ and higher powers,

$$\mathrm{I} = -\frac{\lambda\mathrm{B}^2}{4\,\mathrm{W}\,c_0^2} = -\frac{\lambda}{4\mathrm{W}}\frac{i_0^2}{q^2\epsilon^2\kappa m\, c_0^2}$$

$$= -\frac{\lambda}{8\pi\,\mathrm{W}\,q^2 c^2(\kappa c_0^2)}\frac{i_0^2}{\mathrm{N}}$$

where $\mathrm{N} = \dfrac{m}{2\pi}$ is the frequency.

Therefore

$$\frac{\mathrm{NI}}{i_0^2} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The direct current therefore flows from the small electrode to the large electrode in the electrolyte, and is directly proportional to the square of the strength of the alternating current, and inversely to the frequency.

It is clear that the direct current is the greater, the smaller the value of c is.

It is to be noted that the effect in this case is not a depolarization, but a virtual increase of the polarization.

Special treatment of the case of the
Solution of a Complex Salt.

It is necessary to treat of solutions of complex salts separately. The polarization capacity of a mercury electrode has been worked out in this case theoretically and experimentally by Krüger [*].

In the case of complex salts there is a third element which comes into play in addition to the double-layer of Helmholtz and the effect connected with diffusion, considered by Warburg. This is due to the fact that the mercury ions are formed from complex ions by dissociation when the current brings ions out of solution, and the mercury ions, when brought into solution, disappear to form complex ions, and these ionic reactions do *not take place with infinite velocity*.

Krüger considers three limiting cases:—

(1) The concentration of the mercury salt is very small. In this case the double-layer plays the most important part, as the ion quantity supplied by dissociation is small compared with that required to form the double-layer.

(2) The solution is of medium strength. With increase of concentration, the diffusion and dissociation come into importance. The influence of diffusion, especially that of the undissociated complex, is the important factor.

(3) The limiting case, where the concentration of the salt is very great. If the velocity of reaction were infinite, the electrode would be unpolarizable. This velocity is, however, even if very great, by no means infinitely great, and for frequencies so high as $1000\sim$ to $5000\sim$, probably the velocity of reaction is an important factor.

The three cases are characterized by the theoretical result that the polarization capacity in the first is independent of the frequency, in the second is proportional to $\dfrac{1}{\sqrt{N}}$, and in the third to $\dfrac{1}{N}$. Krüger gives cases in which these three limiting cases are realized or at any rate nearly approached.

In considering the asymmetrical action in the case of complex salts, we may put aside the first limiting case as presenting nothing new, being the same as with typical salts of small concentration.

Also in the second case, it is clear that the corresponding case with the solution of a typical salt is only so far altered, that the diffusion is that of the complex, and not that of the mercury ion. The finiteness of the velocity of reaction has

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* Krüger, *loc. cit.*

no importance in the case where diffusion plays the chief part; and therefore we may suppose that the mercury ions which come into solution practically all disappear instantaneously in the form of the undissociated complex, and with the same rapidity, the mercury ions required for the opposite phase of the current (from electrolyte to electrode) are supplied from the complexes.

Let us suppose that the equilibrium in a reaction

$$Hg_xA_y \underset{\longleftarrow}{\overset{\longrightarrow}{\rule{0pt}{1em}}} x \cdot Hg^{\bullet} + y \cdot A'$$

(where A' represents the anion, and Hg_xA_y the formula of the complex salt) is reached with infinite velocity, and corresponds to a very small mercury ion concentration.

Let c be the mercury ion concentration in the neighbourhood of the small electrode.

c_0 the same in the neighbourhood of the large electrode.

v, v_0 the corresponding concentrations of the undissociated complexes.

Then
$$v = \sigma c^x, \qquad v_0 = \sigma c_0^x$$

by the law of mass action, where σ is a large constant. The concentration of the anion is considered constant on account of the presence of an indifferent electrolyte with the same anion.

The equations now become

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial z^2}, \quad \cdots \quad \cdots \quad (1)'$$

where $z = 0$,
$$\frac{i_0 \sin mt}{q} = q\epsilon\kappa \frac{\partial v}{\partial z}, \quad \cdots \quad \cdots \quad (2)'$$

$$z = \infty, \qquad v = v_0. \quad \cdots \quad \cdots \quad (3)'$$

The equation $(2)'$ is arrived at by supposing that each ion which comes into the solution disappears, and x such ions correspond to one complex molecule.

As before, we have

$$p = \frac{R\theta}{n} \log \frac{c}{c_0} = \frac{R\theta}{nx} \log \frac{v}{v_0} = \frac{\lambda}{x} \log \left\{ 1 + \frac{B'}{v_0} \cos\left(mt + \frac{\pi}{4} \right) \right\},$$

where
$$B' = \frac{i_0}{q\epsilon v \sqrt{\kappa m}}, \quad \text{and} \quad \lambda = \frac{R\theta}{n},$$

$$p = \frac{\lambda}{x} \left\{ \frac{B'}{v_0} \cos\left(mt + \frac{\pi}{4} \right) - \frac{1}{2}\left(\frac{B'}{v_0}\right)^2 \cos^2\left(mt + \frac{\pi}{4} \right) + \ldots \right\}.$$

The direct current I in the direction from large to small electrode in the electrolyte is

$$\frac{1}{\frac{2\pi}{m}}\frac{1}{W.v}\int_0^{\frac{2\pi}{m}} p\,dt = \frac{\lambda}{W.v}\left\{ -\frac{1}{2}\left(\frac{B'}{v_0}\right)\cdot\frac{1}{2^2}\cdot\frac{2}{1} - \frac{1}{4}\cdot\left(\frac{B'}{v_0}\right)^4\cdot\frac{1}{2^4}\cdot\frac{4.3}{1.2} - \cdots\right\}$$

$$I = \frac{-\lambda B'^2}{4\,W.v_0^2.v},$$

if we neglect $\left(\dfrac{B'}{v_0}\right)^4$ and higher powers,

$$= \frac{-\lambda i_0^2}{4\,W.v_0^2 q^2\epsilon^2 x^3\kappa m} = \frac{-\lambda}{8\pi W.q^2\epsilon^2 x^3\kappa v_0^2}\frac{i_0^2}{N};$$

$$\therefore\quad \frac{NI}{i_0^2} = \text{const.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)'$$

where $N = \dfrac{m}{2\pi} =$ the frequency.

We are, of course, led to the same relation as before. The magnitude of the effect depends upon the smallness of the concentration of the complex substance.

With the third case, however, there is a marked difference. If the concentration of the mercury salt is very great, the changes of concentration that take place are negligible. If the velocity of reaction were infinite, there would be no effect —indeed, no polarization would be possible. This is, however, not the case; and changes take place in the concentration of the mercury ions, though not of the undissociated part.

Following Krüger, we take the case of a mercury salt HgA, which is dissociated in very small quantity, according to the equation

$$HgA \; \underset{\longleftarrow}{\overset{\longrightarrow}{}} \; Hg^{\cdot} + A'. \quad . \quad . \quad . \quad . \quad (8)$$
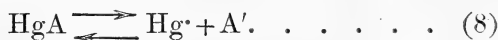
If, now, c is the concentration of the Hg ions and c' that of the undissociated part, then we have the equation of chemical kinetics :

$$-\frac{dc}{dt} = k_1 c^2 - k_2 c', \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where k_1, k_2 are the velocities in the two respective directions of the reaction. A certain very thin layer of thickness δx is supposed, in which the polarization phenomena take place.

The quantity of Hg ion which is brought into the solution by the alternating current in every moment must be equal to that which disappears according to the equation (9).

In this way, Krüger obtains the formula for the polarization:

$$p = \frac{R\theta}{2n} \cdot \log_e \left(1 + \frac{i_0}{\epsilon q \cdot \delta v \cdot k_1 c_0^2} \sin mt\right),$$

where $i_0 \sin mt$ is the alternating current,
$\quad q$ the area of the electrode,
$\quad \epsilon$ the electric charge on a gramme-ion,
and $\quad c_0$ the concentration of the Hg ions in the neighbourhood
$\qquad$ of the large electrode.

The asymmetry of the logarithm gives rise, once more, to
a mean E.M.F., corresponding to a direct current:

$$I = -\frac{R\theta}{2n. W} \left\{\frac{1}{2} K^2 \cdot \frac{1}{2^2} \cdot \frac{2}{1} + \frac{1}{4} \cdot K^4 \cdot \frac{1}{2^4} \cdot \frac{4 \cdot 3}{1 \cdot 2} + \ldots\right\},$$

where $K = \dfrac{i_0}{\epsilon q \cdot \delta v \cdot k_1 c_0^2}$.

The series is convergent, because $K < 1$.

If the alternating current is so weak that the asymmetry
just appears, we can neglect K^4, K^6, &c., and

$$\frac{I}{i_0} = -\frac{R\theta}{8nW} \left(\frac{1}{\epsilon q \, \delta v \cdot k_1 c_0^2}\right)^2 = \text{const.} \quad . \quad . \quad . \quad (10)$$

The direct current is, in this case, *independent of the
frequency.*

Disturbing Effect due to the Asymmetry of the
Current itself.

When the electrolyte was dilute sulphuric acid, it was clear
from the experiments that a disturbing effect of a totally
different nature was present, and that this masked that pre-
viously described. This effect showed itself, unlike the other,
in a permanent increase of the concentration in the neigh-
bourhood of the electrode. There was always an after-effect.
When the alternating current ceased to flow, the galvano-
meter *slowly* went back to zero.

I attribute this disturbing effect to the following cause.
Only a limited number of Hg ions are at hand to carry the
current from electrolyte to electrode. If the number imme-
diately surrounding the electrode does not suffice, it is only
by the slow process of diffusion that the defect may be
made up. The result is that part of the pulse of the current
is carried by hydrogen ions. Before the return pulse of the
current from electrode to electrolyte, some of the hydrogen

has been lost by diffusion. More mercury ions are brought into solution during the one phase of the current than were brought out of it in the other. Hence a real increase of the ionic concentration in the neighbourhood of the small electrode takes place.

It is to be remarked that this effect goes in the opposite direction to the first. Its existence is shown by the presence of an after-effect. Though it is not possible to separate the two effects, it is interesting to note which predominates. I found, with all the electrolytes that I tried, a certain cathodic direct-current polarization, at which the two effects exactly neutralized each other. For lower cathodic polarization (greater Hg-ion concentration) the first effect predominates, and a current flows from small electrode to large in the electrolyte ; and for cathodic polarization higher than this "Nullpoint" (*i. e.* with smaller Hg-ion concentration) the second effect predominates, and the current from the large electrode to the small electrode in the electrolyte is increased by the alternating current. It is clear that both these effects increase as the Hg-ion concentration is diminished, but the second increases more rapidly than the first.

I found the Nullpoint with H_2SO_4 saturated with Hg_2SO_4 at about 0·14 volt ; with KCN, on the other hand, at 0·36 volt; with K_2S at about 0·62 volt cathodic direct-current polarization.

In order to test the results obtained theoretically for the first and more definite effect, it was necessary to use an electrolyte, for which this second effect was inappreciable. Such a case I found to be KCN solution when no direct-current polarization was applied, as was shown by the very small after-effect on shunting out the alternating current.

That the second effect is unimportant in this case is to be attributed to the fact that here the mercury salt in solution is a complex one, and the excess of Hg ions brought into solution mostly disappear in the form of complex molecules, and therefore do not to any great extent effect an increase in the Hg-ion concentration. It may also in part be accounted for by the fact that the solution is alkaline owing to hydrolysis, and hence hydrogen ions are present only to a very small extent.

Experiments made with other electrolytes are described fully in another place *. I will here only bring forward the results with this electrolyte, to show to what extent the results previously obtained hold.

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* *Zeitschr. f. phys. Chemie,* liii. p. 177 (1905).

Apparatus and Experiments.

For the production of the alternating current, I used partly the sine-inductor of Kohlrausch and partly the Dolezalek alternating-current machine, made by Siemens and Halske. In the former, the rotation of a magnet in the form of a small disk inside a coil gives a current of very pure sine form. By use of different weights to drive the clockwork I could obtain currents of frequency 80∽ to 120∽ per sec. With the Dolezalek machine, in which the iron circuit of an electromagnet is almost completed and broken 120 times during each revolution of a wheel and a current induced in a secondary wound round the magnet, a current up to 5000∽ per sec. (10,000 pole changes) could be obtained.

For the measurements of alternating-current strength I used an electrodynamometer of the Kohlrausch type, which gave a double deflexion of 1 mm., read with telescope and scale at a distance of 1 metre, for a current 0.72×10^{-4} amp. For alternating currents too small to measure directly I measured the E.M.F. of the machine, using a measurable current and known resistances, and calculated from the resistances in use at any time the current strength.

The direct currents were measured by galvanometers of moving-coil type. In some cases a direct-reading galvanometer of sensitiveness 1 div. for 5.5×10^{-7} amp. was used, and in others a mirror galvanometer of sensitiveness 1 mm. (double throw) for 4.32×10^{-9} amp.

The electrolyte was contained in a small beaker. A layer of mercury, covering the bottom, formed the large electrode. The electrical connexion with it was made with a platinum wire sealed in a glass tube in the ordinary way.

Dipping in the solution, there was a capillary tube of the form shown in fig. 1. The small limb, about 3 cm. long, had an inner diameter of 2.14 mm. The wider limb had a diam. $1\frac{1}{2}$ cm. The mercury meniscus was from 2 to 3 cm. from the open end of the tube.

Fig. 1.

In some experiments the E.M.F. between the electrodes was directly measured with a Dolezalek quadrant electrometer. Its sensitiveness was 1 mm. for 0.0043 volt.
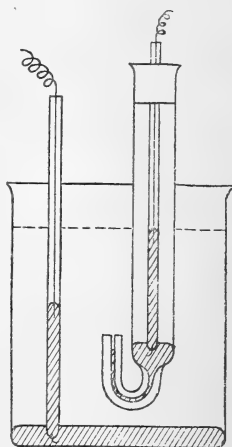
The arrangement is shown in fig. 2.

Fig. 2.
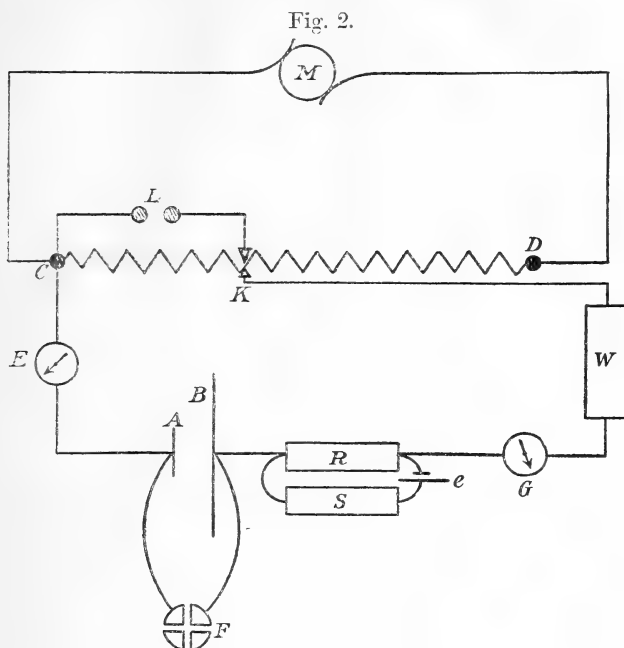
M is the alternating current machine.

E, electrodynamometer.

AB, the solution.

A, the small electrode.

B, the large electrode.

G, the galvanometer.

W, a high resistance.

R and S are resistances, and e a battery, so that an E.M.F. $\frac{R}{R+S}e$ could be included in the circuit. R + S was always 10,000 ohms.

F, the quadrant electrometer.

CD was a resistance-box which amounted to 11,000 ohms. The part CK could be 100, 200, &c. up to 1000 ohms ; or 1000, 2000, &c. up to 11,000 ohms. I could in this way obtain a calculable fraction of the E.M.F. of the machine.

L was a key, so that the alternating current could be shunted in and out of circuit without stopping the machine.

All the resistances were without self-induction. With higher frequencies the resistance W had a very high value (in most cases 100,000 ohms), so that the effective increase of the resistance of the galvanometer and electrodynamometer

due to self-induction could be neglected in comparison with the total resistance.

The experiments, shown in the following tables, were made with normal **KCN** solution without any direct-current polarization. The direct current observed was always in the direction in the electrolyte from small electrode to large. The deflexions of the galvanometer are shown in the third column. The measurements were made twice for each strength of alternating current, the first series with increasing current and the second with decreasing current; both are given to show the constancy of the effect. In the fourth column the direct current I is obtained from the mean of the two deflexions. In the last column $\dfrac{I}{i_0^2}$ is shown with the mean of the values given.

The value of the external resistance W was 100,000 ohms.

TABLE I.

Frequency.	Altern. Current. i_0.	Galvan. Defl.	Direct Current I.	$\dfrac{I}{i_0^2}$.
83	10^{-5} amp. 1·3	22·5 ; 22·1	10^{-8} amp. 19·34	1144 ⎱
	1·82	36 ; 35·8	31·04	940 ⎰ 966
	2·37	53·2 ; 52·6	45·8	814
181	1·59	8·1 ; 8·2	7·06	279·6
	2·23	15·0 ; 14·1	12·56	252·2 247·8
	2·88	23·3 ; 22·6	19·84	239·4
	3·59	33·9 ; ...	29·24	226·4
288	2·25	9·7 ; 8·2	7·68	151·8
	3·15	16·8 ; 15·1	13·72	138·8 138·2
	4·09	26·7 ; 25·7	22·6	135·2
	5·08	37 8 ; ...	32·62	126·8
442	3·10	10·6 ; 9·4	8·64	90
	4·36	17·3 ; 17·5	15·02	81·8 80·2
	5·64	28·3 ; 28·0	24·30	76·6
	7·00	41·0 ; ...	35·4	72·4
587	6·96	18·7 ; 17·1	15·46	31·84
	9·00	28·4 ; 28·1	24·30	29·98 30·02
	11·19	40·9 ; 40·8	35·26	28·22
1028	8·70	13·8 ; 12·8	11·50	15·22
	12·18	24·9 ; 24·5	21 30	14·40 14·00
	15·77	39·6 ; 39·0	33·98	13·70
	19·6	56·6 ; 56·4	48·8	12·72
3100	14·38	11·3 ; ...	9·76	4·72
	23·9	31·8 ; 30 2	26·8	4·70 4·32
	33·5	54·5 ; 54·6	47·2	4·20
	43·4	80·5 ; 79·8	69·2	3·68

All these experiments were done directly after one another.

Table II. shows the results of another set of experiments, all done together, but on another occasion.

The two sets of results are brought together, and the value of $\frac{NI}{i_0{}^2}$ is given in Table III.

<div align="center">TABLE II.</div>

Frequency.	Altern. Current i_0.	Galvan. Defl.	Direct Current I.	$\frac{I}{i_0{}^2}$	
	10^{-5} amp.		10^{-8} amp.		
1700	8·63	13·0; 12·1	10·82	14·52	
	14·31	30·2; 29·3	25·62	12·52	12·14
	20·1	56·7; 51·3	46·6	11·56	
	26·0	77·7; ...	67·2	9·96	
2290	18·18	51 ; 36·5	39·02	11·84	11·06
	25·4	77·3; 76·8	66·40	10·30	
3260	15·0	21 ; 19·5	17·48	7·76	
	25·0	48·3; 47·0	41·0	6·56	6·66
	34·9	80·7; 79·7	69·2	5·68	
5000	21·8	28·2; 26·7	23·62	4·96	
	28·8	45·6; 43·2	38·24	4·60	4·58
	36·1	63·3; 63·0	54·4	4·18	
	50·6	100·7; ...	80·4	3·14	

<div align="center">TABLE III.</div>

Frequency N.	$\frac{I}{i_0{}^2}$.	$\frac{NI}{i_0{}^2}$.
83	966	80200
181	247·8	44800
288	138·2	39800
442	80·2	35400
587	30·02	17500
1028	14·00	14380
3100	4·32	13400
1700	12·14	20600
2290	11·06	25300
3260	6·66	21500
5000	4·58	22820

The difference in value for the two sets shows that the condition of the solution is variable. I found in general that results could only be compared when they were obtained on the same day.

It is seen that only for high frequencies is the relation $\frac{NI}{i_0^2}$ = const. in any measure satisfied.

The sudden change in the magnitude of the effect between $N = 400$ and $N = 600$ appeared so striking, that I repeated the measurements on a new solution. The results are shown in Tables IV. and V. In this series of experiments, the resistance W was 30,000 ohms. For the sake of comparison, I have reduced the observed current I to the value I′ that it would have if the external resistance were 100,000 ohms, as in the former experiments.

TABLE IV.

Freq.	Altern. Current i_0.	Direct Current I′.	$\frac{I'}{i_0^2}$.	Freq.	Altern. Current. i_0.	Direct Current I′.	$\frac{I'}{i_0^2}$.
	10^{-4} amp.	10^{-8} amp.			10^{-4} amp.	10^{-8} amp.	
332	0·205	5·27	125·1	1362	0·563	6·57	20·7
	0·304	11·85	128·1		0·835	14·8	21·2
	0·400	15·9	99·4		1·099	20·6	17·08
518	0·268	5·75	80	2000	0·689	6·91	14·58
	0·398	12·47	78·9		1·02	14·35	13·80
	0·523	18·26	66·9		1·341	21·4	11·92
523	0·319	5·96	58·6	2930	0·835	7·44	10·67
	0·475	12·45	55·2		1·24	17·1	11·1
	0·625	18·9	48·4		1·63	24·2	9·12
758	0·399	6·05	38·0				
	0·591	13·3	38·1				
	0·779	19·1	31·6				
1006	0·466	6·48	29·8	534	0·311	6·75	70
	0·692	13·8	28 8		0·462	13·91	65·4
	0·91	20·2	24·4		0·608	20·2	54·7

TABLE V.

N.	$\frac{I}{i_0^2}$.	$\frac{I'}{i_0^2}$.	$\frac{NI'}{i_0^2}$.
332	126·6	38·0	12600
518	79·4	23·8	12300
523	56·9	17·08	8940
758	38·0	11·4	8940
1006	29·3	8·8	8810
1362	20·9	6·27	8550
2000	14·2	4·26	8510
2930	10·9	3·27	9550
534	67·7	20·3	10850

There is once more to be noticed a sudden change in the neighbourhood of frequency 500.

The absence of any approach to agreement for lower frequencies may be in part due to the fact that the disturbing effect is of greater importance when the frequency is low than when it is high. For a given value of the alternating-current strength, the lower the frequency the more mercury ions are required for the current. On this account the asymmetry of the current itself may be an important factor. The following tables show the direct current through 100,000 ohms as observed for different values of i_0 at low frequencies.

TABLE VI.   TABLE VII.

Frequency 89.			Frequency 118.		
Altern. Current.	Direct Current.	$\dfrac{I}{i_0^2}$.	Altern. Current.	Direct Current.	$\dfrac{I}{i_0^2}$.
amp. 10^{-5}.	amp. 10^{-5}.		amp. 10^{-5}.	amp. 10^{-5}.	
0·93	0·038	4400	1·89	0·0302	846
1·32	0·0506	2900	2·26	0·0336	659
2·28	0·0697	1340	2·82	0·0451	567
3·01	0·0936	1034	3·74	0·0630	450
4·65	0·1195	551	5·54	0·0922	302
5·80	0·1392	414	6·47	0·1075	257
8·37	0·1750	250	7·29	0·1158	218

TABLE VIII.

N = 93. Altern. Current.	Direct Current.	$\dfrac{I}{i_0^2}$.
amp. 10^{-6}.	amp. 10^{-8}.	
4·2	3·4	1950
4·67	4·18	1917
5·25	4·45	1615
6·1	5·71	1535
7·0	6·72	1372
8·4	8·05	1141
10·5	10·88	887

From Tables **VI.** and **VII.** it appears that the lower the

frequency the greater is the variation of $\dfrac{\mathrm{I}}{i_0{}^2}$ with i_0, and Table VIII. shows that by making the current smaller, the variation is greatly reduced. Were it possible to work with still smaller currents, it may well be that a much nearer approach to constancy of $\dfrac{\mathrm{I}}{i_0{}^2}$ at these low frequencies might be found.

Finally, Table IX. shows a test for an after-effect. A comparatively strong alternating current ($3\cdot05 \times 10^{-4}$ amp.) was run for half an hour through the cell, and readings of the galvanometer deflexion were made every 3 minutes. The alternating current was then shunted out and readings taken every $\frac{1}{4}$ minute. There was 40,000 ohms in the circuit.

TABLE IX.

Frequency 680.   Altern. current $3\cdot05 \times 10^{-4}$ amp.

Time. min.	Readings of Galvanometer.		Deflexion.
	Without Alt. Cur.	With Alt. Cur.	
0	677·3	......	......
3	......	475·5	201·8
6	......	485·5	181·8
9	......	478·0	199·3
12	......	482·9	194·4
15	......	488·7	188·6
18	......	497·6	179·7
21	......	507·0	170·3
24	......	507·1	170·2
27	......	513·9	163·4
30	......	514·7	162·6
$30\frac{1}{4}$.........	662·5	......	14·8
$30\frac{1}{2}$.........	672·4	......	4·9
$30\frac{3}{4}$.........	675·0	......	2·3
31	675·6	......	1·7
$31\frac{1}{4}$.........	676·1	......	1·2
$31\frac{1}{2}$.........	676·4	......	0·9
$31\frac{3}{4}$.........	675·2	......	2·1

Even after this long subjection to a strong alternating current, very slight after-effect is to be observed.

By using a capillary of wider bore I found the value of $\dfrac{\mathrm{I}}{i_0{}^2}$ to vary inversely as the fourth power of the diameter, which is also in agreement with the theoretical result.

The Effect with a Gas-charged Platinum Electrode in Sulphuric Acid.

I was not able to realize with complex salts of mercury the case where the effect, arising from the finiteness of the velocity of reaction, is independent of the frequency. It appears with KCN-solutions, from the approach to constancy of $\frac{NI}{i_0{}^2}$, that the velocity of reaction is sufficiently great for the effect due to diffusion to come first into play, even where the solution is comparatively strongly concentrated.

In a case where an asymmetrical effect is large at a very high frequency—at which diffusion must cease to play an important part—one is led, inversely, to consider that the effect, if it arises at all in the way above described, can only do so through the finiteness of some velocity of reaction. Such a case is that of the ordinary electrolytic coherer—a small gas-charged Pt electrode in dilute sulphuric acid.

In the first place, the effect of the above described asymmetry would be a virtual diminution of the gas charge, *i. e.* a depolarization such as is observed.

As for evidence of the reaction which takes place, the following work on the solution of hydrogen in palladium and platinum bears on the subject.

Hoitsema [*] found from the pressure-concentration curve of hydrogen in palladium, that the hydrogen is dissolved at ordinary temperatures in the atomic form and not in the molecular.

A. Winkelmann [†] found that hydrogen diffuses through glowing palladium as H, and not as H_2.

The same author [‡] found an exactly analogous result for hydrogen diffusing through hot platinum.

More recently, O. W. Richardson, Nicol, and Parnell [§] have shown that the dissociation of the hydrogen takes place *in* the platinum, and not outside it.

With regard to polarization experiments, it is interesting to note that M. Wien [||] found for palladium electrodes, covered with Pd-black and strongly charged with hydrogen, the result :—Polarization capacity $\propto \dfrac{1}{\text{Frequency}}$, which

* Hoitsema, *Zeitsch. f. phys. Chemie*, xvii. p. 1 (1895).
† A. Winkelmann, Drude *Annalen*, vi. p. 104 (1901).
‡ A. Winkelmann, Drude *Annalen*, viii. p. 388 (1902).
§ O. W. Richardson, Nicol, and Parnell, Phil. Mag. vii. (1904).
|| M. Wien, Drude *Annalen*, viii. p. 372 (1902).

corresponds with finiteness of velocity of reaction as worked out by Krüger. With platinized platinum electrodes charged with hydrogen, and with Pd electrodes less strongly charged, the result :—

Polarization capacity $\propto \dfrac{1}{\sqrt{\text{Frequency}}}$, which corresponds

with the case of diffusion, was found. The same result was found by Schönherr [*] for unplatinized Pt electrodes, which were kept charged by an applied direct-current E.M.F.

The evidence, therefore, for the reaction $H_2 = 2H$ taking place with finite velocity, and hence giving rise to polari-zation effects, is affirmative with palladium under certain conditions; but not so with platinum, where the effect corre-sponding with diffusion seems to hold.

It is, however, to be added that, at higher and higher frequencies the diffusion effect becomes of less and less importance ; and the effect due to the finiteness of velocity of reaction, though smaller than the diffusion effect at lower frequencies, would eventually at sufficiently high frequencies outweigh it.

Under other conditions with unplatinized platinum elec-trodes, the initial polarization capacity was found inde-pendent of the frequency. This, which corresponds with the Helmholtz double-layer, would introduce no asymmetry. It is thus seen that the nature of the initial polarization capacity may be no safe criterion as to the asymmetry that may arise at higher frequencies.

Experiments with a Platinum Electrode.

I have made experiments with platinum electrodes in 3 norm. solution of sulphuric acid of the same nature as those described with mercury electrodes. The large electrode was a piece of platinized platinum foil about 4 sq. cm. in area. A platinum wire $\frac{1}{10}$ mm. diameter was sealed in the end of a tube and then cut off flush with the glass. This formed the small electrode.

The arrangement and method were the same as before, except that both cathodic and anodic direct-current polari-zation were used. In order that the state of polarization of the small electrode might be determined, I measured the potential-difference between it and a third electrode of zinc, which dipped in a second beaker. This beaker was filled

* Schönherr, Drude *Annalen*, vi. p. 116 (1901).

with the same solution as the first, and an electrical connexion was made between the two solutions by means of an inverted U capillary-tube, also filled with the solution. The E.M.F. measurements were made according to the Poggendorf method. The E.M.F. is given in terms of the H electrode as 0·0 volt.

It was found necessary to polarize the electrode half an hour to an hour long before taking readings in order to obtain regular results. It was also found necessary to keep the alternating current small—otherwise the condition of the electrode appeared to be permanently altered.

The results are shown in Tables X. to XIII.

TABLE X.

Anodic Polarization.   E.M.F. of small electrode 1·7 volt.

Freq.	Altern. Current.	Direct Current.	$\frac{I}{i_0^2}$	Freq.	Altern. Current.	Direct Current.	$\frac{I}{i_0^2}$
	amp. 10^{-4}.	amp. 10^{-7}.			amp. 10^{-4}.	amp. 10^{-7}.	
1391	1·81	2·55	7·8	2600	2·44	2·45	4·11
	2·06	3·71	8·7		2·91	4·22	4·99
	2·33	4·82	8·9	4350	2·35	1·63	2·96
1737	2·27	3·15	6·11		3·08	3·90	4·12
	2·58	4·58	6·9		3·88	6·91	4·6
2020	2·26	2·68	5·25	1770	2·32	3·12	5·8
	2·66	4·24	6·0		2·64	4·23	6·07

TABLE XI.

Anodic Polarization.   E.M.F. of small electrode 1·36 volt.

Freq.	Altern. Current.	Direct Current.	$\frac{I}{i_0^2}$	Freq.	Altern. Current.	Direct Current.	$\frac{I}{i_0^2}$
	amp. 10^{-4}.	amp. 10^{-7}.			amp. 10^{-4}	amp. 10^{-7}.	
2070	4·41	0·104	0·0536	3550	4·62	0·225	0·105
2620	5·05	0·242	0·095		5·3	0·606	0·216
	5·59	0·58	0·186	4040	4·42	0·156	0·08
2990	5·11	0·363	0·139		5·17	0·501	0·188
	5·72	1·1	0·336		5·93	1·93	0·549
3340	5·02	0·277	0·107	2100	4·06	0·121	0·074
	5·66	0·783	0·246		4·49	0·225	0·115

TABLE XII.

Cathodic Polarization.   E.M.F. of small electrode 0·0 volt.

Freq.	Altern. Current.	Direct Current.	$\frac{I}{i_0{}^2}$	Freq.	Altern. Current.	Direct Current.	$\frac{I}{i_0{}^2}$
	amp. 10^{-4}.	amp. 10^{-7}.			amp. 10^{-4}.	amp. 10^{-7}.	
1300	1·93	1·29	3·46	2530	3·28	1·73	1·61
	2·18	2·2	4·59		3·76	2·55	1·80
	2·44	2·43	4·25		4·25	3·26	1·80
1504	2·21	1·495	3·05	2990	2·74	1·24	1·62
	2·50	2·15	3·44		3·29	2 22	2·05
1725	2·54	1·87	2·90		3·83	3·41	2·32
	2·88	2·44	2·95	3920	2·79	0·77	0·99
2060	2·66	1·53	2·16		3·49	2·1	1·94
	3·06	2·02	2·16	1400	2·34	1·64	3·0
	3·46	2·88	2·40		2·61	2·32	3·4

TABLE XIII.

Cathodic Polarization.   E.M.F. of small electrode 0·13 volt.

Freq.	Altern. Current.	Direct Current.	$\frac{I}{i_0{}^2}$	Freq.	Altern. Current.	Direct Current.	$\frac{I}{i_0{}^2}$
	amp. 10^{-4}.	amp. 10^{-7}.			amp. 10^{-4}.	amp. 10^{-7}.	
1380	2·88	1·76	2·12	3320	3·08	1·29	1·36
					3·67	2·6	1·93
1579	2·98	1·75	1·97				
	3·30	3·13	2·88	3620	2·64	0·745	1·07
					3·32	1·885	1·70
1819	3·10	1·75	1·82				
	3·46	2·54	2·18				
2320	3·03	1·24	1·35	4160	2·96	1·115	1·27
	3·47	2·34	1·94		3·74	2·75	1·97
	3·93	4·9	3·18				
2780	3·12	1·21	1·24	1429	2·70	1·071	1·47
	3·68	2·22	1·69		2·99	1·755	1·96

It appears from these results, that it is only when the electrode is practically saturated with the respective gas that there is any approach to constancy of $\frac{I}{i_0{}^2}$ for a given frequency, When the electrode is unsaturated, the direct current increases much faster than a quantity proportional to the square of the alternating current.

It is also to be seen that the diminution of the sensitiveness with increase of frequency is less than in the case of mercury.

Summary.

To summarize the results obtained in this paper :—

(1) It is shown that a perfectly symmetrical alternating current acting at a polarizable electrode leads, according to the osmotic theory of polarization, to an asymmetry of the polarization—in other words to a direct current.

(2) The case of a mercury electrode being worked out gives as a result that the direct current I satisfies the relation $\frac{NI}{i_0^2} = \text{const.}$, where N is the frequency and $i_0 \sin 2\pi Nt$ is the alternating current. The direction of the current corresponds with an effective diminution of the concentration in the neighbourhood of the electrode.

(3) The same relation should hold when the electrolyte is a solution of a complex salt, provided that diffusion from and to the layer of electrolyte surrounding the electrode plays the important part in the polarization.

(4) In cases, however, where the polarization arises (as described by Krüger) from the finiteness of the velocity of reaction by which mercury ions are supplied from the complex molecules, the relation $\frac{I}{i_0^2} = \text{const.}$ holds, being independent of the frequency.

(5) A disturbing effect which masks, in many cases, the above described one is accounted for, and the reasons assigned why in certain cases this disturbing effect is absent, or nearly so.

(6) Experiments, made with KCN solution, show to what extent $\frac{NI}{i_0^2} = \text{const.}$ is found to hold at high frequencies. At low frequencies the relation is totally departed from, and there appears to exist a critical frequency where a sudden change in the phenomenon takes place.

(7) An application to the "electrolytic coherer" is suggested on the supposition that the equilibrium in some reaction such as $H_2 = 2H$ is reached with finite velocity, and an effect corresponding with (4) above takes place.

(8) Experiments with a Pt electrode, both anodically and cathodically polarized, show that the magnitude of the direct-current effect varies at higher frequencies much less with the frequency than is the case with the mercury electrode.

XXVIII. *On the Lateral Vibration of Loaded and Unloaded Bars.* By JOHN MORROW, *M.Sc.* (*Vict.*), *M.Eng.* (*Liverpool*), *Lecturer in Engineering, University College, Bristol* *.

CONTENTS.

Section I. *Introduction and Notation.*

§ 1. A METHOD of calculating the frequency of the lateral vibration of bars has been described in a recent paper, " On the Lateral Vibration of Bars of Uniform and Varying Sectional Area" (see Philosophical Magazine, July 1905).

It is an important feature of this method that it gives, in a simple form, the equation of the elastic central line of the displaced bar, and thus provides data from which the stresses and strains in all parts may be readily calculated. The method lends itself to many cases of loaded bars which have not hitherto been solved, and the present paper, after dealing with some cases of unloaded bars under different end conditions, and bars of negligible mass carrying concentrated loads, gives more particularly the solutions for some important problems of loaded bars of appreciable mass.

It will be seen that a full consideration of the simpler cases treated first very materially lessens the labour involved in solving the more complex problems in which both the load and the mass of the bar itself are taken into account.

These solutions are, in general, obtained by a process of continuous approximation. Each approximation depends on the principle that, at any point in the length of the bar, the curvature is equal to the couple due to the reversed effective forces divided by the flexural rigidity.

To estimate the value of the couple a vibration-curve must be assumed. The above principle then gives an expression for the curvature at all points.

The process of continuous approximation to the exact solution is based on the fact that the expression for the deflexion, as obtained from that for the curvature, is a much closer approximation to the truth than is that originally assumed for the purpose of calculating the effective forces.

* Communicated by the Physical Society : read November 24, 1905.

A reference should be made here to the important papers by Professor Dunkerley [*] and by Dr. Chree [†] on the "Whirling" of Shafts. The relationship of the whirling to the vibrational problem was very clearly brought out by Chree, and under certain circumstances the two problems are identical.

§ 2. The notation used is similar to that of the paper previously cited; x and z are taken parallel, and y perpendicular, to the undisturbed position of the axis, bending occurring in the xy plane.

$E =$ Young's Modulus for the material (assumed homogeneous and isotropic),

$\rho =$ density of material of bar,

$\omega =$ sectional area,

$I = \omega k^2 =$ Geometrical Moment of Inertia of cross-section about the neutral axis,

$l =$ length or span of bar,

$U = \sqrt{\dfrac{E}{\rho}} =$ Velocity of transmission of longitudinal vibrations in the bar,

$y_1\, y_a$ &c. = displacements of given points in the length of the bar,

$M_0\, M_1 =$ bending couples required to fix the ends.

We have also

$$\frac{\ddot{y}_1}{y_1} = \frac{\ddot{y}}{y};$$

and for the frequency

$$N = \frac{1}{2\pi}\sqrt{-\frac{\ddot{y}_1}{y_1}}.$$

The value of N is, however, not always recorded, as it is sufficient to find the expression for $\dfrac{\ddot{y}_1}{y_1}$.

Section II. *Unloaded Bars.*

§ 3. *Uniform Bar Clamped at Both Ends.*—Taking the origin at one end, the terminal conditions are

$$x = 0,\ y = \frac{dy}{dx} = 0\,; \quad x = l,\ y = \frac{dy}{dx} = 0\,; \quad x = \frac{l}{2},\ y = y_1,\ \frac{dy}{dx} = 0\,;$$

* Phil. Trans. Roy. Soc. A 1894, p. 279.

† Phil. Mag. May 1904, p. 504.

and hence the equation to be assumed as the first approximation to the type of displacement is

$$y = \frac{16y_1}{l^2}\left(x^2 - \frac{2x^3}{l} + \frac{x^4}{l^2}\right).$$

The ordinary Euler-Bernoulli theory leads to

$$-\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{M}_0 + \rho\omega\int_0^x (x-z)\,\ddot{y}_z\,dz - \tfrac{1}{2}\rho\omega x\int_0^l \ddot{y}\,dx.$$

Substituting for y, integrating, and determining M_0 by the condition that $\frac{dy}{dx} = 0$ when $x = l$, we find

$$-y = \frac{\cdot 16\rho\omega\ddot{y}_1}{\mathrm{EI}l^2}\Big(\cdot 178{,}571\,l^4x^2 - \cdot 27\,l^3x^3 + \cdot 27\,x^6 - \cdot 238{,}095\frac{x^7}{l}$$
$$+ \cdot 059{,}524\frac{x^8}{l^2}\Big);$$

$$\therefore\quad \frac{\ddot{y}_1}{y_1} = -494\cdot74\frac{\mathrm{EI}}{\rho\omega l^4}.$$

Similarly, using these values of y and $\frac{\ddot{y}_1}{y_1}$, we arrive at a second approximation, in which

$$-y = \frac{\cdot 079158\,\rho\omega\ddot{y}_1}{\mathrm{EI}l^6}\Big(\cdot 35572\,l^8x^2 - \cdot 55113\,l^7x^3 + \cdot 49603\,l^4x^6$$
$$- \cdot 33069\,l^3x^7 + \cdot 05511\,x^{10} - \cdot 03006\frac{x^{11}}{l} + \cdot 00501\frac{x^{12}}{l^2}\Big)$$

and

$$-\frac{\ddot{y}_1}{y_1} = 500\cdot4\frac{\mathrm{EI}}{\rho\omega l^4},$$

agreeing well with the value of $500\cdot6$ obtained for the constant by the exact solution.

§ 4. *Clamped-supported Bar.*—In this case let y_1 be the displacement at such a distance from the fixed end (taken as origin) that $\frac{dy}{dx} = 0$ there. The initial type is then

$$y = 7\cdot6931\,y_1\left(\frac{3}{2}\frac{x^2}{l^2} - \frac{5}{2}\frac{x^3}{l^3} + \frac{x^4}{l^4}\right),$$

and

$$-\mathrm{EI}\frac{d^2y}{dx^2} = \frac{\mathrm{M}_0(l-x)}{l} + \rho\omega\frac{\ddot{y}_1}{y_1}\int_0^x (x-z)\,y_z\,dz - \rho\omega\frac{x}{l}\frac{\ddot{y}_1}{y_1}\int_0^l y(l-x)\,dx.$$

Integrating and determining M_0 and the arbitrary constants by the terminal conditions

$$-y = \frac{\cdot 076931 \, \rho\omega\ddot{y}_1}{EI\,l^4}(\cdot 565476 \, l^6x^2 - \cdot 744048 \, l^5x^3 + \cdot 416 \, l^2x^6$$
$$- \cdot 297619 \, lx^7 + \cdot 059524 \, x^8),$$

and

$$- \frac{\ddot{y}_1}{y_1} = 235\cdot 9 \frac{EI}{\rho\omega l^4}.$$

A small error in the value of x for which $y = y_1$ makes no appreciable difference in the final result.

The next approximation gives

$$-y = \frac{\cdot 018146 \, \rho\omega\ddot{y}_1}{EI\,l^8}(2\cdot 36692 \, l^{10}x^2 - 3\cdot 10194 \, l^9x^3 + 1\cdot 57077 \, l^6x^6$$
$$- \cdot 88577 \, l^5x^7 + \cdot 08267 \, l^2x^{10} - \cdot 03758 \, lx^{11} + \cdot 00493 \, x^{12}) \; ;$$

and putting
$$x = \cdot 5805 \, l,$$

$$- \frac{\ddot{y}_1}{y_1} = 238\cdot 0 \frac{EI}{\rho\omega l^4}$$

and
$$N = \frac{15\cdot 42}{2\pi} \frac{kU}{l^2}.$$

§ 5. *Clamped-Free Bar of Circular Section, Diameter varying as distance from Free End.*—Taking, as previously in similar cases, the origin at the free end,

$$\frac{d^2y}{dx^2} = \frac{\rho}{EI} \frac{\ddot{y}_1}{y_1} \int_0^x \omega y_z (x-z) dz.$$

Let
$$y_z = y_1 \left(1 - \frac{2z}{l} + \frac{z^2}{l^2}\right),$$

and the diameter $d = Ax$. Then

$$- \frac{d^2y}{dx^2} = \frac{16\rho\ddot{y}_1}{EA^2x^4} \int_0^x z^2(x-z)\left(1 - \frac{2z}{l} + \frac{z^2}{l^2}\right)dz,$$

$$-y = \frac{16\rho\ddot{y}_1}{EA^2}\left(\cdot 016 \, l^2 - \cdot 04 \, lx + \cdot 0416 \, x^2 - \cdot 016 \frac{x^3}{l} + \cdot 0027 \frac{x^4}{l^2}\right),$$

and
$$- \frac{\ddot{y}_1}{y_1} = 3\cdot 75 \frac{EA^2}{\rho l^2}.$$

In the second approximation,

$$-\frac{\ddot{y}_1}{y_1} = 4\cdot 5324 \frac{\mathrm{E A^2}}{\rho l^2} \; ;$$

and in the third approximation,

$$-y = \frac{\cdot 696176\,\rho\ddot{y}_1}{\mathrm{E A^2}\,l^4}\Big(\cdot 304931\,l^6 - \cdot 874742\,l^5 x + \cdot 957615\,l^4 x^2$$

$$-\cdot 542880\,l^3 x^3 + \cdot 192,901\,l^2 x^4 - \cdot 044091\,l x^5 + \cdot 006889\,x^6$$

$$-\cdot 000656 \frac{x^7}{l} + \cdot 000033 \frac{x^8}{l^2}\Big),$$

$$-\frac{\ddot{y}_1}{y_1} = 4\cdot 7106\frac{\mathrm{E A^2}}{\rho l^2}, \quad \text{and} \quad \mathrm{N} = \frac{2\cdot 170}{2\pi}\frac{\mathrm{A U}}{l\,l}\,.$$

Kirchhoff* has obtained the solution for the frequency in this case, and his result may be written

$$\mathrm{N} = \frac{2\cdot 179}{2\pi}\frac{\mathrm{A U}}{l}\,.$$

In the Euler-Bernoulli theory of beams it is assumed that the greatest diameter is small compared with the total effective length. When the diameter varies as the distance from one end, therefore, it is necessary that the variation should be small. In other cases, so far as the mathematical theory is concerned, the solution can only be looked upon as a probable approximation.

Section III. *Loaded Bars of Negligible Mass.*

§ 6. When the bar carries a load at some point in its length, and the mass of the bar itself is negligible, it is not necessary to assume a type of displacement. The method then becomes an exact one, and gives at once the true type and frequency of the vibration.

In the following paragraphs, *m* is the mass with which the bar is loaded, and *a* is the distance of its centre of gravity from the point chosen as origin.

§ 7. *Clamped-Free Bar.*—If the origin be at the load and y_1 be the displacement there,

$$-\mathrm{E I}\,\frac{d^2 y}{dx^2} = m\ddot{y}_1 x,$$

and the vibration-curve is

$$-y = \frac{m\ddot{y}_1}{\mathrm{E I}}\Big(\frac{x^3}{6} - \frac{l^2 x}{2} + \frac{l^3}{3}\Big),$$

* See *Berliner Monatsberichte*, 1879, p. 815.

where l is the distance between the load and the fixed end.

$$-\frac{\ddot{y}_1}{y_1} = \frac{3EI}{ml^3}.$$

On the other side of the load the bar is straight.

§ 8. *Bar supported at Both Ends.*—Origin at one end. Mass divides length of bar into segments a and b.

Here, for $x < a$,

$$EI\frac{d^2y}{dx^2} = \frac{b}{l}\,m\ddot{y}_a x\,;$$

whilst, for $x > a$,

$$EI\frac{d^2y'}{dx^2} = m\ddot{y}_a\left(\frac{b}{l}\,x - x + a\right).$$

The constants of integration, expressing the inclinations of the bar at its ends, are obtained from the consideration that both $\frac{dy}{dx}$ and y must have the same value in each equation when $x = a$. The inclinations are, therefore,

$$-\frac{m\ddot{y}_a}{EI}\frac{ab}{l}\left(\frac{a}{6} + \frac{b}{3}\right) \quad \text{and} \quad -\frac{m\ddot{y}_a}{EI}\frac{a}{l}\left(\frac{a^2}{6} + \frac{l^2}{3}\right).$$

Hence

$$y = \frac{m\ddot{y}_a}{6EI}\frac{b}{l}\,(x^3 - 2alx + a^2x),$$

and

$$y' = \frac{m\ddot{y}_a}{6EI}\frac{l-x}{l}\,(a^3 - 2alx + ax^2).$$

In either case,

$$-\frac{\ddot{y}}{y} = \frac{3EIl}{ma^2b^2}.$$

§ 9. *Clamped-Clamped Bar.*—With the same notation as in the preceding case.

If $x < a$,

$$-EI\frac{d^2y}{dx^2} = M_0 + (M_1 - M_0)\frac{x}{l} - m\ddot{y}_a\frac{bx}{l}\,;$$

and if $x > a$ we must add $m\ddot{y}_a(x-a)$ to the above.

By integration we find

$$M_0 = m\ddot{y}_a\frac{ab^2}{l^2} \quad \text{and} \quad M_1 = m\ddot{y}_a\frac{a^2b}{l^2},$$

$$-EIy = \frac{m\ddot{y}_a b^2 x^2}{6l^3}\,[3a(l-x) - bx]$$

and

$$-EIy' = \frac{m\ddot{y}_a a^2}{6l^3}\,[(a+3b)x^3 + 3(a-2b)lx^2 + 3l^2x - al^3].$$

In either case, $x = a$ gives

$$-\frac{\ddot{y}_a}{y_a} = \frac{3EI}{m}\left(\frac{l}{ab}\right)^3.$$

This result and those of §§ 7 and 8 are identical with the corresponding results obtained, otherwise, by Chree.

§ 10. *Supported Bar with two symmetrically placed Concentrated Loads.*—When there are two equal masses, placed at equal distances a from each end of a bar whose mass is negligible, an exact solution can be obtained in a precisely similar manner. Thus, for a bar supported at each end,

For $x < a$,

$$EI\frac{d^2y}{dx^2} = m\ddot{y}_a\,x.$$

$$EI\,y = m\ddot{y}_a\,x\left(\frac{x^2}{6} - \frac{la}{2} + \frac{a^2}{2}\right).$$

For $x = a$ to $(l-a)$,

$$EI\frac{d^2y'}{dx^2} = m\ddot{y}_a\,a,$$

$$EI\,y' = m\ddot{y}_a\,a\left(\frac{a^2}{b} - \frac{lx}{2} + \frac{x^2}{2}\right).$$

The curve between the two masses is a circular arc of radius $EI/m\ddot{y}_a\,a$, and the inclination of the extreme ends is $m\ddot{y}_a\dfrac{a}{2}(a-l)/EI$.

$$\frac{\ddot{y}_a}{y_a} = \frac{6EI}{ma^2(4a-3l)}.$$

§ 11. *Clamped-Clamped Bar with two Concentrated Loads.*—In the previous case, if the ends are clamped, we must subtract M_0 (*i. e.* the couple at each end required to fix the directions) from the expression for $\dfrac{d^2y}{dx^2}$.

We find

$$M_0 = m\ddot{y}_a\frac{a}{l}\,(l-a),$$

$$EIy = m\ddot{y}_a\frac{x^2}{2l}\left(a^2 - la + \frac{lx}{3}\right),$$

$$EIy' = m\ddot{y}_a\frac{a^2}{2l}\left(x^2 - lx + \frac{la}{3}\right),$$

and

$$-\frac{\ddot{y}_a}{y_a} = \frac{6EI\,l}{ma^3(2l-3a)}.$$

§ 12. *Dynamical Method.*—Other methods for the solution of some of the problems of this section have been used by Dr. Chree, but they depend on the assumption of a curve of vibration.

A correct type having been assumed, expressions are obtained for the Kinetic and Potential Energies of the system. Employing these in Lagrange's Equations of Motion the frequency is readily obtained.

Taking, for example, a massless bar carrying a load m, as in § 8. Assume

$$y = \eta bx (l^2 - b^2 - x^2),$$
$$y' = \eta ax' (l^2 - a^2 - x'^2),$$

measuring x and x' from opposite ends of the bar.

The Kinetic Energy

$$T = \tfrac{1}{2} m \dot{y}_a{}^2 - 2ma^4 b^4 \dot{\eta}^2.$$

The Potential Energy of Bending, V,

$$= \tfrac{1}{2} EI \left\{ \int_0^a \left(\frac{d^2 y}{dx^2} \right)^2 dx + \int_0^b \left(\frac{d^2 y'}{dx'^2} \right)^2 dx' \right\}$$

$$= 6EI a^2 b^2 (a+b) \eta^2.$$

The Lagrangian Equation

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\eta}} \right) - \frac{\partial T}{\partial \eta} + \frac{\partial V}{\partial \eta} = 0$$

becomes

$$4ma^4 b^4 \ddot{} + 12EI\, a^2 b^2 (a+b) \eta = 0 ;$$

$$\therefore \quad \frac{\ddot{\eta}}{\eta} = - \frac{3EI\, l}{ma^2 b^2}.$$

Section IV. *Loaded Massive Bars.*

§ 13. When the mass of the bar, in addition to that of the concentrated load, is taken into account, the expressions for the elastic curve and the frequency are more complicated. If, however, the position of the load and the ratio of the masses are given, the solutions are simple.

§ 14. *Clamped-Free Massive Bar, Load at End.*—In this case, if we assume

$$y = y_1 \left(1 - \frac{4}{3} \frac{x}{l} + \frac{1}{3} \frac{x^4}{l^4} \right),$$

the equation

$$-\mathrm{EI}\frac{d^2y}{dx^2} = \rho\omega\int_0^x \ddot{y}_z(x-z)dz + mx\ddot{y}_1$$

gives

$$-\mathrm{EI}y = \rho\omega\ddot{y}_1\Big(\cdot08194l^4 - \cdot11269841l^3x + \cdot0416x^4 - \cdot01\frac{x^5}{l}$$

$$+ \cdot00019841\frac{x^8}{l^4}\Big) + m\ddot{y}_1(\cdot16x^3 - \cdot5l^2x + \cdot3l^3)$$

$$-\frac{\ddot{y}_1}{y_1} = \frac{\mathrm{EI}}{\cdot08194\rho\omega l^4 + \cdot3ml^3}.$$

The next approximation gives

$$-\mathrm{EI}y = \frac{\rho\omega\ddot{y}_1 10^{-3}}{\cdot08194\rho\omega l^4 + \cdot3ml^3}\bigg\{\rho\omega\Big(6\cdot63084l^8 - 9\cdot12718l^7x$$

$$+ 3\cdot41435l^4x^4 - \cdot93915l^3x^5 + \cdot02480x^8 - \cdot00367\frac{x^9}{l}$$

$$+ \cdot00002\frac{x^{12}}{l^4}\Big) + m(26\cdot1905l^7 - 36\cdot1l^6x + 13\cdot8l^3x^4$$

$$- 4\cdot16l^2x^5 + \cdot1984x^7)\bigg\} + m\ddot{y}_1(\cdot3l^3 - \cdot5l^2x + \cdot16x^3).$$

Whence

$$-\frac{\ddot{y}_1}{y_1} = \mathrm{EI}\bigg\{\rho\omega l^4\frac{\cdot00663084\rho\omega l + \cdot0261905m}{\cdot08194\rho\omega l + \cdot3m} + \cdot3ml^3\bigg\}^{-1}$$

§ 15. *Supported Massive Bar, Load in any position.*—Let the load divide the bar into segments a and b. At each end

$$y = \frac{d^2y}{dx^2} = 0;$$

and when $x=a$, $y=y_a$.

Hence, for the first approximation

$$y = \frac{y_a}{\mathrm{A}}(l^3x - 2lx^3 + x^4),$$

where, for brevity,

$$\mathrm{A} = l^3a - 2la^3 + a^4.$$

For $x < a$,

$$\mathrm{EI}\frac{d^2y}{dx^2} = \frac{b}{l}m\ddot{y}_ax + \frac{\rho\omega}{2}\frac{\ddot{y}_a}{y_a}x\int_0^l y_z dz - \rho\omega\frac{\ddot{y}_a}{y_a}\int_0^x (x-z)y_z dz.$$

When $x > a$ we must add $-m\ddot{y}_a(x-a)$ to the above expression. Determining the constants of integration by

equating the values for $\dfrac{dy}{dx}$ and y when $x=a$, we get for $x<a$,

$$y=\frac{\ddot{y}_a}{\mathrm{EI}}\left\{\frac{m}{6l}(lx^3-ax^3-a^3x-2al^2x+3a^2lx)+\frac{\rho\omega}{\mathrm{A}}(-\cdot01011905l^7x\right.$$
$$\left.+\cdot016\dot{l}^5x^3-\cdot008\dot{3}l^3x^5+\cdot00238095lax^7-\cdot00059524x^8)\right\}.$$

Whilst for $x>a$ the first term in the brackets is

$$\frac{m}{6l}(la^3-a^3x-ax^3-2al^2x+3alx^2).$$

In either case

$$\frac{\ddot{y}_a}{y_a}=\mathrm{EI}.\mathrm{P};$$

where

$$\mathrm{P}^{-1}=\frac{ma^2l^2}{3l}+\frac{\rho\omega}{\mathrm{A}}(-\cdot01011905l^7a+\cdot016\dot{l}^5a^3-\cdot008\dot{3}l^3a^5$$
$$+\cdot00238095la^7-\cdot00059524a^8).$$

§ 16. In the second approximation, for $x<a$

$$\mathrm{EI}\frac{d^2y}{dx^2}=\frac{b}{l}\,m\ddot{y}_ax-\rho\omega\frac{\ddot{y}_a}{y_a}\int_0^x(x-z)y_zdz$$
$$+\rho\omega\frac{\ddot{y}_a}{y_a}\frac{x}{l}\left[\int_0^a y\,(l-x)dx+\int_a^l y'(l-x)dx\right].$$

If $x>a$, the second expression on the right hand must be written

$$-\rho\omega\frac{\ddot{y}_a}{y_a}\left[\int_0^a(x-z)y_zdz+\int_a^x(x-z)y_z'dz\right],$$

and the term $-m\ddot{y}_a(x-a)$ must be added.

By the method already indicated we find the inclinations of the ends to be

at $x=0$:

$$\frac{dy}{dx}=\frac{\ddot{y}_a}{\mathrm{EI}}\left[\frac{m}{l}\left(\frac{a^2l}{2}-\frac{a^3}{6}-\frac{al^2}{3}\right)+\frac{\rho\omega\mathrm{P}}{\mathrm{A}}\left\{\cdot00010386\rho\omega l^{11}\right.\right.$$
$$\left.\left.+\frac{m\mathrm{A}}{6l}\left(-\frac{17a^7}{420}+\frac{la}{30}+\frac{a^5l^2}{60}-\frac{a^3l^4}{45}+\frac{4al^6}{315}\right)\right\}\right];$$

at $x=l$:

$$\frac{dy'}{dx}=\frac{\ddot{y}_a}{\mathrm{EI}}\left[\frac{m}{l}\left(-\frac{a^3}{6}-\frac{al^2}{3}\right)+\frac{\rho\omega\mathrm{P}}{\mathrm{A}}\left\{\cdot00010386\rho\omega l^{11}\right.\right.$$
$$\left.\left.+\frac{m\mathrm{A}}{6l}\left(-\frac{17a^7}{420}+\frac{a^5l^2}{60}-\frac{a^3l^4}{45}+\frac{4al^6}{315}\right)\right\}\right]$$

and the equations of the central line

$$y = \frac{\ddot{y}_a}{EI}\left[\frac{mbx}{6l}(x^2 - 2la + a^2) - \frac{\rho\omega P}{A}\left\{\rho\omega 10^{-4}(-1{\cdot}0386l^{11}x + 1{\cdot}7085l^9 x^3\right.\right.$$

$$-{\cdot}8433l^7 x^5 + {\cdot}1984l^5 x^7 - {\cdot}0276l^3 x^9 + {\cdot}0030lx^{11} - {\cdot}0005x^{12})$$

$$+ \frac{mA}{6l}\left[(l-a)\frac{x^7}{840} - \left(\frac{a^2}{6} - \frac{al}{2} + \frac{l^2}{3}\right)\frac{ax^5}{20} - \left(\frac{a^4}{20} - \frac{a^3l}{4} + \frac{a^2l^2}{3} - \frac{2l^4}{15}\right)\frac{ax^3}{6}\right.$$

$$\left.\left.\left. - \left(\frac{la^6}{30} - \frac{17a^7}{420} + \frac{a^5l^2}{60} - \frac{a^3l^4}{45} + \frac{4al^6}{315}\right)x\right]\right\}\right].$$

$$y' = \frac{\ddot{y}_a}{EI}\left[\frac{ma}{6l}(l-x)(a^2 - 2lx + x^2) - \frac{\rho\omega P}{A}\left\{\rho\omega 10^{-4}(-1{\cdot}0386l^{11}x + 1{\cdot}7085l^9 x^3\right.\right.$$

$$-{\cdot}8433l^7 x^5 + {\cdot}1984l^5 x^7 - {\cdot}0276l^3 x^9 + {\cdot}0030lx^{11} - {\cdot}0005x^{12})$$

$$+ \frac{mA}{6l}\left[-\frac{ax^7}{840} + \frac{alx^6}{120} - \left(\frac{a^2}{6} + \frac{l^2}{3}\right)\frac{ax^5}{20} + \frac{a^3lx^4}{24} - \left(\frac{a^4}{20} + \frac{a^2l^2}{3} - \frac{2l^4}{15}\right)\frac{ax^3}{6}\right.$$

$$\left.\left.\left. + \frac{a^5lx^2}{40} + \left(\frac{17}{420}a^7 - \frac{a^5l^2}{60} + \frac{a^3l^4}{45} - \frac{4al^6}{315}\right)x - \frac{17}{420}a^7l\right]\right\}\right].$$

Putting $x = a$ in either of these, we get the following relation between y_a and $\ddot{y}_a$:

$$-\left(\frac{\ddot{y}_a}{y_a}\right)^{-1} = \frac{1}{EI}\left[\frac{ma^2b^2}{3l} + \rho\omega P\left\{(-1{\cdot}0386l^{11}a + 1{\cdot}7085l^9a^3 - {\cdot}8433l^7a^5\right.\right.$$

$$+ {\cdot}1984l^5a^7 - {\cdot}0276l^3a^9 + {\cdot}0030la^{11} - {\cdot}0005a^{12})\frac{\rho\omega 10^{-4}}{A}$$

$$\left.\left. + \frac{ma^2}{90l}({\cdot}3392857a^6 + {\cdot}5178572a^5l - 1{\cdot}3a^4l^2 + {\cdot}6a^2l^4 - {\cdot}1904762l^6)\right\}\right].$$

§ 17. In particular, if $a = \dfrac{l}{2}$,

$$A = \frac{5}{16}l^4, \quad \text{and} \quad P^{-1} = -\frac{ml^3}{48} - {\cdot}0103051\rho\omega l^4;$$

hence

$$-\frac{\ddot{y}_a}{y_a} = EI\left[\frac{ml^3}{48} + \frac{3{\cdot}306\rho\omega l + 7{\cdot}436m}{322{\cdot}038\rho\omega l + 651{\cdot}042m}\rho\omega l^4\right]^{-1};$$

and further, if the masses of the load and bar be equal,

$$-\frac{\ddot{y}_a}{y_a} = \frac{31{\cdot}375\,EI}{ml^3}.$$

§ 18. *Clamped-Clamped Massive Bar, Load in any position.*

The end conditions lead to the preliminary assumption

$$y = y_a\left(\frac{l}{ab}\right)^2\left(x^2 - \frac{2x^3}{l} \mp \frac{x^4}{l^2}\right).$$

For $x < a$,

$$-\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\frac{\ddot{y}_a}{y_a}\int_0^x (x-z)y_z\,dz$$

$$-\frac{\rho\omega x}{2}\frac{\ddot{y}_a}{y_a}\int_0^l y_x\,dx - m\ddot{y}_a\frac{bx}{l}.$$

If $x > a$, add $m\ddot{y}_a(x-a)$.

By putting $x = a$, as in previous cases, we find

$$\mathrm{M}_0 = m\ddot{y}_a\frac{ab^2}{l^2} + \cdot 00357143\frac{\rho\omega\ddot{y}_a\,l^6}{a^2b^2},$$

$$\mathrm{M}_1 = m\ddot{y}_a\frac{a^2b}{l^2} + \cdot 00357143\frac{\rho\omega\ddot{y}_a\,l^6}{a^2b^2}\,;$$

and hence for $x < a$,

$$-\mathrm{EI}y = \rho\omega\ddot{y}_a\left(\frac{l}{ab}\right)^2\left(2\cdot\dot{7}x^6 - 2\cdot38095\frac{x^7}{l} + \cdot59524\,\frac{x^8}{l^2} - 2\cdot\dot{7}l^3x^3\right.$$

$$\left. + 1\cdot78572\,l^4x^2\right)10^{-3} + m\ddot{y}_a\frac{bx^2}{6\,l^3}(3abl - l^2x - abx + a^2x)\,;$$

and for $x > a$ the latter portion of the expression is changed to

$$m\ddot{y}_a\frac{a^2}{6l^3}(3lx^3 - 2ax^3 - 6l^2x^2 + 3alx^2 + 3l^3x - al^3).$$

When $x = a$, either of the above gives

$$-\frac{\ddot{y}_a}{y_a} = \mathrm{EI}\,.\,\mathrm{P},$$

where

$$\mathrm{P}^{-1} = \frac{\rho\omega}{b^2}\,10^{-3}(1\cdot78572\,l^6 - 2\cdot\dot{7}\,al^5 + 2\cdot\dot{7}\,a^4l^2 - 2\cdot38095\,a^5l$$

$$+ \cdot59524\,a^6) + \frac{m}{3}\left(\frac{ab}{l}\right)^3.$$

§ 19. In the next approximation, for $x < a$:

$$\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\frac{\ddot{y}_a}{y_a}\int_0^x (x-z)y_z\,dz - \frac{\rho\omega}{l}\frac{\ddot{y}_a}{y_a}x\int_0^a y_x(l-x)dx$$

$$-\frac{\rho\omega}{l}\frac{\ddot{y}_a}{y_a}x\int_0^l y'_x(l-x)dx - \frac{m\ddot{y}_a vx}{l}\,;$$

$$= \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\mathrm{P}\ddot{y}_a\left[\frac{\rho\omega l^2 10^{-4}}{a^2b^2}(-\cdot33068l^7x + 1\cdot48809l^4x^4\right.$$

$$-1\cdot3\dot{8}l^3x^5 + \cdot49603x^8 - \cdot33068\frac{x^9}{l} + \cdot06614\frac{x^{10}}{l^2}\right)$$

$$+ \frac{m}{6l^3}\left\{\left(\frac{3}{20}a^2l - \frac{l^3}{20} - \frac{a^3}{10}\right)x^5 + \left(\frac{al^3}{4} - \frac{a^2l^2}{2} + \frac{a^3l}{4}\right)x^4\right.$$

$$\left.\left. - \left(\frac{3}{20}a^2l^5 - \frac{7}{20}a^3l^4 + \frac{a^4l^3}{4} - \frac{a^5l^2}{20}\right)x\right\}\right] - m\ddot{y}_a\frac{bx}{l}.$$

Similarly for $x > a$:

$$-\mathrm{EI}\frac{d^2y'}{dx^2} = \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\frac{\ddot{y}_a}{y_a}\left[\int_0^a (x-z)y_z dz + \int_a^x (x-z)yz'dz\right]$$

$$-\rho\omega\frac{\ddot{y}_a}{y_a}\frac{x}{l}\left[\int_0^a y_x(l-x)dx + \int_a^l y'_x(l-x)dx\right] - m\ddot{y}_a\left(\frac{bx}{l} - x + a\right);$$

$$= \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\mathrm{P}\ddot{y}_a\left[\frac{\rho\omega l^2}{a^2 b^2}10^{-4}\left(-\cdot 33068 l^7 x\right.\right.$$

$$\left. + 1\cdot 48809 l^4 x^4 - 1\cdot 38 l^3 x^5 + \cdot 49603 x^8 - \cdot 33068\frac{x^9}{l} + \cdot 06614\frac{x^{10}}{l^2}\right)$$

$$+ \frac{m}{6l^3}\left\{\left(\frac{3}{20}a^2 l - \frac{a^3}{10}\right)x^5 + \left(\frac{a^3 l}{4} - \frac{a^2 l^2}{2}\right)x^4 + \frac{a^2 l^3 x^3}{2} - \frac{a^3 l^3 x^2}{2}\right.$$

$$\left.\left. - \left(\frac{3}{20}l^3 - \frac{7}{20}al^2 - \frac{a^3}{20}\right)a^2 l^2 x - \frac{a^5 l^3}{20}\right\}\right] - m\ddot{y}_a\left(\frac{bx}{l} - x + a\right).$$

Integrating and equating when $x = a$, we find

$$\mathrm{M}_0 = \rho\omega\mathrm{P}\ddot{y}_a\left[\cdot 0000071145\frac{\rho\omega l^{10}}{a^2 b^2} + \frac{m}{6l^3}\left(\frac{a^7 l}{140} - \frac{a^6 l^2}{30} + \frac{a^5 l^3}{20}\right.\right.$$

$$\left.\left. - \frac{11}{210}a^3 l^5 + \frac{a^2 l^6}{35}\right)\right] + m\ddot{y}_a\frac{ab^2}{l^2};$$

$$\mathrm{M}_1 = \rho\omega\mathrm{P}\ddot{y}_a\left[\cdot 0000071145\frac{\rho\omega l^{10}}{a^2 b^2} + \frac{m}{6l^3}\left(-\frac{a^7 l}{140} + \frac{a^6 l^2}{60} - \frac{13}{420}a^3 l^5\right.\right.$$

$$\left.\left. + \frac{3}{140}a^2 l^6\right)\right] + m\ddot{y}_a\frac{a^2 b}{l^2}.$$

Whence the form of the centre-line is

$$-\mathrm{EI}y = \rho\omega\mathrm{P}\ddot{y}_a\left[\rho\omega\left(\frac{l}{ab}\right)^2 10^{-6}\left(3\cdot 5573 l^3 x^2 - 5\cdot 5113 l^2 x^5\right.\right.$$

$$\left. + 4\cdot 9603 l^4 x^6 - 3\cdot 3069 l^3 x^7 + \cdot 5511 x^{10} - \cdot 3006\frac{x^{11}}{l} + \cdot 0501\frac{x^{12}}{l^2}\right)$$

$$+ \frac{m}{6l^3}\left\{(3a^2 l - l^3 - 2a^3)\frac{x^7}{840} + (al^3 - 2a^2 l^2 + a^3 l)\frac{x^6}{120} + (7a^6 l - 2a^7\right.$$

$$\left. - 35a^4 l^3 + 52a^3 l^4 - 22a^2 l^5)\frac{x^3}{840} + (3a^7 l - 14a^6 l^2 + 21a^5 l^3 - 22a^3 l^5\right.$$

$$\left.\left. + 12a^2 l^6)\frac{x^2}{840}\right\}\right] + m\ddot{y}_a\left\{\frac{ab^2}{2}\frac{x^2}{l^2} - (2a+l)\frac{b^2}{6}\frac{x^3}{l^3}\right\}$$

and

$$-\mathrm{EI}y' = \rho\omega\mathrm{P}\ddot{y}_a \left[\rho\omega\left(\frac{l}{ab}\right)^2 10^{-6} \left(3\cdot5573l^8x^2 + 5\cdot5113l^7x^3 + 4\cdot9603l^4x^6 \right.\right.$$

$$-3\cdot3069l^3x^7 + \cdot5511x^{10} - \cdot3006\frac{x^{11}}{l} + \cdot0501\frac{x^{12}}{l^2} + \frac{m}{6l^3}\left\{ (3a^2l - 2a^3)\frac{x^7}{840} \right.$$

$$+ (a^3l - 2a^2l^2)\frac{x^6}{120} + \frac{a^2l^3x^5}{40} - \frac{a^3l^3x^4}{24} + (52a^3l^4 - 22a^2l^5 - 2a^7 + 7a^6l)\frac{x^3}{840}$$

$$\left.\left.+ (3a^7l - 14a^6l^2 - 22a^3l^5 + 12a^2l^6)\frac{x^2}{840} + \frac{a^6l^3x}{120} - \frac{a^7l^3}{840}\right\}\right]$$

$$+ \frac{m\ddot{y}_a}{6l^3}\left\{ (3a^2l - \dot{2}a^3)x^3 + (3a^3l - 6a^2l^2)x^2 + 3a^2l^3x - a^3l^3 \right\}.$$

When $x = a$ these give

$$-\left(\frac{\ddot{y}_a}{y_a}\right)^{-1} = \frac{\rho\omega\mathrm{P}}{\mathrm{EI}}\left[\rho\omega\frac{l^2}{b^2} 10^{-6}\left(3\cdot5573l^8 - 5\cdot5113l^7a + 4\cdot9603l^4a^4 \right.\right.$$

$$\left.- 3\cdot3069l^3a^5 + \cdot5511a^8 - \cdot3006\frac{a^9}{l} + \cdot0501\frac{a^{10}}{l^2}\right)$$

$$\left.+ \frac{ma^4}{1260l^3}(3l^6 - 11l^5a + 13l^4a^2 - 2l^3a^3 - 7l^2a^4 + 5la^5 - a^6)\right]$$

$$+ \frac{a^3l^3}{3l^3}\frac{m}{\mathrm{EI}}.$$

§ 20. As a particular case, if the load be at the centre

$$\mathrm{P}^{-1} = \cdot0020213\rho\omega l^4 + \cdot005208\dot{3}ml^3$$

and

$$-\frac{\ddot{y}_a}{y_a} = \mathrm{EI}\left[\frac{ml^3}{192} + \frac{4\cdot0398\rho\omega l + 10\cdot076m}{2021\cdot3\rho\omega l + 5208\cdot3m}\rho\omega l^4\right]^{-1}.$$

If the masses of load and bar be equal we have a further simplification to

$$-\frac{\ddot{y}_a}{y_a} = \frac{139\cdot65\,\mathrm{EI}}{ml^3}.$$

Section V. *Practical Formulæ for Loaded Bars.*

§ 21. The formulæ obtained in the last section for the frequency of the lateral vibrations of loaded bars of appreciable mass are too cumbersome for general use. In every case, however, given the position and mass of the concentrated load and the mass of the bar (assumed uniform throughout

its length), the frequency is given by a formula of the type

$$N = \frac{1}{2\pi} \sqrt{-\frac{\ddot{y}_a}{y_a}},$$

and the ratio $\ddot{y}_a/y_a$ is given by

$$-\frac{\ddot{y}_a}{y_a} = \frac{\beta EI}{\rho \omega l^4}.$$

In the following paragraphs the values of β are given for all positions of the load and for ratios of mass of load to mass of bar varying from zero to 1·0.
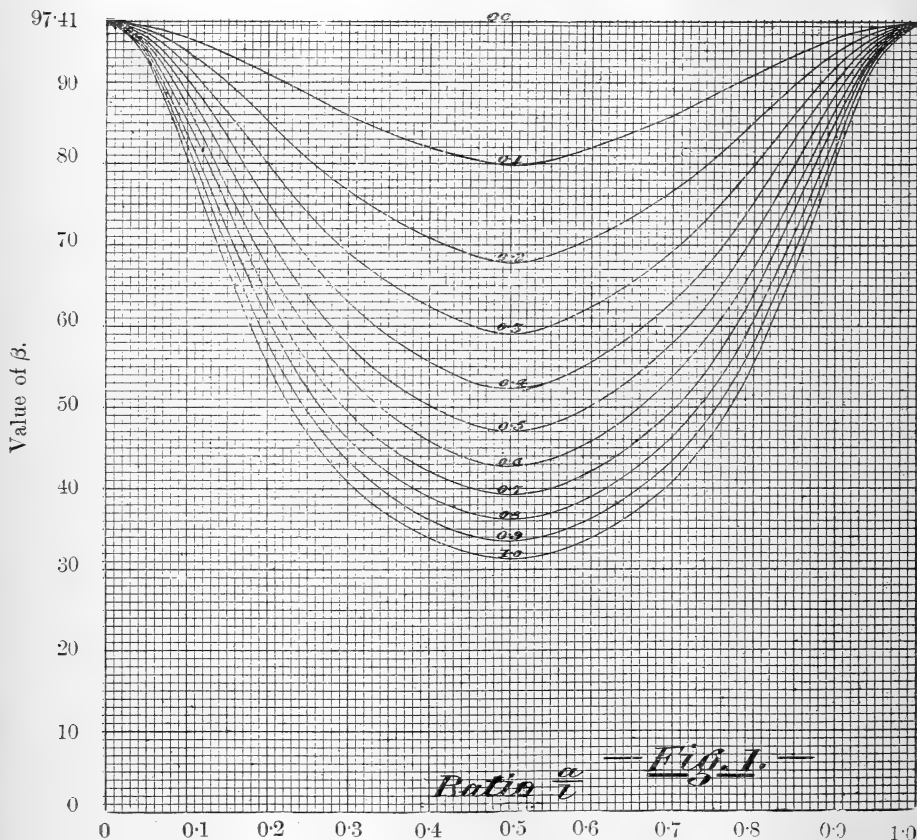
§ 22. *Bar supported at both ends.*—In this case the values of β have been obtained from the last equation of § 16. They are given in Table I. and represented graphically in figure 1. If the ratio of mass of load to that of bar cannot be expressed

exactly in tenths, further interpolation is required for accuracy. This may easily be obtained graphically by plotting the values of β given by the curves for the required position of the load.

In the tables l is the effective length of the bar, and a the distance of the centre of gravity of the load from either end. The numbers on the curves represent the ratio of mass of load to that of effective length of bar.

TABLE I.

Values of β for Bars Supported at Each End.

Ratio. Mass of Load. Mass of Bar.	Values of β for different positions of Load.				
	$a/l=\cdot1$.	$a/l=\cdot2$.	$a/l=\cdot3$.	$a/l=\cdot4$.	$a/l=\cdot5$.
0·1	95·56	90·98	85·91	82·03	79·80
0·2	93·74	85·20	76·71	70·85	67·73
0·3	91·95	80·02	69·22	62·35	58·99
0·4	90·21	75·36	63·01	55·68	52·31
0·5	88·69	71·17	57·81	50·30	47·03
0·6	86·86	67·39	53·91	45·86	42·74
0·7	85·24	63·96	49·56	42·15	39·18
0·8	83·67	60·84	46·25	38·99	36·17
0·9	82·15	57·99	43·35	36·27	33·60
1·0	80·66	55·39	40·78	33·91	31·38

§ 23. *Bar Clamped at Both Ends.*—The values of β are here found from the last equation of § 19. They are given in Table II. and fig. 2.

TABLE II.

Values of β for Clamped-Clamped Bar.

Ratio. Mass of Load Mass of Bar.	Values of β for different positions of Load.				
	$a/l=\cdot1$.	$a/l=\cdot2$.	$a/l=\cdot3$.	$a/l=\cdot4$.	$a/l=\cdot5$.
0·1	499·9	480·9	443·9	411·2	399·0
0·2	497·9	460·8	396·7	347·9	331·3
0·3	495·9	441·3	357·4	301·1	283·0
0·4	493·9	422·6	324·6	265·2	247·0
0·5	491·8	404·7	296·9	236·7	219 0
0·6	489·6	387·8	273·3	213·8	196·7
0·7	487·3	371·9	253·0	194·8	178·5
0·8	484·9	357·0	235·4	178·9	163·3
0·9	482·6	342·9	217·8	165·4	150·6
1·0	480·1	329·8	206·4	153·7	139·7

Fig. 2.

Value of β.

Ratio $\frac{a}{l}$
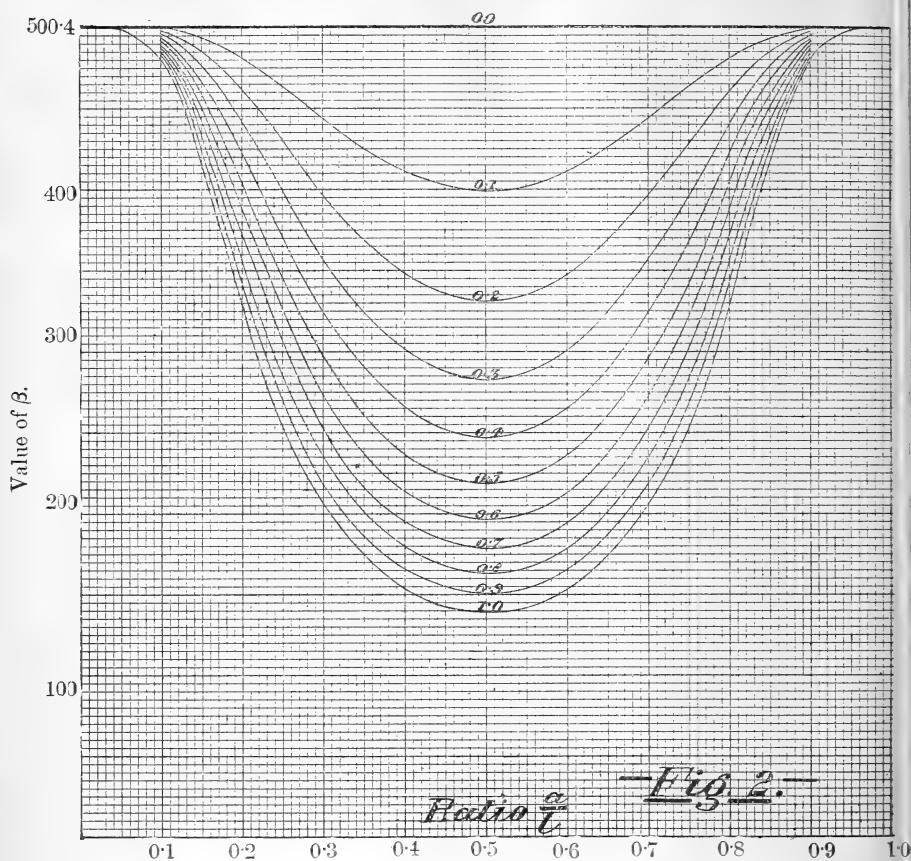
Section VI. *Correction for Rotatory Inertia.*

§ 24. Hitherto the terms depending on the angular motion, both of the concentrated masses and of the sections of the bars themselves, have been neglected. This is equivalent to supposing the inertia of each element to be concentrated at its centre.

The effect of the angular acceleration of the bar is sufficiently small to be neglected in practice, but in the case of a concentrated mass the additional terms may be important.

There are various methods of procedure. An important approximate one being to assume in the first place that the type of vibration is unaltered by rotatory inertia and then to find the correction due to this cause.

A general method has been given by Lord Rayleigh. It is applied to a special case in the next paragraph, in which

    I' = Moment of Inertia of concentrated mass about an axis perpendicular to the plane of bending ($=mk'^2$).
    T and V are the Kinetic and Potential Energies of the system.

§ 25. For a Fixed-Free Massless Bar (*cf.* Rayleigh's 'Sound,' Art. 183). Origin at fixed end and mass m at free end, the equation of the bar is

$$y = (3y_1 - l\theta)\left(\frac{x}{l}\right)^2 + (l\theta - 2y_1)\left(\frac{x}{l}\right)^3,$$

where y_1 and θ are the values of y and $\dfrac{dy}{dx}$ at the point where the load is attached. This equation is deduced from

$$EI\frac{d^2y}{dx^2} = m\ddot{y}_1(l-x) + I'\ddot{\theta}.$$

Integrating, and determining $\ddot{y}_1$ and $\ddot{\theta}$ from

$$EIy_1 = m\ddot{y}_1\frac{l^3}{3} + I'\ddot{\theta}\frac{l^2}{2} \Bigg\}$$
$$EI\theta = m\ddot{y}_1\frac{l^2}{2} + I'\ddot{\theta}l \Bigg\}$$

we get the desired result.

Following Rayleigh's solution, the equations of motion are

$$m\ddot{y}_1 + \frac{2EI}{l^3}(6y_1 - 3l\theta) = 0 \Bigg\}$$
$$I'\ddot{\theta} + \frac{2EI}{l^3}(-3ly_1 + 2l^2\theta) = 0 \Bigg\}$$

whence

$$-\frac{\ddot{y}}{y_1} = \frac{2EI}{lI'}\left\{1 + \frac{3k'^2}{l^2} \pm \sqrt{1 + \frac{3k'^2}{l^2} + \frac{9k'^4}{l^4}}\right\},$$

answering to the two different periods.

§ 26. The solution can be simplified if we assume to start with that the effect of I' is small. This is usually the case in practice, and the method has been fully investigated by Dr. Chree.

Adopting this simpler method (*cf.* Chree, *l. c.* p. 511, § 9)

the vibration curve in § 7 can be written

$$y = y_1 \left(\frac{x^3}{2l^3} - \frac{3x}{l} + 1 \right),$$

and we can take

$$\theta = \left[\frac{dy}{dx} \right]_{x=0} = - \frac{3y_1}{2l}.$$

Then, since

$$T = \tfrac{1}{2} m \dot{y}_1^2 + \tfrac{1}{2} I' \dot{\theta}^2 = \tfrac{1}{2} m \dot{y}_1^2 + \frac{9}{8} \frac{I'}{l^2} y_1^2,$$

and

$$V = \tfrac{1}{2} EI \int_0^l \left(\frac{d^2 y}{dx^2} \right)^2 dx = \frac{3}{2} \frac{EI}{l^3} y_1^2,$$

the equation of motion

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{y}_1} - \frac{\partial T}{\partial y_1} + \frac{\partial V}{\partial y_1} = 0$$

becomes

$$m \ddot{y}_1 + \frac{9}{4} \frac{I'}{l^2} \ddot{y}_1 + \frac{3EI}{l^3} y_1 = 0 ;$$

$$\therefore \quad - \frac{\ddot{y}_1}{y_1} = \frac{3EI}{ml^3 \left(1 + \frac{9}{4} \frac{k'^2}{l^2} \right)}.$$

§ 27. The same problems worked by the methods of this paper do not involve Lagrange's Equations and are as follows :—

Taking the origin at the free end,

$$- EI \frac{d^2 y}{dx^2} = m \ddot{y}_1 x - I' \ddot{\theta}.$$

Whence

$$- EI y_1 = m \ddot{y}_1 \frac{l^3}{3} - I' \ddot{\theta} \frac{l^2}{2} \left. \right\}$$

$$- EI \theta = - m \ddot{y}_1 \frac{l^2}{2} + I' \ddot{\theta} l \left. \right\}$$

Putting $\dfrac{\ddot{\theta}}{\theta} = \dfrac{\ddot{y}_1}{y_1}$ we get

$$\frac{ml^2}{6} \left(\frac{\ddot{y}_1}{y_1} \right)^2 + 2 \left(\frac{1}{l} + \frac{ml}{3I'} \right) EI \frac{\ddot{y}_1}{y_1} = - \frac{2}{I'} \left(\frac{EI}{l} \right)^2 ;$$

$$\therefore \quad - \frac{\ddot{y}_1}{y_1} = \frac{2EI}{I'l} \left(1 + \frac{3I'}{ml^2} \pm \sqrt{1 + \frac{3I'}{ml^2} + \frac{9I'^2}{m^2l^4}} \right).$$

Or, for an approximate solution, starting with

$$-\mathrm{EI}\frac{d^2y}{dx^2} = m\ddot{y}\ x - \mathrm{I}'\ddot{\theta},$$

we may put

$$\theta = -\frac{3}{2}\frac{y_1}{l}.$$

Whence

$$-\mathrm{EI}y = \ddot{y}_1(x-l)^2\left\{\frac{m}{6}(x+2l)+\frac{3\mathrm{I}'}{4l}\right\},$$

and

$$-\frac{\ddot{y}_1}{y_1} = \frac{3\mathrm{EI}}{ml^3\left(1+\frac{9}{4}\frac{k'^2}{l^2}\right)}.$$

§ 28. When the mass of the bar is taken into account, and Rayleigh's method is used, the kinetic energy is

$$\mathrm{T} = \frac{1}{2}\left\{m\dot{y}_1^2 + \mathrm{I}'\left[\frac{d^2y}{dx\,dt}\right]^2 + \int\rho\omega\dot{y}^2dx + \int\rho\omega k^2\left(\frac{d^2y}{dx\,dt}\right)^2dx\right\},$$

where the part in square brackets is to be taken for the position of the load, and those under the integral signs are between the limits. The potential energy of bending is, as before,

$$\mathrm{V} = \frac{1}{2}\int\mathrm{EI}\left(\frac{d^2y}{dx^2}\right)dx.$$

In general the evaluation of these, though perfectly simple, is tedious. The values of y and $\frac{dy}{dx}$ are taken from earlier parts of this paper, and the energy expressions then used in the Lagrangian Equation just as in the example already given.

§ 29. It is, however, simpler merely to add the terms due to Rotatory Inertia to the previous solutions. A general investigation of these terms would occupy too much space, but in a numerical example the work is not difficult.

Referring to § 14 for the case of a loaded massive bar, the term to be added to the value of $-\mathrm{EI}\frac{d^2y}{dx^2}$ for the rotatory inertia of m is

$$-\mathrm{I}\ \frac{d^2}{dt^2}\left(\frac{dy}{dx}\right)_{x=0};$$

and for an approximate solution we can write

$$\left(\frac{dy}{dx}\right)_{x=0} = -\frac{4}{3}\frac{y_1}{l}.$$

Hence the term to be added to $-\mathrm{EI}y$ is

$$\frac{4}{3}\frac{\mathrm{I}'}{l}\left(\frac{x^2}{2}-lx+\frac{l^2}{2}\right);$$

and for a first approximation,

$$-\frac{\ddot{y_1}}{y_1}=\frac{\mathrm{EI}}{\cdot08194\,\rho\omega l^4+\cdot3\,ml^3+\cdot6\,\mathrm{I}'l}.$$

This example is sufficient to indicate the procedure to be adopted in all such cases.

University College, Bristol,
        October 1905.

XXIX. *On the Range of Stokes's Deep-Water Waves.* By R. F. Gwyther [*].

THE waves with which I propose to deal are the waves of finite amplitude of which the investigation was initiated by Sir George Stokes [†] in his paper on the Theory of Oscillatory Waves, and continued in the Supplement to that paper. The object of the paper is to establish the correctness of the opinion expressed in the paper quoted (p. 227) " After careful consideration I feel satisfied ... that we may approach as near as we please to the form in which the curvature at the vertex becomes infinite, and the vertex becomes a multiple point where the two branches ... enclose an angle of 120°." The method of the Supplement is adhered to as closely as is convenient. The result is to establish that the velocity in all waves of the series, small as well as great, is represented by a function which possesses poles which as the amplitude increases approach nearer to the fluid surface, and in the limiting form is identical with that investigated by Mr. Michell [‡] in his paper on the " Highest Wave in Water," in which the poles lie on the water-surface.

§ 1. Making a change in the notation used by Stokes in the Supplement, in order to make the analytical form of the velocity more obvious, I write

$$x+iy=\frac{\phi+i\psi}{c}-i\Sigma\frac{h_n}{nk}e^{\frac{ink(\phi+i\psi)}{c}},$$

where n is an integer, and k is a constant defining the wave-length. Also let $\psi=0$ define the free surface of the water.

Then the velocity is given by

$$qe^{i\omega} = u + iv = \frac{1}{\dfrac{d(x-iy)}{d(\phi-i\psi)}} = \frac{c}{f\left(\dfrac{\phi-i\psi}{c}\right)}$$

where

$$f\left(\frac{\phi+i\psi}{c}\right) = 1 + \Sigma h_n e^{\frac{ink(\phi+i\psi)}{c}}.$$

At the free surface we have $\psi = 0$, and it will be convenient in this case to write, in place of $f\left(\dfrac{\phi+i\psi}{c}\right)$,

$$F\left(\frac{ik\phi}{c}\right) = 1 + \Sigma h_n e^{\frac{ink\phi}{c}}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

The condition for uniform pressure along the free surface requires that, when $\psi = 0$,

$$gy - \tfrac{1}{2}q^2 = \text{constant}.$$

The determination of the successive values of h_n resulting from this condition is only a matter of labour. I have carried the determination to the sixth order, as an extended number of terms is important as the basis of the future argument.

Stating results only, I find

$$kc^2/g = 1 + h^2 + \frac{7}{2}h^4 + \frac{229}{12}h^6,$$

$$h_2 = 2h^2 + h^4 + \frac{29}{6}h^6,$$

$$h_3 = \frac{9}{2}h^3 + \frac{19}{4}h^5,$$

$$h_4 = \frac{32}{3}h^4 + \frac{313}{18}h^6,$$

$$h_5 = \frac{625}{24}h^5,$$

$$h_6 = \frac{324}{5}h^6, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

where I have written h in place of h_1.

Also if we write 2α to denote the height of the wave from trough to crest, I find

$$k\alpha = h + \frac{3}{2}h^3 + \frac{163}{24}h^5. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3)$$

If we write out in full the resulting expression for $F\left(\dfrac{ik\phi}{c}\right)$, the successive terms will be seen to show a marked resemblance to those in the expansion of

$$\left(1 - 3he^{\frac{ik\phi}{c}}\right)^{-\frac{1}{3}}.$$

In order to examine this resemblance more closely, I expand $\left\{F\left(\dfrac{ik\phi}{c}\right)\right\}^{-3}$ on the assumption of convergence, and find

$$\left\{F\left(\frac{ik\phi}{c}\right)\right\}^{-3} = 1 - 3he^{\frac{ik\phi}{c}} - \left(3h^4 + \frac{29}{2}h^6\right)e^{\frac{2ik\phi}{c}}$$

$$+ \left(\frac{1}{2}h^3 - \frac{9}{4}h^5\right)e^{\frac{3ik\phi}{c}} + \left(h^4 - \frac{7}{6}h^6\right)e^{\frac{4ik\phi}{c}} + \frac{15}{8}h^5 e^{\frac{5ik\phi}{c}}$$

$$+ \frac{18}{5}h^6 e^{\frac{6ik\phi}{c}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is on the greatly increased convergency produced in the series by this step that the argument depends. It is clear that there is a value of μ, not very widely different from $3h$, for which $1 - \mu e^{\frac{ik\phi}{c}}$ is a factor of the right-hand side of (4).

For more precision, write

$$3h = \alpha\mu + \beta\mu^3 + \gamma\mu^5$$

and determine α, β, γ so that the right-hand side of (4) may vanish approximately when $1/\mu$ is put in place of $e^{\frac{ik\phi}{c}}$, *i. e.* so as to satisfy

$$1 - \frac{3h}{\mu} - \frac{3h^4 + \dfrac{29}{2}h^6}{\mu^2} + \ldots\ldots = 0.$$

This leads to the equations

$$1 - \alpha + \frac{1}{54}\alpha^3 + \frac{1}{81}\alpha^4 + \frac{5}{648}\alpha^5 + \frac{2}{402}\alpha^6 = 0,$$

$$\beta\left(1 - \frac{1}{18}\alpha^2 - \frac{4}{81}\alpha^3\right) = -\left(\frac{1}{27}\alpha^4 + \frac{1}{108}\alpha^5 + \frac{7}{4574}\alpha^6\right),$$

$$\gamma = -\left(\frac{4}{27}\alpha^3\beta + \frac{29}{1458}\alpha^6\right).$$

These give approximately

$$\alpha = 1\cdot054,$$
$$\beta = -\cdot068,$$
$$\gamma = -\cdot015; \qquad \ldots \ldots \quad (5)$$

and these values lead to

$$h = \cdot351\,\mu - \cdot023\,\mu^3 - \cdot005\,\mu^5. \quad \ldots \ldots \quad (6)$$

This value of h increases with increasing fractional values of μ, and reaches the value $\cdot323$ approximately when $\mu = 1$.

Since the hypothesis of convergence holds good for any value of μ finitely less than unity, we may take this to give approximately the greatest admissible value of h.

Using the numerical values (5) again, we find when $\mu < 1$,

$$\left\{ \mathrm{F}\left(\frac{ik\phi}{c}\right) \right\}^{-1} = \left(1 - \mu\,e^{\frac{ik\phi}{c}}\right)^{\frac{1}{3}} \times \mathrm{U} \qquad . \quad (7)$$

where

$$\mathrm{U} = 1 - (\cdot018\mu - \cdot023\mu^3 - \cdot005\mu^5)\,e^{\frac{ik\phi}{c}} - (\cdot019\mu^2 - \cdot009\mu^4)\,e^{\frac{2ik\phi}{c}}$$
$$- (\cdot011\mu^3 - \cdot003\mu^5)e^{\frac{3ik\phi}{c}} - \cdot006\mu^4\,e^{\frac{4ik\phi}{c}} - \cdot003\mu^5\,e^{\frac{5ik\phi}{c}} \cdots .$$

The expression for the velocity ($q e^{i\omega}$) will possess a series of poles, generally above the water, indicated by

$$\mu\,e^{\frac{ik(\phi+i\psi)}{c}} = 1.$$

Using the numbers in (2) and (3), I find

$$kc^2/g = 1 + \cdot123\,\mu^2 + \cdot038\,\mu^4 + \cdot019\,\mu^6$$

and

$$k\alpha = \cdot351\,\mu + \cdot041\,\mu^3 + \cdot019\,\mu^5.$$

If we write $k = 2\pi/\lambda$, we get

$$\frac{2\alpha}{\lambda} = \cdot112\,\mu + \cdot013\,\mu^3 + \cdot006\,\mu^5, \quad \ldots \quad (8)$$

which gives for the height of the greatest rounded wave $2\alpha/\lambda = \cdot131$ approximately. This value is not greatly in defect of the ratio, viz. $\cdot142$, found by Mr. Michell for the pointed wave. There appears therefore to be no stage, while μ is finitely less than unity, at which the expressions cease to give rounded waves of the same general character.

§ 2. In order to demonstrate that in the critical stage when

$\mu=1$, this method leads to expressions identical with those of Mr. Michell, I return to the general free-surface condition, and, for convenience, write ζ in place of ϕ/c.

On differentiating the free-surface condition in respect to ζ, we get an equation of the form

$$\frac{g}{i}\{F(ik\zeta)-F(-ik\zeta)\}=c^2\frac{d}{d\zeta}\{F(ik\zeta)F(-ik\zeta)\}-1\,;$$

and in this I write

$$F(ik\zeta)=(1-\mu\,e^{ik\zeta})^{-\frac{1}{3}}U-1,$$
$$F(-ik\zeta)=(1-\mu\,e^{-ik\zeta})^{-\frac{1}{3}}V-1,$$

where U and V are rapidly converging series of the type

$$1+C_1e^{ik\zeta}+C_2e^{2ik\zeta}+\ldots,$$

so that

$$\frac{g}{i}\left\{\frac{1}{(1-\mu e^{ik\zeta})^{\frac{1}{3}}U}-\frac{1}{(1-\mu e^{-ik\zeta})^{\frac{1}{3}}V}\right\}$$
$$=c^2\frac{d}{d\zeta}\{(1-\mu e^{ik\zeta})^{\frac{1}{3}}(1-\mu e^{-ik\zeta})^{\frac{1}{3}}UV\}.\quad\ldots\quad(9)$$

This may be put in the form

$$-\frac{g}{i}\left\{\left(\frac{1-\mu\,e^{ik\zeta}}{1-\mu\,e^{-ik\zeta}}\right)^{\frac{1}{3}}U-\left(\frac{1-\mu\,e^{-ik\zeta}}{1-\mu\,e^{ik\zeta}}\right)^{\frac{1}{3}}V\right\}$$

$$=c^2\left\{\frac{(1-\mu e^{ik\zeta})^{\frac{1}{2}}(1-\mu e^{-ik\zeta})^{\frac{1}{2}}}{2}\frac{d}{d\zeta}(U^2V^2)\right.$$

$$\left.+\frac{2k\mu}{3}\frac{\sin k\zeta}{(1-\mu e^{ik\zeta})^{\frac{1}{2}}(1-\mu e^{-ik\zeta})^{\frac{1}{2}}}U^2V^2\right\}\quad.\quad(10)$$

which becomes, when $\mu=1$, identical, except for the change in units, with the equation which Mr. Michell has employed to obtain the form of the Highest Wave in Water.

If, however, the left-hand side of (9) is rationalized the right-hand side is also found to be rational, and this form may be found more convenient whether μ is less than or equal to unity. The values of the constants in U and V are, of course, different in these two cases.

The simplification on which I have relied would appear to be applicable to all cases of irrotational waves whether in shallow or in deep water, and it will probably be found that such waves in either case will be related to poles, as I have shown to be the case with deep-water waves.

XXX. *On a Non-leaking Glass Tap.*
By A. P. Chattock *.

I T is a comparatively simple matter to prevent the leakage
of gas between the inside of a glass tap and the atmo-
sphere ; careful grinding and greasing being usually sufficient
without the additional security of mercury traps outside the
ends of the plug. But to prevent the passage of gas from
one side of the plug to the other is not so easy.

As is well known, when the plug revolves the edges of the
hole bored through it tend to remove part of the grease, and
thus to leave a trace in the latter which is only too likely to
connect the opposite sides of the tap; and this result, though
rendered less likely, is not by any means certainly prevented
by boring the hole obliquely.

An ingenious tap, in which the plug is hollow and contains
mercury, was described by Dr. Milner in this Magazine
(July 1903, p. 78). His tap requires no grease, and abso-
lutely prevents the passage of gas past the plug ; but this is
in one direction only ; and the tap is besides somewhat
elaborate. It is the object of this communication to show
that by a modification of Dr. Milner's idea it is possible to
construct a simple tap which shall prevent the passage of
gas in either direction, provided the ordinary use of grease is
permissible.
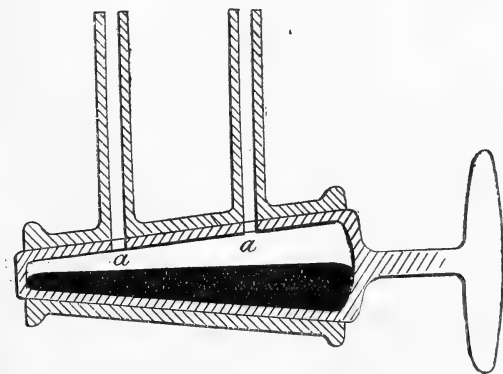
The figure shows the arrangement in section, the mercury
being marked black. It will be seen that if a leak occurs
the gas may get into the plug ; but it cannot get out again
when the holes *a, a* are below the mercury.

* Communicated by the Author.

2 C 2

Several of these taps, constructed by Messrs. Baird and Tatlock, have been in use for some time in the writer's laboratory with very satisfactory results. The only case in which trouble may occur is in opening connexion between vessels containing gases at very different pressures, as there is then a chance of blowing the mercury out of the tap; but this is quite easy to avoid by opening the tap slowly.

University College, Bristol.

XXXI. *Tensile Overstrain and Recovery of Aluminium, Copper, and Aluminium-Bronze.* By ARTHUR MORLEY, *M.Sc., Professor of Applied Mechanics, University College, Nottingham, and* G. A. TOMLINSON, *B.Sc.**

[Plate VII.]

THE limits of tensile stress within which the strain entirely disappears with the removal of the stress are usually called the elastic limits. For most hard metals this range of stress is considerable, and the strain takes place very quickly, and disappears quickly after the removal of the stress. In the case of some other metals such as aluminium and copper, the range is smaller and above it some of the strain appears slowly, the material showing the effect of " creeping " for long intervals of time after the first application of the stress.

The limits within which the strain produced is proportional to the stress are, for most metals, the same as the limits of elasticity just defined. The range of proportionality of strain to stress in aluminium as measured by a good tensile extensometer is small : from very low loads (usually much below 2 tons per square inch) the strain is found to increase more quickly than the stress provided sufficient time is allowed for the strain to develop. The stretch modulus (Young's) of elasticity usually quoted for aluminium is obtained from the average rate of strain over a moderate range of stress.

Although for moderate loads the strains increase out of proportion to the stress, it is found that two tests on the same piece of material made with a short interval between them give practically identical readings of strain, provided the material has not been " overstrained," that is strained beyond the yield point,—a stress considerably above the elastic limit, and not sharply defined in aluminium—at which the material stretches at an enormously greater rate than below it, and above which the yielding is mainly plastic. After overstraining, however, the material behaves very differently under the action of small loads, the strains

* Communicated by the Authors.

increasing at a still greater rate and deviating further from the law of proportionality to stress, as may be seen most readily by plotting load-extension diagrams. As the load is applied, the material exhibits a strain which is increased by "creeping" more or less slowly.

Professor Ewing has found * that this overstrained state in the case of iron and steel gradually disappears, and that the material approaches to its original condition after a sufficient interval of time. Further, Mr. Muir has found † that this process of recovery is greatly accelerated by raising these materials to such moderate temperatures as 100° C.

After consulting Prof. Ewing and receiving a favourable reply, we decided to find to what extent overstrained aluminium recovered its original properties after intervals of time and after subjection to moderate heat, and to investigate other properties bearing on this plastic condition. The general result of these experiments was to show that, as in the case of iron and steel, recovery of aluminium takes place with rest, and is brought about more quickly by the application of heat.

The recovery with lapse of time is nearly, but not quite, complete after 14 days, while Muir ‡ found that steel shows recovery to greater hardness than the original in 17 days.

The heated specimens, although they recover quickly, do not show after 14 days so complete a recovery as the unheated ones.

Experiments with copper gave practically the same results immediately after overstrain as before it, showing that if the properties are altered by overstrain, recovery is very rapid.

Specimens of aluminium-bronze (Al 10 per cent., Cu 90 per cent.) were also examined in a similar way. The recovery with lapse of time is slow, but under the influence of a moderate degree of heat it is very rapid.

The effect of mechanical vibration on overstrained aluminium appears to be inappreciable, but in the case of aluminium-bronze vibration has on an overstrained specimen the effect of augmenting the breakdown of its original properties and retarding its recovery.

The effect of prolonged loading upon tensile strength of aluminium has been investigated by Prof. E. Wilson and by Mr. J. Gavey §, who pointed out that a load of two-thirds of the nominal breaking load caused rupture when applied

* Proc. Roy. Soc. vol. lviii. 1895.
† Phil. Trans. Roy. Soc. vol. cxciii. 1899.
‡ Phil. Trans. Roy. Soc. vol. cxciii. 1899.
§ Journal of Inst. Elec. Engineers, No. 154, vol. xxxi. Feb. 1902.

continuously for about three weeks. Our overstraining loads
are beyond this amount, being but little below the values
necessary to cause fracture in a very short time.

Materials and Apparatus.

The aluminium on which the tests were carried out was
all cut from a $1\frac{1}{4}$ inch round rolled bar supplied by the
British Aluminium Company, the composition of which is
given as

$$
\begin{array}{lll}
\text{Al} & 99 \cdot 55 & \text{per cent.} \\
\text{Fe} & 0 \cdot 31 & ,, \\
\text{Si} & 0 \cdot 14 & .,
\end{array}
$$

The bar, the section of which was found by a micrometer
gauge to be $1 \cdot 251$ inch diam. and very uniform, was tested
as received, without machining of any kind. The tensile
strength of turned specimens was found to be from 9 to 10
tons per square inch.

The tests were carried out on the 50-ton Wicksteed single-
lever testing-machine, fitted with hydraulic traverse, in the
Engineering Laboratory of University College, Nottingham.
Any given load can on this machine be applied and read
accurately to $\frac{1}{500}$ ton. The extensions were measured over
a length of 8 inches by means of the latest type of Ewing
extensometer [*], and with this instrument it was possible to
measure correctly to $\frac{1}{50,000}$ inch.

The application and adjustment of the load to a fixed value
usually occupied from 15 to 20 seconds, but on account of
the creeping which occurs, even in the new metal, and more
particularly in the overstrained specimens, it was found
desirable to take extensometer readings at regular intervals
of one minute.

For aluminium an overstraining load of 11 tons was
chosen, this being well above the yield-point and causing a
permanent extension of about $\frac{1}{10}$ inch on 8 inches and a
contraction of diameter from $1 \cdot 251$ to $1 \cdot 245$ inches. The
heating of the specimens was carried out in a cylindrical
bath of water 20 inches high and 9 inches diameter, the
temperature of which was kept at 99° C. by passing in
steam from a $\frac{1}{2}$-inch steam-pipe under the control of a valve.
The cooling to the temperature of the atmosphere took place
in a large rectangular tank under a water-tap. In order to
bring out clearly the difference between two load-extension

[*] Proc. Roy. Soc. vol. lviii. 1895.

diagrams of the same or of different specimens, Prof. Ewing's geometric device of "shearing back" the curves * has been adopted. The extensions in figs. 1 to 5 have been all reduced by the equivalent of 6 extensometer divisions, *i. e.* by 0·0012 inch per ton of load. This makes the use of a large scale possible without occupying a very large space.

The results given below are selected from a larger number of experiments made in this investigation.

Results.

After a number of preliminary trials to determine suitable ranges of loading, and intervals of time for moderate degrees of recovery after overstrain, specimens marked No. 8 and No. 9 were tested as follows :—

No. 9 was loaded to 8·5 tons, and readings of the extensometer were taken after each increment of $\frac{1}{2}$ ton. The extensometer was then practically detached, being allowed to swing freely by the upper clips only, the instrument being held together by means of a clamp plate and the lower clips being disengaged; the load was then increased to 11 tons and then entirely removed after causing a permanent extension of $\frac{1}{10}$ inch. The results are shown in Table I. columns 1, 2, 3 and 4, the extensometer readings being in $\frac{1}{50,000}$ inch units.

The specimen was then allowed to rest in the machine free from load for an interval of 20 minutes. The extensometer was then re-adjusted on the specimen and a test made in the same way as before to a load of 5·5 tons. The results are given in columns 1, 2, 5, and 6 of Table I.

The 3rd and 4th tests were made, beginning at intervals of one hour and 5 days respectively from the time of overstraining. The results are given in Table I.

Fig. 1 (Pl. VII.) shows the load-extension curves for the first overstrain (A), and for the 2nd, 3rd, and 4th loadings (B, C, and D respectively). The recovery with lapse of time is clearly shown. The origin of curves B, C, and D is different from that of A for the sake of clearness.

After an interval of 14 days from the time of first overstraining the specimen was again tested and overstrained and re-tested to a load of 5·5 tons at the same intervals of time as before : the results are given in Table II.†

---

* Phil. Trans. Roy. Soc. vol. cxciii. 1899.
† One test in this series was unavoidably interrupted.

TABLE I.—Specimen No. 9.

Time in Minutes.	Load in Tons.	1st Loading. Nov. 10, 1904.		2nd Loading. 20 mts. after overstrain.		3rd Loading. 1 hour after overstrain.		4th Loading. 5 days after overstrain.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0·0	200	...	200	...	200	...	200	...
1	0·5	236	36	236	36	236	36	236	36
2	1·0	273	37	273	37	271	35	271	35
3	1·5	308	35	312	39	308	37	308	37
4	2·0	344	36	353	41	348	40	345	37
5	2·5	378	34	393	40	389	41	381	36
6	3·0	415	37	437	44	430	41	417	36
7	3·5	454	39	482	45	472	42	456	39
8	4·0	492	38	528	46	515	43	495	39
9	4·5	529	37	574	46	558	43	532	37
10	5·0	568	39	619	45	602	44	569	37
11	5·5	607	39	663	44	646	44	608	39
12	6·0	647	40						
13	6·5	687	40						
14	7·0	734	47						
15	7·5	784	50						
16	8·0	846	62						
17	8·5	926	80						
19	11·0								

TABLE II.—Specimen No. 9 (Second Overstrain).

Time in Minutes.	Load in Tons.	1st Loading. Dec. 8, 1904.		2nd Loading. 20 mts. after overstrain.		3rd Loading. 1 hour after overstrain.		4th Loading. 5 days after overstrain.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0	200	...	200	...	...	...	200	...
1	0·5	236	36	237	37	...	...	235	35
2	1·0	270	34	275	38	...	...	271	36
3	1·5	306	36	313	38	...	...	306	35
4	2·0	342	36	353	40	...	...	341	35
5	2·5	380	38	394	41	...	...	377	36
6	3·0	420	40	434	40	...	...	414	37
7	3·5	458	38	475	41	...	...	452	38
8	4·0	494	36	516	41	...	...	491	39
9	4·5	532	38	560	44	...	...	530	39
10	5·0	570	38	604	44	...	...	570	40
11	5·5	610	40	647	43	...	...	611	41
12	6·0	650	40						
13	6·5	691	41						
14	7·0	734	43						
15	7·5	781	47						
16	8·0	832	51						
17	8·5	893	61						
19	11·0								

A third overstrain and similar series of tests were made starting 28 days after the first overstrain, with the result shown in Table III.

TABLE III.—Specimen No. 9 (Third Overstrain).

Time in Minutes.	Load in Tons.	1st Loading. Dec. 8, 1904.		2nd Loading. 20 mts. after overstrain.		3rd Loading. 1 hour after overstrain.		4th Loading. 5 days after overstrain.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0	200	...	200	...	200	...	200	...
1	0·5	235	35	239	39	238	38	237	37
2	1·0	272	37	278	39	277	39	274	37
3	1·5	308	36	316	38	314	37	312	38
4	2·0	343	35	356	40	354	40	349	37
5	2·5	380	37	396	40	393	39	387	38
6	3·0	416	36	437	41	434	41	426	39
7	3·5	452	36	479	42	475	41	466	40
8	4·0	490	38	524	45	517	42	506	40
9	4·5	529	39	567	43	559	42	546	40
10	5·0	570	41	611	44	603	44	587	41
11	5·5	610	40	656	45	646	43	630	43
12	6·0	651	41						
13	6·5	696	45						
14	7·0	742	46						
15	7·5	790	48						
16	8·0	842	52						
17	8·5	902	60						
19	11·0								

Specimen No. 8 was cut from the same bar as No. 9, and was tested in exactly the same way and at the same intervals as No. 9, except that between the first and second and between the second and third loadings in each series the specimen was subjected to a temperature of 99° C. for 3 minutes by means of a water-bath, and then cooled to the temperature of the atmosphere. The results of the three series of tests are given in Tables IV., V., and VI.

Fig. 2 shows the load-extension curve for the first overstrain of specimen No. 8 (A), and the three subsequent tests B, C, and D in respective order. That the recovery due to the heating is much quicker than for specimen 9 is obvious from a comparison of figs. 1 and 2.

Figs. 3, 4, and 5 also exhibit the quicker recovery under the influence of heat for the three overstrains at such intervals that recovery from one was practically complete before the next took place. The full lines represent the behaviour of specimen No. 9 recovering without heat (see Tables I., II.,

TABLE IV.—Specimen No. 8.

Time in Minutes.	Load in Tons.	1st Loading. Nov. 3, 1904.		2nd Loading. After 3 mts. at 99° C. and 20 mts. after overstrain.		3rd Loading. After 3 more mts. at 99° C. and 1 hour after overstrain.		4th Loading. 5 days after overstrain.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0	200	...	200	...	200	...	200	...
1	0·5	236	36	237	37	236	36	235	35
2	1·0	270	34	273	36	271	35	270	35
3	1·5	305	35	310	37	307	36	305	35
4	2·0	340	35	346	36	341	34	340	35
5	2·5	376	36	382	36	378	37	375	35
6	3·0	413	37	420	38	414	36	411	36
7	3·5	450	37	459	39	451	37	447	36
8	4·0	487	37	497	38	490	39	485	38
9	4·5	525	38	536	39	530	40	524	39
10	5·0	563	38	576	40	570	40	564	40
11	5·5	602	39	620	44	611	41	604	40
12	6·0	642							
13	6·5	686							
14	7·0	729							
15	7·5	777							
16	8·0	829							
17	8·5	890							
19	11·0	(say5000)							

TABLE V.—Specimen No. 8 (Second Overstrain).

Time in Minutes.	Load in Tons.	1st Loading. Nov. 17, 1904.		2nd Loading. After 3 mts. at 99° C. and 20 mts. after overstrain.		3rd Loading. After 3 more mts. at 99° C. and 1 hour after overstrain.		4th Loading. 5 days after overstrain.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0	200	...	200	...	200	...	200	...
1	0·5	235	35	236	36	236	36	236	36
2	1·0	270	35	273	37	272	36	273	37
3	1·5	305	35	309	36	308	36	309	36
4	2·0	340	35	346	37	343	35	345	36
5	2·5	375	35	382	36	380	37	383	38
6	3·0	412	37	419	37	418	38	421	38
7	3·5	449	37	456	37	457	39	459	38
8	4·0	486	37	496	40	496	39	497	38
9	4·5	525	39	536	40	536	40	536	39
10	5·0	564	39	578	42	577	41	575	39
11	5·5	603	39	622	44	619	42	614	39
12	6·0	644	41						
13	6·5	685	41						
14	7·0	727	42						
15	7·5	777	50						
16	8·0	829	52						
17	8·5	889	60						
19	11·0	(say5000)							

TABLE VI.—Specimen No. 8 (Third Overstrain).

Time in Minutes.	Load in Tons.	1st Loading. Dec. 1, 1904.		2nd Loading. After 3 mts. at 99° C. and 20 mts. after overstrain.		3rd Loading. After 3 more mts. at 99° C. and 1 hour after overstrain.		4th Loading. 5 days after overstrain.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0	200	...	200	...	200	...	200	...
1	0·5	236	36	236	36	236	36	237	37
2	1·0	273	37	270	34	272	36	273	36
3	1·5	310	37	307	37	310	38	311	38
4	2·0	345	35	343	36	349	39	348	37
5	2·5	384	39	380	37	387	38	385	37
6	3·0	424	40	419	39	426	39	422	37
7	3·5	465	41	459	40	465	39	460	38
8	4·0	504	39	500	41	505	40	500	40
9	4·5	544	40	543	43	546	41	540	40
10	5·0	585	41	588	45	589	43	580	40
11	5·5	626	41	636	48	633	44	621	41
12	6·0	673	47						
13	6·5	722	49						
14	7·0	768	46						
15	7·5	822	54						
16	8·0	884	62						
17	8·5	956	72						
19	11·0								

and III.), and the broken lines represent that of specimen No. 8 recovering under the application of heat, as in Tables IV., V., and VI. Fig. 3 is plotted from Tables I. and IV., and represents the first overstrain of the two specimens and their recovery. Fig. 4 is from Tables II. and V. for the second series of tests, and fig. 5 is from Tables III. and VI. for the third series. The order of letters A, B, C, D gives the order of tests in every case.

Rate of Time Recovery.

The rate at which a newly overstrained piece of aluminium recovers its original properties with lapse of time, during comparatively short intervals, was investigated by experiments with a piece (No. 14) of the same bar from which specimens Nos. 8 and 9 were cut. It was overstrained in exactly the same manner as before, and then a load of four tons was applied at intervals and quickly removed. The extensometer was read at each application of the 4-ton load, the reading for no load being as before 200.

The manner in which these readings decreased is shown in

Table VII. and in fig. 6, which has been plotted from the table. The readings were taken immediately after the application of the 4-ton loads, so that little creeping took place.

TABLE VII.

Time in Minutes.	Load in Tons.	Extensometer-reading in $\frac{1}{50,000}$ inch units.
...	0	200
0	4	502
5	4	493
28	4	483
46	4	482
61	4	480
131	4	476
266	4	470

For this reason the readings are less than for the 4-ton load in the test after overstrain of specimen No. 9.

Rate of " Creeping " during Loading.

The early curvature of the load-extension curves for specimens Nos. 8 and 9 in their original condition is probably largely due to that part of the extension which takes time to develop and which is generally called " creeping." The rate at which " creeping " took place at two different loads, 3 tons and 5·5 tons, was observed both before and after overstraining ; the specimen used (No. 12) was cut from the same bar as Nos. 8 and 9, and was overstrained by a load of 11 tons. The results are given in Tables VIII., IX., X., and XI., which are plotted in fig. 7 ; the broken lines show the rate of creeping before overstrain has taken place, and the full lines show the corresponding rates after overstrain. The extensometer-reading, with no load on the specimen, was 200 in each case, and the extension was measured on a length of 8 inches.

On specimen No. 15 an attempt was made to separate, as far as possible, the plastic extensions due to " creeping," from the elastic extensions which develop immediately after the load is applied. For this purpose the jockey weight of the testing-machine was set in succession to the necessary positions before the loads were applied to the specimen by the hydraulic ram. After this the load was very quickly applied,

the extensometer-reading taken, and the pull almost entirely removed again, the total time of application being under 10 seconds. The results of tests of this kind, before and after

TABLE VIII.

Load 3 tons. Before overstrain.

Time in Minutes.	Reading of Extensometer in $\frac{1}{50,000}$ inch units.
0	414
5	424
10	429
22	433
47	438
72	440
157	445

TABLE IX.

Load 3 tons. After overstrain.

Time in Minutes.	Reading of Extensometer in $\frac{1}{50,000}$ inch units.
0	420
5	433
10	438
17	441
25	445
30	447
35	449
41	450
164	469

TABLE X.

Load 5·5 tons. Before overstrain.

Time in Minutes.	Reading of Extensometer in $\frac{1}{50,000}$ inch units.
0	596
7	610
15	618
21	622
30	629
35	631
48	637
68	644
139	662
188	671
203	673

TABLE XI.

Load 5·5 tons. After overstrain.

Time in Minutes.	Reading of Extensometer in $\frac{1}{50,000}$ inch units.
0	626
1	639
5	653
10	662
15	669
22	675
32	682
37	686
52	694
167	731
197	738
230	743

overstrain, are shown in Table XII. and are plotted in fig. 8. For convenience in setting the load without loosening the wedge grips which hold the specimen, the load was reduced to 0·1 ton instead of zero. The curve A represents the material before overstrain and the condition immediately after an overstraining load of 11 tons is shown by curve B. In these tests the permanent set (or rather extension, which does not disappear in 10 minutes from the removal of the load) begins

to show itself at about 5-tons load. Curve A shows that the material obeys the law of proportionality of strain to stress when the load is applied quickly, within about the same range. Curve C, to five times the previous scale, and not "sheared back," shows the permanent set. From a comparison with fig. 1 and Table I. it appears then, that in such a material the term "limit of elasticity" cannot be defined without some reference to time, unless it be understood that an indefinitely long time is to be allowed for extension and contraction to take place.

Effects of Vibration.

A specimen was overstrained in exactly the same manner as No. 8, and then tested again. It was then subjected to a number of sharp blows from a hammer both endwise and transversely, and immediately tested again. The result showed that the overstrained state was unaltered by the vibration, and subsequent tests showed that this treatment did not in any way affect the rate of recovery from overstrain.

TABLE XII.—Specimen No. 15.

Load in Tons.	Extensometer-reading before overstrain.	Diff.	Permanent Set.	Extensometer-reading after overstrain.	Diff.
0·1	200	...	...	200	...
1·0	261	...	0	261	...
1·5	295	34	0	297	36
2·0	330	35	0	336	39
2·5	365	35	0	376 (?)	40
3·0	400	35	0	411	35
3·5	437	37	0	451	40
4·0	472	35	0	490	39
4·5	506	34	0+	532	42
5·0	541	35	0+	573	41
5·5	576	35	1	611	38
6·0	616	40	2	656 (?)	45
6·5	653	37	4	695	39
7·0	691	38	9	738	43
7·5	733	42	10		
8·0	771	38	10		
8·5	822	51	43		

Aluminium-Bronze.

A length of 1¼ inch round bar (specimen No. 2) was turned down to a diameter of 1·118 for a distance of 10 inches and was then tested and overstrained by a load of 37 tons, which

gave it a permanent strain of $\frac{1}{10}$ inch and reduced its diameter to 1·109 inches. The specimen was then tested after intervals of 20 minutes, 1 hour, and 5 days; being heated for 3 minutes to 99° C. before the first two tests after the overstrain. The results are shown in Table XIII., from which the broken-line curves in fig. 9 have been plotted in the order A, B, C, D. Examination will show that recovery is complete after the first heat treatment, the material showing slightly more hardness than before overstrain.

TABLE XIII.—Specimen No. 2 (Aluminium-Bronze).

Time in Minutes.	Load in Tons.	1st Loading. Oct. 19, 1905.		2nd Loading. After 3 mts. at 99° C. and 20 mts. after overstrain.		3rd Loading. After 3 more mts. at 99° C, and 1 hour after overstrain.		4th Loading. 5 days after overstrain.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0	200	...	200	...	200	...	200	...
1	1	260	60	262	62	262	62	262	62
2	2	325	65	327	65	328	66	325	63
3	3	389	64	390	63	392	64	388	63
4	4	453	64	455	65	456	64	453	65
5	5	520	67	519	64	522	66	518	65
6	6	587	67	586	67	588	66	585	67
7	7	655	68	654	68	653	65	652	67
8	8	724	69	721	67	722	69	719	67
9	9	794	70	789	68	790	68	787	68
10	10	865	71	859	70	861	71	857	70
11	11	938	73						
12	12	1011	73						
13	13	1087	76						
16	37								

Specimen No. 2 was then again overstrained, by a load of 38 tons, giving a further permanent extension of $\frac{1}{10}$ inch, and was retested at the same intervals as before, but without heating, with the results given in Table XIV. The full line curves in the order A, B, C, D, fig. 9, have been plotted from Table XIV. in consecutive order. They show that with lapse of time the aluminium-bronze recovers its original properties much more slowly at ordinary atmospheric temperatures than at 99° C. The " shearing back," in figs. 9 and 10, is at the rate of 5·5 extensometer divisions, *i. e.* 0·0011 inch, per ton of load.

Finally, the effect of vibration on overstrained aluminium-bronze was investigated. From results of tests made on a trial

specimen which needed considerable hammering to remove it from the wedge grips *, it was suspected that vibration had a similar effect to that of further overstraining. This proved to be the case, as is obvious from Table XV. and fig. 10, the curves A, B, C, and D of which are from the figures in columns 3, 5, 7, and 9 of the table. The difference between curves B and C shows the effect produced by vigorously tapping the specimen, endwise and transversely, with a hand hammer.

TABLE XIV.—Specimen No. 2 (Aluminium-Bronze).

Time in Minutes.	Load in Tons.	1st Loading. Nov. 9, 1905.		2nd Loading. 20 minutes after overstrain.		3rd Loading. 1 hour after overstrain.		4th Loading, 5 days after overstrain.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0	200	...	200	...	200	...	200	...
1	1	259	59	262	62	262	62	262	62
2	2	321	62	329	67	331	69	325	63
3	3	387(?)	66(?)	397	66	400	69	391	66
4	4	448	61	468	71	470	70	458	67
5	5	516	68	538	70	540	70	525	68
6	6	581	65	609	71	611	71	593	68
7	7	650	69	681	72	682	71	661	68
8	8	718	68	755	74	754	72	731	70
9	9	787	69	830	75	828	74	800	69
10	10	858	71	905	75	901	73	871	71
12	38								

TABLE XV.—Specimen No. 3 (Aluminium-Bronze).

Time in Minutes.	Load in Tons.	1st Loading. Nov. 16, 1905. After heating to 99° C.		2nd Loading. 15 minutes after overstrain.		3rd Loading. After vibration, and 30 minutes after overstrain.		4th Loading. Nov. 22, 1905, after 3 minutes at 99° C.	
		Reading.	Diff.	Reading.	Diff.	Reading.	Diff.	Reading.	Diff.
0	0	200	...	200	...	200	...	200	...
1	2	326	126	330	130	340	140	326	126
2	4	465	139	471	141	481	141	462	136
3	6	605	140	615	144	627	146	600	138
4	8	747	142	761	146	776	149	741	141
5	10	895	148	911	150	928	152	888	147
7	38								

* In obtaining results, shown in Table XIV., the specimen was left untouched in the machine between successive tests.

XXXII. *A New Improved Type of Chronograph.* By
ROBERT LUDWIG MOND, *M.A.*, *F.R.S.E.*, *and* MEYER
WILDERMAN, *Ph.D.*, *B.Sc.**

[Plate VIII.]

IN the course of a physico-chemical investigation we made
some years ago a chronograph measuring time to 0·1
second was required. We procured one of the best make
with a revolving drum of 60 cm. circumference making one
revolution per minute, and subjected it to a careful investi-
gation. We found on calibrating it with a clock making
electrical contacts automatically every minute, that not only
was the absolute value of each revolution not equal to a
minute, but that the differences in time between two contacts
proved to be very irregular and greatly different. On
further comparing four curves obtained successively one
after another with the same instrument, the clock having
been wound up afresh every time, we found that the form
and shape of the succession of the contacts of each spiral was
not always the same, so that no correction table could be
applied in connexion with the above instrument. Since it
is exceedingly difficult to get a large revolving drum to
move at the above speed with great regularity, it was evident
that unless a simultaneous calibration of the curve is made for
every second or two seconds, the above type of instrument
cannot be used even for moderately accurate measurements
of time.

Since no fault could be found with the workmanship of
the instrument, it was evident that the reasons for these
irregularities had to be looked for in the principle upon
which the instrument for high speeds was constructed.
This it was not difficult to discover. It is a well-known fact
that very little is required to stop a clock, and still less to
make its movements irregular. By making a clock drive a
drum with a speed of one revolution per minute instead of
24 hours, the clock has to do about 1000 to 1500 times more
work, and the solution of the problem for chronographs of
higher speed could therefore be only effected if this enormous
increase of work of the clock could be correspondingly
counterbalanced by a reduction in the work the clock has to
do per each revolution.

The principles upon which our chronograph was worked
out can be summarized as follows :—(1) Instead of moving
the heavy drum, thick screw, and the one or two carriages

* Communicated by the Authors.

with writing-pens, we keep them all stationary and only move the light spindle with the writing-pen, thus diminishing the load upon the clock arising from the work transmitted by the clock. (2) We reduce the friction of the different parts in motion to that of the moving spindle only, and further reduce that by the use of friction wheels or balls, &c. (3) We avoid the reactions of inertia, as far as possible, by a perfect balance of the moving part of the instrument—the spindle.

Experiments for several years showed the intricacy and the minuteness of conditions to be carried out simultaneously in the construction of a chronograph for high speeds before a satisfactory result could be obtained. We shall for this reason describe and explain all those details upon which the successful construction and use of our chronograph depends.

Figure I. (& Pl. VIII. fig. 1) represents a horizontal type of
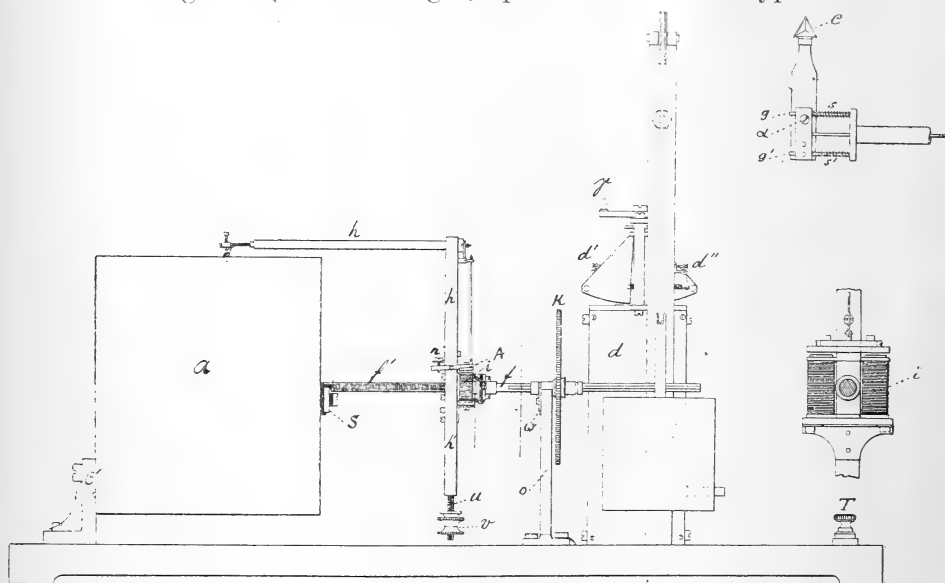
Figure I.

our chronograph, as made for us by Messrs. Sanger, Shepherd and Co. *d* is the driving clock, *a* is the stationary drum 60 cm. circumference fixed (in exact horizontal position) to the bracket (C′). The horizontal spindle (*f*) carrying the pen-arm (*h*) of the writing-pen (*c*) and the electromagnet (*i*) is counterbalanced by the nut (*v*) which is adjustable on the screw (*u*). The horizontal spindle passes at one end through the boss of a pinion wheel (K), driven by the wheel (*l*)

of the clock, thus describing a circle round the drum. A strip projecting from the spindle forms a long feather, and a corresponding featherway in the boss of the pinion, 2 mm. wide, renders the spindle capable of axial movement with regard to the pinion, caused by the screw of the spindle (f') and the half-nut (S), fixed to the drum and insulated from the same by means of the ebonite strip. Owing to this double action of the spindle, the writing-pen describes a spiral on the drum. The instrument shown in fig. I. gives a spiral $50 \times 60 = 3000$ cm. long, the circumference of the drum being 60 cm. Since the writing pen can easily make one revolution per minute or in 10 seconds, we have 10 mm. or 60 mm. for registration of one second. The line of our pen being very thin, we can read well to 0·2 mm. or even less, *i. e.* 1/50 or 1/300 of a second, provided, however, means are found to make electrical contacts to the same degree of accuracy; a problem which, it seems to us, will give much trouble if a conscientious scientific investigation should be properly attempted. Since in our type of chronograph the drum is stationary and a great extension in its height and width means only an extension of the length of the arm of the spindle, which means only a small increase of load and almost no increase in the friction or the reaction of inertia, we are thus able to construct instruments with still greater length of spiral and of the same accuracy.

The boss of the spindle rests on two little friction wheels (w) fixed to the support (o), and is kept in position by the slight exterior circumferential groove on the boss with which the friction wheels engage ; the friction of the spindle at this end is thus reduced to two points only. The screw end of the spindle has only to guide the pen which writes with almost no friction (the spindle must be naturally placed exactly in the centre of the drum). It is made thin for this reason, and the half-nut (s) contains only a little more than a line of the thread just sufficient to move the spindle securely forward. In this manner the friction at this end of the spindle was also reduced to the smallest possible amount.

It will be perceived that the wheel (K) is not fixed to the spindle. This enables us to exchange without trouble one pair of wheels for another giving a smaller or greater speed (say one revolution of the spindle in one minute or in ten seconds), so as to get the speed which is most suitable for the investigation of the given physical, physico-chemical, physiological or meteorological phenomenon, &c. To allow of the use of the exchange wheels an adjustable fly is provided for the instrument.

Our adjustable fly is an improved type of Breguet governor. Its purpose is a double one : first, to counteract all small irregularities of speed ; secondly, to adjust the absolute value of the time of each revolution when different pairs of wheels are used or when the springs in the natural course of time vary in elasticity.   As to the first purpose, it should be remarked that only when the irregularities in the movement of the clock independent of the governor are very small, is the governor effective ; when the variations are too great the governor serves no purpose and is a decoration only.   The adjustment of the absolute values of the speed of each revolution is usually made as follows.   When the wheels, &c. of the instrument are properly chosen, the final adjustment is usually effected by employing flexible spirals of a given material and of a certain number of turnings.   This adjustment can, however, only be a rough one, and what is still worse, even if a perfect adjustment is once effected it is certain not to remain perfect after being in use for some time owing to the variation in the flexibility of the springs, to the effect of slow oxidation, moisture, fatigue, &c.   So also when exchange-wheels are used, the speed is not in proportion to the number of teeth in the wheel, as is often assumed, because the work to be spent in moving the clock parts and the spindle is not directly proportional to the speed of each revolution; *i. e.*, each pair of wheels must have its own governor which is to be specially arranged for the given speed.   To meet all the above requirements, the springs of the governor are fixed as shown in fig. II. (p. 399), and by turning the screws d' and d'' they can be considerably increased or diminished in tension ; the adjustment of speed can therefore be carried out with great sensitiveness and without difficulty.

At the end of the cross arm (h) a screw (u) is arranged, which together with the nut (v) forms a balance-weight at the end of the cross arm.   This arrangement gives an adjustable and extremely accurate means for securing an exact balance of all parts of the spindle while in position, which, besides the clock, is the only movable part of the instrument. This is also a point of great consideration in chronographs of great speed, because greater speed does not necessarily mean a corresponding greater accuracy.   When a clock moves different parts of an instrument, not only does the clock move those parts, but those parts also move the clock. Since it is exceedingly difficult to get the wall of a hollow cylinder of a uniform thickness throughout, it being a sufficiently difficult task to get its upper surface true alone, the drum is never or almost never completely counterbalanced

in all its parts; its movement is, owing to inertia, for this reason not of a uniform character, and this upsets the regularity of the movement of the clock. In our instrument the parts f and f' themselves form the axis, and the two cross-pieces (h and h') with the nut (v) form the arms of a balance carrying the weights to be adjusted to each other by means of the screw and nut. It is of no little importance for the accuracy of the spiral obtained, that in our instrument this result is produced by one moving part only, while in other instruments it is produced by several moving parts, which in the nature of things can never be made to be absolutely dependent one upon another.

We now pass to the description of our improved means for operating and working the writing-pen. After many trials and modifications the following arrangement was adopted. A bell-crank is provided at the end of the cross arm, one of its arms being connected by a rigid link to the armature (A) of the magnet, while the other is connected in a similar manner through the axial hole in the pen-arm to the penholder itself. The end of the pen-arm carries a small cross-head from which two guides (g and g') parallel to the arm project. A cross-piece to the centre of which the operating link is connected slides on these guides, its motion being controlled by light flexible springs (s and s'), and to the cross-piece a flexible strip carrying the pen is secured. The pressure with which the pen bears upon the drum is regulated by a small screw (a) acting between the flexible strip and the cross-piece. In our newest type of chronograph arrangements are made to raise the pen from the drum, when the paper is changed, so that the pressure of the pen upon the drum, once adjusted, should always remain absolutely the same. For this purpose a little plate carrying the writing-pen with the screw is hinged to the cross-piece. This arrangement provides a very steady support for the pen, making vibrations of the penholder impossible and keeping the pen in all positions of the spindle at exactly the same distance from the drum (instead of resting on it), and this enables the pen to be brought just so near to the drum as is necessary to trace a spiral on the same, *i. e.* with the least possible friction. To the arm carrying the pen a reservoir containing a considerable quantity of ink may be fixed, from which the ink flows into the pen through a capillary tube, or a capillary tube containing a cotton thread. By this means the pen never becomes dry, and there is no necessity for repeated cleaning and filling with ink.

The writing-pen is actuated electrically. The current

enters the terminal (T), which is in electrical contact with
the base of the instrument, it passes through the support to
the friction wheels, driving spindle, electromagnet, the screw,
and then, by way of the insulated half-nut, from here it passes
by the return wire to the terminal (T') which is insulated
from the base of the instrument.

There is another distinctive feature in the arrangement of
our pen for registration, and this is that our pen can be used
at the same time both for registration and calibration. The
length of the two guides (g and g'), the greater flexibility
of the two flexible springs (s and s'), allow the pen to slide
on the guides long distances, and for this the electromagnet
(i) must also be adjusted at its screw (n). By connecting the
terminals (T and T') with a battery, and by using the
pendulum of a clock or other suitable calibrating arrange-
ments in the circuit for making contacts, we are able by
inserting a corresponding resistance to adjust the current
passing through the electromagnet so as to get for calibration
small deviations of the pen from the normal spiral as desired
each second, or better, each half minute or minute. By
having the terminals (T and T') at the same time connected
with another battery sending the current in the same
direction, and by having the electrical key or the automatic
contact of the rain-gauge, anemometer, speed indicator, &c.,
we can so adjust the voltage and the resistance as to get
a greater current through the electromagnet and a corre-
spondingly greater deviation of the writing-pen from its
normal course, thus registering the observed or recorded
phenomenon at the same time. The results obtained for
calibration and registration cannot possibly be mistaken for
one another even if both contacts should happen to be effected
simultaneously, because two intervals of calibration cannot
possibly with our instrument be mistaken for one, and the
longer line of registration cannot be covered by the shorter
one of calibration. Not only does this dispense with a
second carriage and pen, but also leads to more accurate
results, because we register and calibrate straight on one and
the same spiral, not on two separate ones, and we have not
subsequently to transfer the calibration from one spiral to the
other. For the special purposes, where two different kinds
of phenomenon are to be recorded simultaneously, e. g. wind
and rain, or where two observers have to record simultaneously
the same phenomenon, such as the transit of a star in
astronomical work, the instrument can be provided with two
pens.

The great regularity and perfection of motion of the spindle

in our chronograph attained in the manner described above is best seen from the fact that not the least sound (except from the fly) can be heard when the instrument is working, while in instruments of the older type of a similar speed harsh sounds both in the clock and in other parts of the instrument are heard, indicative of the irregularity of motion.

In figure II. (& Pl. VIII. fig. 2) the vertical type of our chronograph is given. The principles and details are the

Fig II
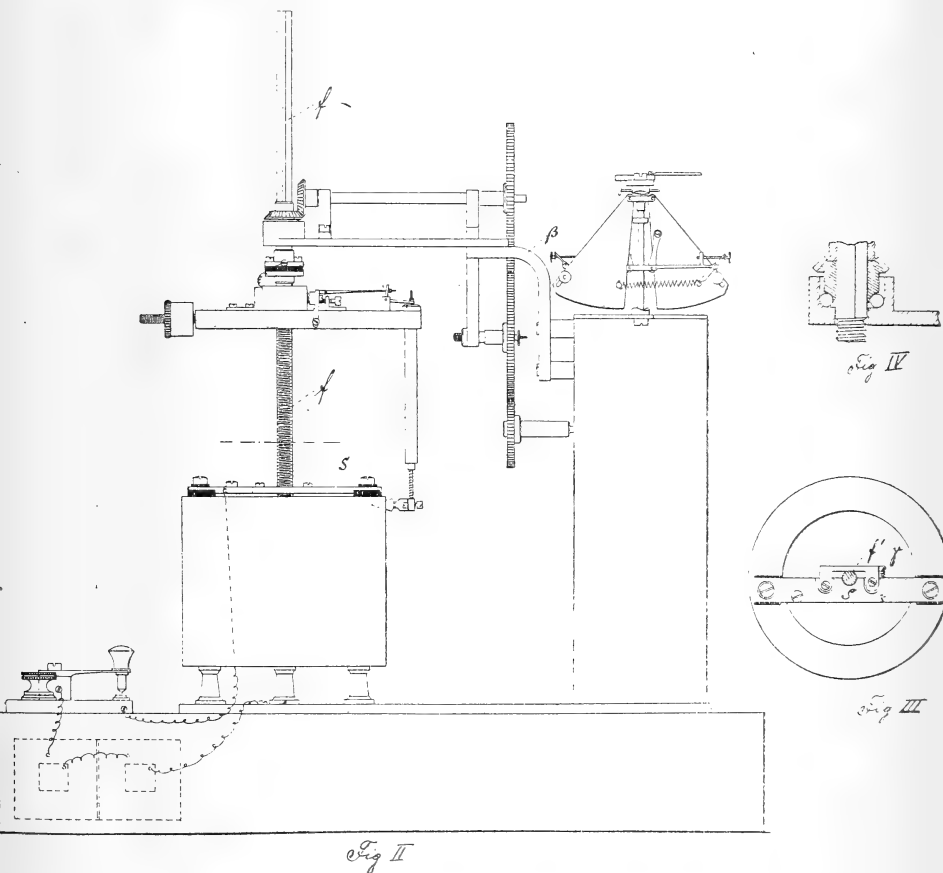
same here as in the horizontal type and are indicated by the same letters. Here the spindle is arranged with its driving end upwards, the weight of the pen-carrying system being taken by the half nut (S) on the drum.

A detail of this portion is given in figure III., a hinged

clip (γ), provided with spring and clamping screw, serving to hold the spindle (f') in operative engagement with the half-nut. The driving is effected through spur-wheels, supported by the bracket (β); the bevel-wheel, as shown in detail in figures II., III., and IV., being arranged to run on ball bearings. The arrangements of the pen-carrying system are similar to that described with reference to the horizontal type of instrument, the electrical connexions being also substantially the same.

The vertical type is an improvement on the horizontal type, though usually made of a smaller size, since it allows continuous reading of the records on the drum, can stand more knocking about, is easier to be accurately constructed, and gives also better results.

When the whole spiral is completed, the instrument stops automatically by itself. In the horizontal type we raise then the spindle (ff') from the half-nut (s), and push it through the boss of pinion (K) back to its former position, stopping first the governor by means of (γ'). In the vertical type, we open clip (γ), release the screw of the spindle from the half-nut, and push it up to its former position.

Both types are set into motion or stopped, whenever desired, by stopping or releasing the governor by means of γ'.

While costly and complicated arrangements for calibration each second are indispensable for instruments with revolving drums, all we require for very accurate work with our instrument where the movement of the pen is very regular is to calibrate it, using an ordinary $\frac{1}{5}$ sec. or $\frac{1}{10}$ sec. watch and electric key at longer intervals, say each minute (at the same time as we make our research observations), and in this case we do not even require to exactly adjust the absolute value of a revolution, having only to use a factor. The same or almost the same error in calibration falls upon a minute instead of upon a second, and the results obtained in this manner are much more accurate and satisfactory. Since no work can be done for nothing, any extra work of the pen spent on calibration is not conducive to the accuracy of the results obtained, and it is well that we are able to avoid frequent calibration.

Up to the present we have tried a great number of arrangements for calibration, some of them very complicated and difficult. The great accuracy and regularity in the movement of the writing-pen, as well as the great length of the spiral, make our vertical instrument a very handy and easily manageable recording calibrator for other instruments

with revolving drums, where those have to be used in view of the nature of the research (physiological, physical, &c.) in spite of their inaccuracy. All that is required for this is to fix to the axis of the revolving drum a suitable wheel with projections making electrical contacts for our chronograph, at as frequent intervals as is desirable, and to calibrate the value of each revolution of the pen of our instrument with an ordinary watch, at a greater time interval.

The accuracy of the instrument as a time indicator and the length of the spiral allows of its wide application for all sorts of scientific and technical purposes : *e. g.* by a corresponding variation in the accessory instruments for more frequent contacts a considerably more accurate anemometer, speed indicator, &c., is easily arranged, allowing a very much greater accuracy than has hitherto been practicable. We shall illustrate this only on the two instruments mentioned, *i. e.* on the anemometer and the speed indicator.

The anemometer has to indicate the miles of wind in the unit of time, *i. e.* $\frac{F}{t}$. In the present instrument two clocks are usually used : one moves the drum with the speed of one revolution in 24 hours (this gives t) ; the other raises the writing-pen and allows it to drop, thus extending the line for contacts several times (this gives F). Since the length of the curve for time is very short, the registration of t is very defective, $\frac{F}{t}$ must be defective also. With our instrument the revolving pen makes a contact on our spiral for say each F miles of wind. The spiral is about 50 times longer than with ordinary instruments and is used both for the indication of time and for the contacts. The length of line is about 10 times longer for F and about 50 times longer for t.

The speed indicator has to give the number of revolutions (of an electromagnet, dynamo, gas-engine, steam-engine, motor car, &c.) in the unit of time, *i. e.* $\frac{S}{t}$. In all present instruments time is given by a revolving drum. The indication of time (t) is very wrong, especially where large strips of paper are rolled and unrolled from drums and drawn by a clock (in the first instance on account of enormous friction). The number of revolutions is given by a rising fly—this (s) is again wrong, because the rise of the fly is not directly proportional to the speed. Then the pen writes not in straight lines, but in arcs. The result is that we get an idea of the speed, but no quantitative measurement of it, and we have no means of knowing exactly the speed at any given

time owing to the interference of too many sources of error. With our instrument we arrange that a contact should be made by the dynamo, gas-engine, motor car, &c. &c., at convenient intervals, say each 1 or 10 or 100 revolutions. The distances between the contacts of the spiral on paper ruled in mm. to indicate the time gives also at a glance the character of the varying speed, but the same unit divided by the distances between two contacts read in mm., gives at the same time also exactly the speed between the time intervals indicated on the paper. The greater length of spiral also allows of this variation being observed more frequently and with greater accuracy.

London, August 1905.

XXXIII. *The Comparison of Electric Fields by means of an Oscillating Electric Needle.* By DAVID OWEN, *B.A.* (*Camb.*), *B.Sc.*(*Lond.*), *Lecturer in Physics, Birkbeck College, London* *.

EXPERIMENTS on fields of force due to electric charges suffer as compared with the corresponding measurements in magnetic fields by reason of the impracticability of obtaining an "electric needle" corresponding to the oscillating magnetic needle. But while it is true that a permanently electrified needle cannot be obtained, it is possible to have one that has equal charges of opposite sign induced by the field. This paper contains an account of experiments on the use of such a needle for the measurement of electric fields, steady or alternating, and for the experimental illustration of some of the laws of electrostatics.

Theory.

Consider a cylindrical needle supported at its centre by a fibre whose torsional control is negligible, placed with its axis horizontal in a uniform horizontal field of strength F. The needle will move so as to set itself parallel to the field. If disturbed from that position a restoring couple will come into play. This couple will be proportional, both to the induced charge, *i. e.* to the "pole-strength," and to the field. As the former is proportional to the latter, it follows that the couple will vary as the square of the field-strength. It will diminish as the angle between the axis of the needle and the direction of the field increases, and will clearly be zero when the two are at right angles. In the case of an

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ellipsoid the couple is proportional to the sine of twice the angle θ between the long axis and the field. Assuming that law to hold for a cylindrical needle, we may write the couple equal to $a\mathrm{F}^2 \sin 2\theta$, where a is some constant. If θ is small we have

$$\frac{\text{Couple}}{\text{Angular displacement}} = \frac{a\mathrm{F}^2 \sin 2\theta}{\theta} = 2a\mathrm{F}^2 \text{ (constant),}$$

in which case the vibrations of the needle will be isochronous, the time being given by

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{2a\mathrm{F}^2}} \quad \cdots \quad \cdots \quad \text{(I)}$$

where $\mathrm{I} = $ moment of inertia of the needle.

Writing $\mathrm{T} = \dfrac{1}{\mathrm{N}}$, we may denote the frequency of vibration by

$$\mathrm{N} = b\mathrm{F}, \quad \cdots \quad \cdots \quad \cdots \quad \text{(II)}$$

where b is a constant depending on the form, size, and mass of the needle. Thus it appears that the strengths of uniform fields are directly in proportion to the frequencies of such a needle oscillating in them.

Of course if the controlling couple due to the suspending fibre is sufficient to take into account, this may be done by observing the frequency N_0 when the electric field is nil. Denoting the frequency actually observed in field by N', we have

$$\mathrm{N} = \sqrt{\mathrm{N}'^2 - \mathrm{N}_0^2} = b\mathrm{F}. \quad \cdots \quad \cdots \quad \text{(II)}'$$

This law was tested by establishing an electric field between a pair of parallel circular plates, each 12 cms. in diameter, kept at a constant distance apart. The difference of potential between them was obtained by means of a Kelvin voltaic pile, whereby any voltage from 400 volts downwards could be applied. The volts were measured by a Kelvin multicellular electrostatic voltmeter reading to 400 volts. The vibrations were observed by the help of a plane mirror placed below the needle, as shown in fig. 1, which represents the apparatus used. The image of the needle in the mirror was viewed through a telescope, in the focal plane of which an image of the needle appears vertical in the position of rest. The vertical cross-wire is set to bisect it.

The time of a counted number of oscillations was obtained by means of a stop-watch reading to fifths of seconds, sufficient passages being allowed to admit of the whole period of time measured being from one to two minutes.

The apparatus was enclosed in a balance-case in order to avoid draughts. Observations were restricted within an amplitude of oscillation of about 5°, in which case the error due to finite amplitude is negligible.

Fig. 1.

Kelvin Pile.

Reading Telescope.
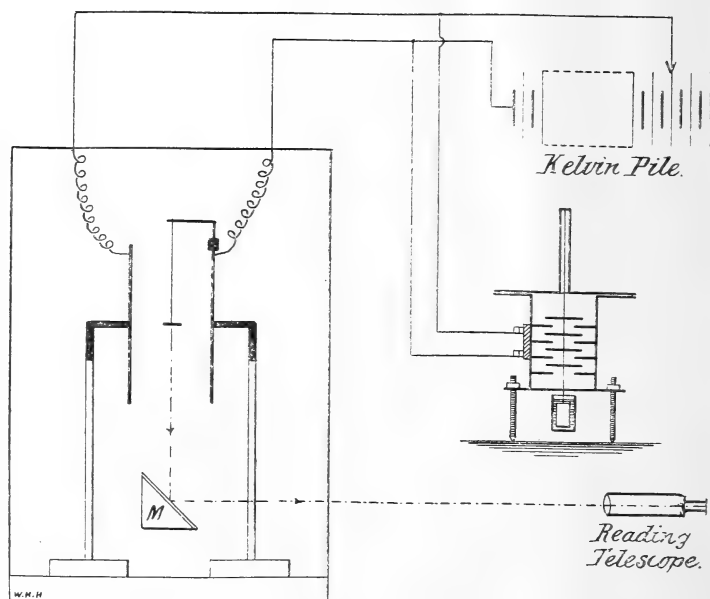
The needle used was of copper wire, 1·4 cms. long and 2 mms. in diameter, suspended by a quartz fibre about one-tenth of a mm. in diameter and 8 centimetres long. The plates were at a distance of 3 cms. apart. The results gave as expected a straight-line law connecting V and N.

Disturbance of the Field due to the presence of the Needle.

The effect of the presence of such a needle in an electric field is in every case to produce a concentration of the field near the needle. Further, if the charged conductor forming one of the boundaries of the field is sufficiently near the needle, the distribution of the charge on that boundary will be altered. This of course prevents the application of the electric needle for the accurate comparison of the electric forces *near* the surfaces of conductors of different forms and curvatures. It was considered desirable before proceeding

further to investigate the nature of the disturbance caused, and to determine its bearing on the applicability of the needle for exact quantitative measurements of field.

A case of importance is that of the field between two parallel metal plates. The question may be put thus : if an electric needle be placed in an infinite uniform field, at what distance does the disturbing influence of the needle cease to be appreciable? I know of no mathematical treatment for the case of a short cylindrical needle such as was used above. The following experiment was therefore performed. The parallel plates already described were used, and the arrangement of apparatus was as in fig. 1. The plates could be moved perpendicular to themselves so as to alter their distance d apart. A constant potential-difference of 400 volts was maintained between them. The needle used was of aluminium, of the following dimensions :—

$$\text{length} = 1\cdot 526 \text{ cms.,}$$
$$\text{diameter} = \cdot 101 \quad \text{,,}$$
$$\text{and} \quad \text{mass} = \cdot 0329 \text{ grams.}$$

The distance d was varied between the limits of $6\cdot 04$ cms. and $1\cdot 92$ cms., and values of N for these and several intermediate values of d were determined. The needle, suspended by a quartz fibre, was set centrally between the plates in each case. The results may be tabulated as follows :—

$$T_0 = 18\cdot 80 \text{ secs.}$$

No. of exp.	d in cms.	T in secs.	N' per sec.	$N = \sqrt{N'^2 - N_0^2}$	$N \times d.$
1	6·04	7·20	·1392	·1282	·775
2	5·33	6·40	·1562	·1468	·784
3	4·82	5·82	·172	·163	·785
4	4·40	5·34	·1866	·178	·783
5	4·03	5·00	·200	·193	·778
6	3·00	3·69	·271	·2656	·797
7	2·53	3·04	·329	·325	·822
8	2·325	2·72	·368	·364	·844
9	2·07	2·32	·433	·430	·885
10	1·92	2·04	·490	·487	·916

These numbers are represented graphically in fig. 2, where products of frequency × distance apart of plates $(N \times d)$ are plotted against distance apart of plates (d). From this curve it may be seen that beyond $d = 4\cdot 5$ cms., or say exceeding three times the length of the needle, the disturbing

effect of the needle is inappreciable, and the product $N \times d$ remains constant. This may be seen further by the aid of a

Fig. 2.

Length of Needle = 1·526 cms.
Diameter of Needle = ·101 cms.

Frequency
× dist. apart
of plates.
($N \times d$)

Distance apart of plates in cms.
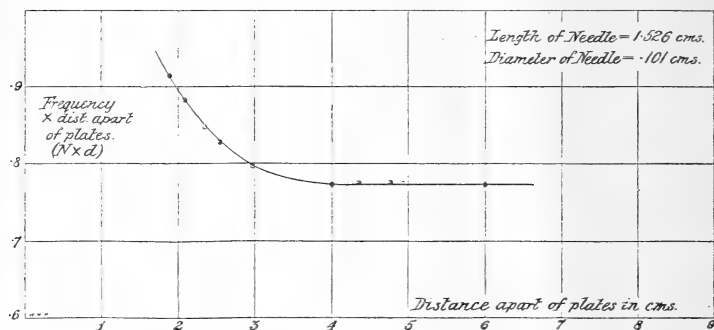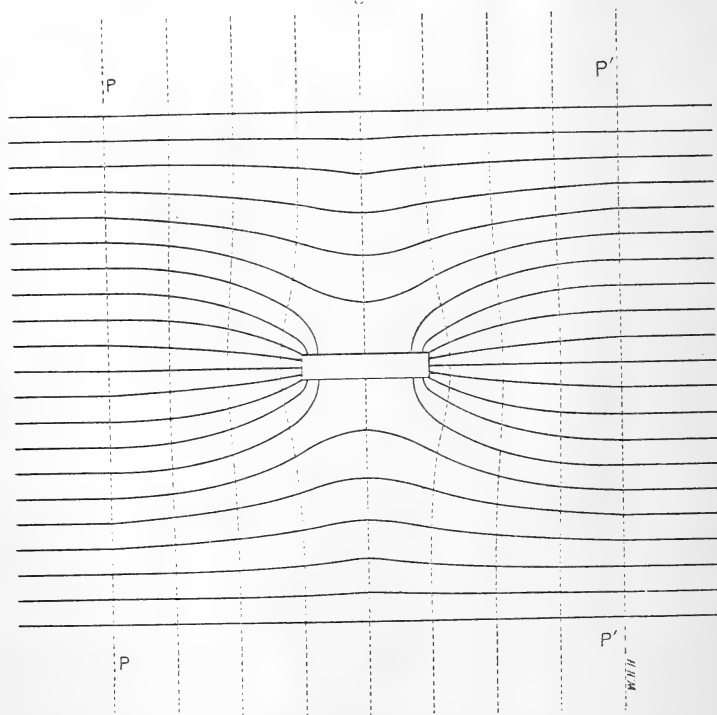
Fig. 3.

diagram (fig. 3) representing the lines of force and the equipotentials near the needle when the axis of the latter is along the direction of the undisturbed field.

The lines of force are shown as full lines, the equipotentials as broken lines. The equipotentials near the needle are much curved, but the effect diminishes rapidly on going away from the needle, so that the equipotentials PP and P'P' are practically straight. Beyond these the equipotential surfaces are a series of parallel planes, and the field is uniform and of the same value as if the needle were not present.

The result suggests the possibility of determining by the use of an electric needle the specific inductive capacity K of a dielectric. For if a pair of parallel plates kept at fixed potential-difference be used to establish a uniform field, their distance apart well exceeding the limiting distance at which the disturbing effect of the needle is appreciable ; and if between one of the plates and the critical plane a parallel slab of the dielectric be introduced, the effect is to increase the field in the air space by an amount involving K in a simple and well-known way. Of this nothing further will be said in this paper, as careful experiments are being undertaken dealing with the determination of specific inductive capacity by such a method.

In regard to the disturbance introduced in other cases, it is important to notice that for a given arrangement of conductors and given position of the needle, variations of field due to alteration of potential will be measured by the frequency of the needle, the ratio of frequencies being unaffected by the distortion of the field. For example, in the case of the field between a pair of parallel plates at a distance apart only slightly exceeding the length of the needle; though the frequency of the needle is greater, as shown above, than in an infinite field of the same strength, yet it increases exactly in proportion to the difference of potential between the bounding plates.

The choice of Needles.

It is desirable that the needle should be as small as possible, to reduce the disturbing effect; and sensitive, for the measurement of as small fields as possible.

To investigate this latter condition six cylindrical needles were made of aluminium, with dimensions as follows :—

Three needles of diam. of 1 mm.; lengths $1\frac{1}{2}$ cms., 1 cm., ·5 cm.
   ,,   ,,   ,,   ·5 mm.;   ,, $1\frac{1}{2}$,, 1 ,, ·5 ,,

A similar set was afterwards made in brass.

These needles were drilled centrally at right-angles to their length with a hole of about ·3 mm. in diameter, into which was fitted a suspending wire of the same diameter, and of

length varying from 5 to 10 mms. To this wire was cemented
a quartz fibre about $\frac{1}{15}$ mm. in diameter, and about 8 cms.
long, the upper end of the fibre being cemented to a supporting
piece attached to one of the plates. These needles were sup-
ported in succession between the plates, which were kept at
the same distance apart and at the same difference of potential
throughout the experiment. The time of vibration of each
was thus determined :—The dimensions and mass of the
needles being known, the moments of inertia could be calcu-
lated, and by equation (I) the couple on each when in a
specified field could be determined.

No high degree of accuracy was to be expected where the
moment of inertia of such small needles was involved. The
results given below, however, show sufficiently the effect of
alteration of length and of diameter. The frequency came
out less for the brass than for the aluminium needles, just as
calculated from their relative density. The values of the
frequency in column 4 are for needles of aluminium.

Dependence of Frequency on Dimensions of Needle.
Electric field = 100 volts per cm.

Length (cms.).	Diameter (cms.)	1000 × Couple (dyne-cms.)	Frequency (per sec.) N.	Relative couples.	Relative frequencies.
1·526	·101	9·82	·197	100	100
1·103	·1025	4·42	·233	45	118
·510	·1025	·643	·275	6·54	140
1·500	·050	·693	·337	70·5	171
1·000	·050	·253	·371	25·7	178

These numbers show that although the couple on the
needle increases rapidly with length and slowly with diameter,
yet the effect on frequency is more than counterbalanced by
the increase in the moment of inertia. Thus whilst, as the
first three rows of numbers above show, the length of needle
decreased in the proportion of about 3·1, the couple decreased
in the proportion of about 15 : 1, whilst the frequency went
up in the proportion of only 1 : 1·4.

Hence, while as the needle gets smaller the disturbing
effect on the field rapidly diminishes, its frequency of vibration
increases, though slowly. This is a favourable circumstance
to be borne in mind in experiments such as that suggested
above for the determination of specific inductive capacity.

Where the disturbance is not of importance, and it is desired to show the vibrations, either directly or by projection in a lantern, the needle may with advantage be large. Aluminium needles of $1\frac{1}{2}$ cms. length and 1 mm. diameter have been found very satisfactory ; brass needles should be of smaller diameter. For strong fields the needles may be still heavier, and the suspension may be of silk threads consisting of several single-fibres. There is thus little danger of breakage of the suspension, where, as for purposes of lecture demonstration, the needle has to be moved about and may be more or less roughly used.

Applications.

The electric needle may be regarded as a measurer of *electric force*, and be applied to test the field-strength due to any arrangement of conductors, effect of alteration of their relative position, or of the potential, &c. In some cases exact results will not be obtained owing to the disturbance introduced by the needle itself. In others the disturbing effect of the needle does not enter. Of the first class the following example may be cited :—

Comparison of the electric field at different points on the surface of any conductor.

The case of a pear-shaped conductor (fig. 4) was examined.

Fig. 4.
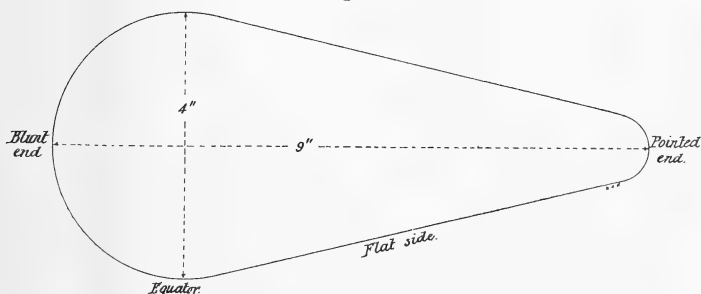
Four positions on it were chosen, namely, at the pointed end and blunt end, at the equator, and opposite the middle of the flat side (where the curvature is single). The needle used was $1\frac{1}{2}$ cms. long. Its nearer end was set in each case

at a distance from the conductor equal to half the length of needle. The results were as follows :—

Position.	Time of oscillation T (secs.).	Relative frequency.
Pointed end	2·55	1
Blunt end	3·20	·8
Equator..............	3·25	·78
Flat side	3·85	·66

For a couple of spherical conductors at a distance from other conductors the following numbers were obtained :—

Diameter of sphere.	T.	Relative frequencies.	Electric force (theoretical).
12 cms.	2·40 secs.	1	1
3　,,	1·23　,,	1·96	4

These comparative numbers indicate the increase of electric force F with curvature. By means of a proof-plane the increase of surface-density σ of charge with curvature may be shown. In both cases the numbers suffer in accuracy owing to the disturbance introduced by test needle and proof-plane respectively.

Such experiments as the above are best performed by attaching the conductor by a fine wire to the inner coating of a charged Leyden jar. The potential then changes only very slowly with time.

It may be interesting to add that the potential of the pear-shaped conductor, as determined by an electric needle, was 6200 volts in the case to which the numbers above refer.

In the following two cases the disturbing effect of the needle is of no significance :—

(1) Wherever the variation of electric force with alteration of potential is to be found, no motion of the bounding conductors or of position of needle taking place. The electric force as measured by the frequency of the needle is proportional to the difference of the potential.

(2) To prove that inside a charged conductor (not enclosing an insulated charged conductor) the electric force is zero. Though this theorem is one of fundamental importance, it is

not easy to demonstrate it experimentally. In the electric needle we have a means of doing this of a simplicity corresponding to that which the proof-plane affords for showing that the charge inside the conductor is zero.

As the needle is lowered into the conductor, *e.g.* a metal can or a cylinder of wire netting, the frequency of the needle falls until when well inside its value is the same as when the conductor is discharged.

On the Shielding Effect of Dielectrics.

Having mounted a needle within a glass specimen-tube in order to do away with the effects of air-currents, it was observed that even when in the strongest electric fields the needle was quite unaffected. The tube acted as if it were a conductor. As this effect might be due simply to the unclean condition of the surfaces, the experiment was repeated with a carefully cleansed glass lamp-chimney, with the same result. It appears then, that owing to the conductivity of the glass, slight as it is, a closed glass vessel screens electric force from its interior, just as if it were a conductor.

The same result was found for a sheet of mica of thickness $\frac{1}{12}$ mm. bent into a cylinder. A mica cylinder in ordinary air screens the interior completely (when placed in a steady field, as in fig. 5). If heated over a hot plate and quickly transferred to the field so as to surround a needle swinging in it, there is seen to be electric force at the needle ; but it quickly dies down, and in 20 or 30 seconds the needle once more vibrates in its own natural period, depending on the torsional couple of the quartz suspension.

This conductivity of the mica appears to be a surface effect, probably due to the deposition of moisture from the atmosphere. For if the mica sheet be immersed for some time in melted paraffin-wax at a temperature close on 200° C., then taken out and allowed to cool, the sheet (made into a cylinder as before) is found to be capable of transmitting the field for some time. Bnt even in this case the field within the cylinder falls off with time. Thus in one case a needle within a paraffined mica cylinder gave 12 vibrations in 10 seconds at first, but four minutes later only 8 vibrations were counted in the same time, the fall of frequency being according to the exponential law. Electrical separation has slowly taken place in the mica. If now the plates between which the field was produced be discharged, the cylinder should be left charged, and an electric field should exist in the interior. This is found to be the case. The field dies out with time, as in the experiments in the constant

field. Another way of showing this effect of minute conductivity is to introduce the cylinder into a steady field, leave it there for some time, and then turn it about its axis through 180°. The field inside the cylinder is seen to be for the moment greater than that of the original field, for on the latter is superposed the field due to the separated charges on the walls of the cylinder. With cylinders of thicker sheets of mica similarly treated with paraffin-wax, the rate of falling off of the interior field is found to be greater. These experiments indicate an appreciable though exceedingly small conductivity in the mica itself.

Experiments on the shielding effect of ordinary white paper were made. Cylinders of paper screen the interior completely from the surrounding field. But on heating for a short time over a hot plate the shielding effect vanishes ; in about one minute, however, the frequency of the needle within is found to have fallen considerably, and a minute later is the same practically as in zero field. Dry paper then acts as a perfect insulator. But by the absorption of moisture when exposed to the open air it quickly gains appreciable conductivity.

With paraffined paper the shielding power was nil, and remained nil for days in the open air of the room. Of all the materials tried this alone permitted for any length of time the undiminished transmission of electric force. Thus by applying a thin coating of paraffin wax to a sheet of dielectric, and then observing its shielding effect on an electric needle, we have a convenient means of testing the conductivity of the dielectric, which removes all doubts as to surface-action. This effect of the high insulating, non-hygroscopic, quality of paraffin-wax is made use of in the experiments on mica above described.

Experiments with alternating fields.—As with steady fields glass and mica proved sufficiently conducting to screen off electric effects, experiments were made with these materials, using alternating electric fields. The alternating voltage was obtained by the use of the Musical Arc. Mr. Duddell, I may say here, very kindly advised me as to the conditions for obtaining high voltages by this means. The connexions were as shown in fig. 5.

The alternating voltage on the self-induction P in the shunt circuit to the arc was found to be about 200 volts. This was transformed up by means of a second coil S, to the terminals of which the parallel plates were connected. In this way any voltage up to 1000 volts could be maintained between the plates. The frequency of alternation of this

voltage was found, by means of a sonometer and standard fork, to be 1700 per second. The needle was allowed to vibrate in the field with

      (*a*) glass cylinder surrounding it :
      (*b*) cylinder taken away.

Fig. 5.
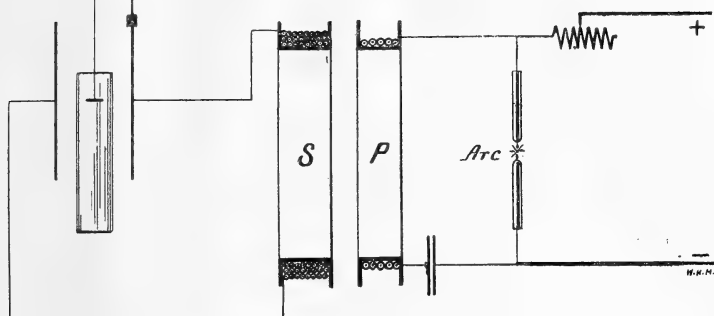
The time of vibration was found to be the same in each case, namely 1·83 secs. The glass cylinder had a diameter of 5 cms., the thickness of wall being 1·4 mm. With such dimensions the weakening of the field inside the cylinder due purely to the specific inductive capacity effect should only be small. With a cylinder of thicker glass, but diameter only slightly exceeding the length of the needle (1 cm.), the frequency of vibration was distinctly less in case (*a*) than in case (*b*). With mica cylinders the same result was obtained.

These experiments emphasise the necessity of using alternating fields in all cases where from the measurement of the force experienced by a mass of a dielectric its specific inductive capacity is to be determined. In steady fields, unless the specific conductivity of the material be quite nil, the force will ultimately be the same as for a conductor. This consideration will account for the high values of k obtained in some of the experiments of Boltzmann (see Gray's Abs. Measmts. vol. i. p. 465).

Use of Electric Needle to measure Volts.

A pair of insulated parallel metal plates with a needle suspended centrally between them forms a simple means of measuring high voltages. For we have

    volts across plates = frequency of needle × a constant.

The constant may be determined at a low voltage by any voltmeter available. The highest voltage measurable is limited by the difficulty of counting the frequency of the

needle if this exceeds 4 or 5 per second. But the range may be extended by opening out the plates; for as already shown the frequency of the needle for constant difference of potential is inversely as the distance apart of the plates, provided that the minimum distance used exceeds three times the length of the needle.

The constant k is most simply defined as the volts across the plate required to produce a frequency of oscillation of 1 per sec. for a distance apart of the plates equal to 1 cm. Then if

d = distance apart of plates,
N = frequency,
V = voltage between plates,

we have

$$V = kd \times N. \quad \ldots \ldots \ldots \text{(III)}$$

An instance of this application has been given already Another may be added :—

The difference of potential yielded by a Wimshurst machine was measured by charging a Leyden jar, the outer coating being connected to one pole, the inner to the other pole, the poles being kept well apart. The jar was connected with the pair of parallel plates, which were at a distance of 10·5 cms. apart. The frequency of the needle in the field between them was 2·04 per sec. The needle used had a constant (reduced to d = 1 cm.) of 1300 volts. Applying formula III, we have

$$V = 1300 \times 10·5 \times 2·04$$
$$= 27800 \text{ volts.}$$

The machine, it may be added, was working badly, giving a spark between balls of 2 cms. and 5 cms. diameter respectively, Leyden jars being in, of something less than 1 cm. length.

These experiments were made in the laboratories of the Birkbeck College. I would record my thanks to the Principal for facilities for carrying them out.

XXXIV. *On the Resolving Power of Spectroscopes.*
By H. Morris-Airey, *M.Sc.*[*]

THE object of this note is to call attention to a factor which appears to have been neglected in the consideration of the resolving power of spectroscopes.

It has been shown by Rayleigh that, using an indefinitely narrow slit, the resolving power of a grating-spectroscope

depends only on the total number of lines on the grating which are utilized. In practice, however, we frequently make use of slits so wide that the theoretical resolving power is far from realized. This case was considered by Schuster, who introduced the idea of " purity " of spectra.

If light containing two homogeneous radiations differing in wave-length by $\delta\lambda$ falls on the slit of a spectroscope, the condition for resolution of the corresponding lines has been shown by Schuster to be

$$\frac{\delta\lambda}{\lambda} = \frac{2\lambda + \psi d}{\lambda R},$$

where d is the width of the slit, R the theoretical resolving power as defined by Rayleigh, and ψ is the angle subtended at the slit by the diameter of the collimator-lens.

This expression appears to contain no factor depending on the position of the grating or prism; but a simple consideration shows that the purity becomes greater as the angle of incidence of the light on the grating increases.

When the slit has a finite width we can consider the light falling on the grating to consist of a number of beams of parallel rays making slightly different angles with the normal to the grating.

Let ϕ be angle of incidence and θ the angle which the corresponding diffracted beam makes with the normal.

If e is the grating space, we have the usual relation

$$e(\sin \phi - \sin \theta) = \pm n\frac{\lambda}{2};$$

from which $$d\theta = \frac{\cos \phi}{\cos \theta} d\phi,$$

and $$\frac{d\theta}{d\lambda} = \frac{n}{2e \cos \theta} \text{ for any value of } \phi.$$

The diffracted beams will thus make slightly different angles with the normal, the total variation being $d\theta$. This variation can be made as small as we please by increasing the value of ϕ so that in the limit when the light falls on the grating at grazing incidence the diffracted beams corresponding to any particular wave-length and order all make the same angle with the normal, and will be received in the telescope under nearly the same conditions as if the collimator-slit had been very narrow. Further, $\frac{d\phi}{d\lambda}$ is also increased, so that the total effect is to greatly increase the purity of the spectrum.

This result is really included in Schuster's formula when we examine the meaning of the term ψ.

Hitherto this has always been interpreted to mean in every case the angle subtended by the collimator-lens at the slit. But it is clear that by tilting the prism or grating sufficiently, only part of the beam passing through the lens is effective. Reference to the proof of the formula * shows that in that case it is only the width of the effective portion which should be considered in determining the value of the angle ψ.

Hence that angle really depends on the position of the prism or grating, and may in the limit be reduced to zero. The smaller ψ is, the greater is the purity, and ultimately by reducing its value the full resolving power is reached with a great width of slit.

This is confirmed by experiment, and I find that with a small Thorp grating containing between 1300 and 1400 lines it is easily possible to separate the sodium D-lines even when the slit is half a centimetre wide ; in fact in most cases the collimator can be entirely omitted.

The application of this device is limited by the great loss of light by reflexion at the surface of the grating when large angles of incidence are used.

Cases, however, do occur where the principle may be used with advantage. For example, it is very convenient in the photometry of arc-lamps when using the Vierordt colour-photometer, where large errors are introduced, notably in the red end of the spectrum, owing to differences in the purity of the two spectra under comparison.

Physical Laboratory, Manchester.

XXXV. *Quantitative Relation between Molecular Conductivities.*
By Philip Blackman†.

THE formation of a salt MX from the acid HX and the base MOH may be represented by the equation

$$H^{\cdot} + X' + M^{\cdot} + OH' = M^{\cdot} + X' + HOH.$$
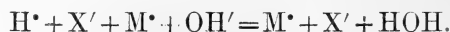
The initial and final electrical states differ by the expression $H^{\cdot} + OH' = H.OH$; hence, we should expect in the formation of the salt the disappearance of electrical conductivity equivalent to that required by the above equation.

This quantity of electrical conductivity will depend on

---

(1) the temperature, (2) the molecular concentration, and (3) the nature of the acid. Supposing these three conditions be fixed, there ought to be a constant quantity for the electrical conductivity corresponding to $H^\cdot + OH' = H.OH$.

Let μv_{HX}, μv_{M_1OH}, μv_{M_2OH}, μv_{M_1X}, μv_{M_2X}, represent the molecular conductivities (all measured at the same molecular concentration v, and at the same temperature) of the acid HX, of the bases M_1OH and M_2OH, and of the salts M_1X and M_2X respectively. Then, according to the above,

$$\mu v_{HX} + \mu v_{M_1OH} = \mu v_{M_1X} + K,$$
$$\mu v_{HX} + \mu v_{M_2OH} = \mu v_{M_2X} + K,$$

where K is a constant.

Hence $\quad \mu v_{HX} + \mu v_{M_1OH} - \mu v_{M_1X} = \mu v_{HX} + \mu v_{M_2OH} - \mu v_{M_2X}.$

The following values, calculated from the data in the 'Physikalisch-Chemische Tabellen, von Landolt und Börnstein' *, illustrate the above :—

v (at 18°) =	1.	2.	10.	32.	100.	1000.	1024.	10,000.
$\mu v\frac{1}{2}H_2SO_4+\mu vKOH \ -\mu v\frac{1}{2}K_2SO_4$ ...	287	300	317	418	387	422	468	470
$\mu v\frac{1}{2}H_2SO_4+\mu vNaOH-\mu v\frac{1}{2}Na_2SO_4$...	284	297	305	420	381	413	481	461
$\mu v\frac{1}{2}H_2SO_4+\mu vLiOH-\mu v\frac{1}{2}Li_2SO_4$....	268	282	314	419	390	...	482	
$\mu vHNO_3+\mu vKOH \quad -\mu vKNO_3$......	374	...	423	492	441	439	485	490
$\mu vHNO_3+\mu vNaOH \quad -\mu vNaNO_3$ ...	364	...	411	491	428	429	494	480
$\mu vHNO_3+\mu v\frac{1}{2}BaO_2H_2-\mu v\frac{1}{2}BaN_2O_6$.	...	...	...	473	...	...	481	
$\mu vHCl+\mu vKOH \quad -\mu vKCl$	358	...	418	490	439	437	489	485
$\mu vHCl+\mu vNaOH \quad -\mu vNaCl$........	358	...	408	492	433	425	490	479
$\mu vHCl+\mu v\frac{1}{2}BaO_2H_2-\mu v\frac{1}{2}BaC_2$......	...	...	...	470	...	...	490	
$\mu vHCl+\mu vLiOH \quad -\mu vLiCl$	344	...	...	493	442	...	491	
$\mu vHI+\mu vKOH \ -\mu vKI$..............	...	...	426					
$\mu vHI+\mu vNaOH-\mu vNaI$	...	...	427					
$\mu vHI+\mu vLiOH-\mu vLiI$................	...	...	411					
$\mu vHIO_3+\mu vKOH \ -\mu vKIO_3$....	...	...	...	442	...	...	482	
$\mu vHIO_3+\mu vNaOH-\mu vNaIO_3$	...	...	...	441	...	...	481	

The last equation, by subtracting the common term μv_{HX} and then by transposing, becomes

$$\mu v_{M_1OH} - \mu v_{M_2OH} = \mu v_{M_1X} - \mu v_{M_2X}.$$

The equation

$$\mu v_{M_1OH} - \mu v_{M_2OH} = \mu v_{M_1X} - \mu v_{M_2X},$$

* The table, which was first drawn up from an earlier edition of Landolt-Börnstein, has been corrected by means of the latest (1905) edition recently published.

it will be at once noticed, furnishes a means of calculating the molecular conductivities of (1) insoluble or unstable bases, and (2) insoluble or unstable salts, the molecular conductivities of which cannot be determined by direct measurement.

Similarly, it can be shown that

$$\mu v_{M_1 OH} - \mu v_{M_2 OH} = \mu v_{M_1 X_1} - \mu v_{M_2 X_1}$$
$$= \mu v_{M_1 X_2} - \mu v_{M_2 X_2}$$
$$= \ \ldots \ =$$

Hence
$$\mu v_{M_1 X} - \mu v_{M_2 X} = \mu v_{M_1 X_1} - \mu v_{M_2 X_1}$$
$$= \mu v_{M_1 X_2} - \mu v_{M_2 X_2}$$
$$= \mu v_{M_1 X_3} - \mu v_{M_2 X_3}$$
$$= \ \ldots \ldots$$
$$= \text{constant.}$$

This final equation, it will be observed, is identical with the hitherto unexplained fact discovered by Kohlrausch (and further extended by Ostwald, see *Lehrb. der Alg. Chemie*, ii.).

East London Technical College,
        London, E.

XXXVI. *Notices respecting New Books.*

Ions, Electrons, Corpuscules. Mémoires réunis et publiés par H. ABRAHAM et P. LANGEVIN. Paris : Gauthier-Villars. 1905. Premier Fascicule, pp. xvi + 512. Second Fascicule, pp. 513–1138.

IF any proof were needed of the extent to which the modern theory of electrons dominates physical science, it might, not inappropriately, be regarded as furnished by the publication of the two large volumes under review. The Société Française de Physique, which has already done good work in rendering accessible to numerous readers collections of important memoirs on various branches of physics (issued under the title of *Collection de Mémoires sur la Physique*), has performed another most useful service by collecting all the scattered papers dealing with the rise and development of the electronic idea, and arranging them in a form which renders it an easy matter to look up the particular contributions towards any branch of the subject made by different investigators.

Originally, the work was to have been edited by the late M. Alfred Potier (to whose memory it is dedicated), but owing to

his death the editorship passed into the hands of Messrs. Abraham and Langevin—two well-known investigators who have themselves made important contributions to the electronic theory. The arrangement of the matter is alphabetical as regards the authors' names, and chronological as regards the contributions of each author. At the beginning of the book is provided a table of contents which practically forms a subject index, a separate author index being provided at the end of the book. This arrangement greatly facilitates the task of finding any information the reader may be in search of, and enhances the value of the book as a work of reference.

It is safe to say that no worker in this most fascinating branch of physics can afford to be without a copy of this timely and important publication.

La Séparation Electromagnétique et Electrostatique des Minerais. Par DÉSIRÉ KORDA. Paris: L'Éclairage Électrique. 1905. Pp. 219.

THE subject of electromagnetic separators is one which has not before been treated in a systematic manner, and the present valuable little monograph should prove of considerable interest to mining engineers. The application of purely electrostatic methods to the separation of different kinds of ore is of comparatively recent origin, and has not as yet attained to anything like the importance which may be claimed for electromagnetic methods. The author devotes a few pages to the electrostatic method, and then gives a detailed account of the practically important electromagnetic separators. Then follow chapters on the magnetic properties of minerals and the theory of magnetic separators; the treatment of the latter subject is such as to form an elementary treatise on electromagnetism for readers ignorant of the subject, with a view to rendering the book self-contained. A good account is next given of the various magnetic separating plants at present in existence, and of the results obtained with them. The preliminary treatment of the ore before it passes into the separator is then dealt with, and in a concluding theoretical chapter the author gives a brief outline of paramagnetic substances other than iron.

Essais des Matériaux. Notions Fondamentales Relatives aux Déformations Elastiques et Permanentes. Par H. BOUASSE, Professeur de Physique à l'Université de Toulouse. Paris: Gauthier-Villars. 1905. Pp. 150.

THIS volume is the first of a series of small text-books to be published under the general title of " Bibliothèque de l'Elève Ingénieur." It deals in a very general manner with the theory of strains and stresses, and with the leading principles which underlie the construction of various testing-machines. It is not intended to be a mere laboratory guide, nor is any attempt made to describe the technical details of commercial testing-machines.

The task accomplished by the author is, indeed, a much more formidable one, and much more likely to be useful to the reader than a purely descriptive work. The treatment is characterized by vigour and freshness combined with clearness. To a large extent the author's method is based on a study of the stress-strain diagram and its characteristics in various cases. No very advanced mathematical methods are used, although the reader is assumed to have a knowledge of the calculus and of the simpler forms of differential equations. The subject is throughout approached from the purely physical rather than technical standpoint, and the rough-and-ready methods of the purely technical man come in for a good deal of severe comment. We could desire no better introduction to the difficult subject of strains and stresses than this masterly exposition, which augurs well for the success of the new series of text-books.

Elektromagnetische Schwingungen und Wellen. Von Dr. JOSEF RITTER VON GEITLER, a. o. Professor der Physik an der K.K. Deutschen Universität, Prag. Mit 86 eingedruckten Abbildungen. Braunschweig : F. Vieweg und Sohn. 1905. Pp. viii + 154.

THIS monograph forms No. 6 of the series entitled *Die Wissenschaft*, now in course of publication. The subject of electromagnetic waves has always had a singular fascination for all interested in the development of physical science, and the simple and clear non-mathematical account of it contained in the monograph under review should appeal to a very wide circle of readers. The author deals with the subject in the order of its historical development, beginning with a brief review of the old action-at-a-distance theories, Faraday's ideas regarding strains and stresses in a medium, Clerk Maxwell's brilliant theoretical developments of those ideas, and their practical realization in the epoch-making discoveries of Hertz. The remaining half of the volume deals with the methods of detecting electromagnetic waves, with their relation to light waves, and with their practical applications in wireless telegraphy.

Dr. J. Fricks Physikalische Technik. Siebente vollkommen umgearbeitete und stark vermehrte Auflage von Dr. OTTO LEHMANN, Professor der Physik an der technischen Hochschule in Karlsruhe. Erster Band: Zweite Abteilung. Mit 1905 in den Text eingedruckten Abbildungen. Braunschweig : F. Vieweg und Sohn. 1905. Pp. xx + 1002.

SOME time ago we had occasion to review Part I. of Vol. I. of this valuable work, and now Part II, completing the volume, is before us. The subjects dealt with are measurements of length, mass, and time ; various forms of apparatus illustrating the laws of statics ; experiments on the general properties of solids, liquids, and gases ; the measurement of temperature and construction of

thermometers; the phenomena connected with the change of physical state; calorimetry, hygrometry, and heats of combination; the laws of dynamics; waves and wave-motion; hydrodynamics, the dynamics of gases, and thermodynamics.

In a brief review such as the present, it would be impossible to do justice to the wealth of learning contained in this work. Not only are full directions given for the successful carrying out of lecture experiments on a large scale and the construction of various forms of apparatus, but the work abounds in most interesting passages relating to the history of physics and the gradual evolution of different types of instruments. In many instances defects are pointed out in existing forms of apparatus as usually constructed, and suggestions are made for improvements; the work is thus one which may very profitably be studied by manufacturers of physical apparatus. Special mention must be made of the numerous excellent illustrations which adorn the book. Teachers of physics desiring to become proficient in the art of lecture illustration, or having to prepare estimates of laboratory equipment, would do well to turn to this unique publication for information on these points.

XXXVII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 192.]

December 20th, 1905.—J. E. Marr, Sc.D., F.R.S., President, in the Chair.

THE following communications were read :—

1. 'The Highest Silurian Rocks of the Ludlow District.' By Miss Gertrude L. Elles, D.Sc., and Miss I. L. Slater, Newnham College, Cambridge.

After an introduction dealing with previous work in the district, the authoresses adopt the following classification of the beds :—

		Feet	
III. TEMESIDE GROUP.	B. Temeside or *Eury-pterus*-Shales.	110 to 120	Zone of *Lingula cornea* and *Eurypterus*.
	A. Downton-Castle or Yellow Sandstone.	30 to 50	Zone of *Lingula minima*.
II. UPPER LUDLOW GROUP.	B. Upper Whitcliffe or *Chonetes*-Flags.	150 to 160	Zone of *Chonetes striatella*.
	A. Lower Whitcliffe or *Rhynchonella*-Flags.	110 to 120	Zone of *Rhynchonella nucula*.
I. AYMESTRY GROUP.	B. Mocktree or *Dayia*-Shales.	40 to 150	Zone of *Dayia navicula*.
	A. Aymestry or *Conchidium*-Limestones.	75 to 250	Zone of *Conchidium Knightii*.

515 to 850

A brief outline-description of the main subdivisions is first given, as they appear when followed from Ludlow southward to Overton, eastward to Caynham Camp, westward to Downton-on-the-Rock, and northward to Bromfield, and also near Onibury and Norton. The main tectonic features of the district appear to be due to the superposition of Armorican movements in rocks with a Caledonian trend, held by some rigid mass to the north, presumably the Longmynd massif. A detailed description is then given of the succession, as seen at the following localities :—River Teme, Wigmore Road, Deerhouse Bank, Caynham inlier, the Teme and north-east of the Castle at Downton, Downton-Castle inlier, Mocktree, and near Onibury on the Craven-Arms Road, the Onibury-Norton Lane, and at Norton. The paper closes with a detailed list of fossils obtained by the authoresses, supplemented by the collection in the Ludlow Museum.

2. ‘ The Carboniferous Rocks at Rush (County Dublin).’ By Charles Alfred Matley, D.Sc., F.G.S. With an Account of the Faunal Succession and Correlation. By Arthur Vaughan, B.A., D.Sc., F.G.S.

Rocks of the Carboniferous Limestone Series are exposed along 5 miles of coast near Rush, Loughshinny, and Skerries, in County Dublin. The present paper deals only with the beds near Rush, in the southern portion of this tract, where about 2500 feet of the series are exposed, without allowing for gaps in the succession. The upward sequence is (on the whole) from south to north, and the range is from the Upper *Zaphrentis*- to the Upper *Dibunophyllum*-Zone.

The Rush Slates are the lowest beds, 1380 feet thick, but their base is not visible. They consist of black and dark-grey, well-cleaved argillaceous, and less perfectly-cleaved calcareous, slates ; and they contain bands and nodules of limestone. The peculiar outcrop of some of the limestone-bands is described, and instances of cataclastic structure are noticed. The characteristic fossil is *Zaphrentis* aff. *Phillipsii*.

The Rush Conglomerate-Group succeeds the Rush Slates, after a short interval of passage-beds. It is 500 feet thick, and consists of well-bedded alternations of conglomeratic, pebbly, and sandy limestones, with shales and calcareous flaggy beds. Ordovician and Silurian rock-fragments abound in them, together with many inclusions of Carboniferous Limestone. The group is shown to be of the same age as the Pendine *Syringothyris*-conglomerate and the volcanic rocks of Weston-super-Mare, and its existence indicates that the movement and disturbance in Mid-Avonian times extended over a considerable area.

The beds above the conglomerates are mainly limestones and calcareous shales. They are thrown into numerous sharp folds,

and are occasionally inverted.   The highest beds seen (*Cyathaxonia*-Beds) are correlated with the Eastern Gower or Oystermouth Limestone of the South-Western Province; but the fauna agrees still more closely, and is identical, with that of the highest Avonian beds of the Midlands of England, at Parkhill, Wetton, Thorpe Cloud, etc.   The disappearance by solution of a considerable thickness of limestone is described.

A list is given of the fossils from a large number of horizons in the Rush Series (which is divided into the *Zaphrentis*-, *Megastoma*-, and *Cyathaxonia*-Beds), as well as of the fauna of the Curkeen-Hill Limestone, near Loughshinny, the horizon of which is assigned to the Upper *Dibunophyllum*-Zone, probably below the *Cyathaxonia*-Beds.

The palæontological section deals only with brachiopods and corals.   In that part which deals with the brachiopods the interrelationship of the various members of the more important gentes is discussed in considerable detail.   In the part which is devoted to the corals a new subgenus is suggested, and four new species are described.

January 10th, 1906.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

The following communications were read :—

1. 'The Clay-with-Flints : its Origin and Distribution.'   By Alfred John Jukes-Browne, B.A., F.G.S.

Until recently the Clay-with-Flints has been regarded as being, in the main, a residue from the slow solution of the Chalk.   This was the explanation proposed by Mr. W. Whitaker in 1864, although he admitted that the deposit included some material derived from the Eocene.   Writing in 1865, Mr. T. Codrington thought that an overlying stratum of clay or loam was essential to the formation of Clay-with-Flints.   Lastly, Charles Darwin in 1881 seems to have taken it for granted that it was solely a residue from the Chalk.   Of late years, the opinion has been growing that it consists very largely of material derived from the Eocene.

The present paper is devoted to an examination of the facts, with the view of ascertaining whether the Clay-with-Flints could possibly be derived from the Chalk, or whether the theory of its derivation from the Eocene is confirmed by more detailed enquiry. The author first describes its composition, noting that unbroken flints are not everywhere abundant, that broken angular flints are common, and green-coated flints are not rare ; finally, that if the clay is washed it always yields a residue of sand, composed chiefly of rounded quartz-grains with some of iron-oxide, and both apparently derived from the Eocene sands.

The thickness of the accumulation is next discussed, especially

with reference to sheets of it that lie on fairly-even floors. In such positions it varies from 2 to 12 feet in depth, and large areas occur where it must have an average depth of 3 or 4 feet. The products resulting from artificial solution of chalk are then considered, and a series of analyses is given, from which the average amount of insoluble residue existing in the four lower zones of the Upper Chalk is deduced. Experiments have been made by Mr. William Hill to determine the relative weights of a cubic foot of Upper Chalk and a cubic foot of Clay-with-Flints, in order that allowance might be made for the difference in calculating the quantity of clay which would be left by the solution of a given quantity of chalk. The result shows that 100 cubic feet of the *Micraster-coranguinum*-Chalk will produce only 1·2 cubic feet of clay, and the solution of the *Marsupites*- and *Micraster-coranguinum*-Zones to the extent of 200 feet over any part of the area would only yield clay enough to make a layer 2 feet deep. Lastly, it is shown that the quantity of flints in the Upper Chalk is so much greater than the quantity of clay, that the natural residue could not form a clay-with-flints. Thus, solution of 100 feet of *Micraster-coranguinum*-Chalk would yield a bed of flints about 7 feet thick, and only enough clay to fill up the interstices between the nodules.

The next section is devoted to the distribution of the Clay-with-Flints, and its stratigraphical relations to the Chalk on the one hand, and to the Eocene on the other. In dealing with this part of the subject, details are restricted to the areas lying west and north-west of the London Basin and to the wide area between the London and Hampshire Basins.

From these several lines of investigation the author concludes :—
(1) That the Clay-with-Flints cannot have been formed from mere solution of the Upper Chalk. (2) That all its components, except the unbroken and angular flints, could have been furnished by the Reading Beds. (3) That the positions occupied by it are such that no great thickness of Chalk can have been destroyed to form it, the tracts being seldom more than 30 or 40 feet below the local plane of the Eocene base, or the presumed level of that plane.

Finally, an attempt is made to explain the manner in which the Clay-with-Flints was formed ; and the theory adopted is that the outlying Eocene tracts, which were in existence during late Pliocene time, were broken up and spread out by the severe climatic conditions of the Glacial Period. In post-Glacial time little has been added, but much removed by erosion.

2. ' On Footprints from the Permian of Mansfield (Nottinghamshire).' By George Hickling, B.Sc.
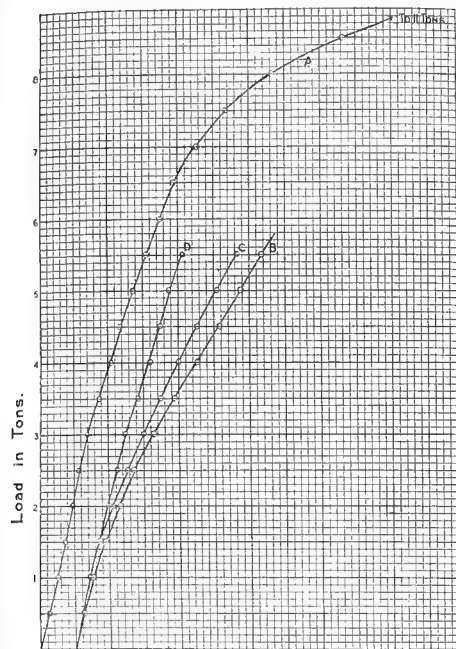
FIG. 1.

FIG. 2.

FIG. 3.

FIG. 4.

FIG. 5.

Load in Tons.

EXTENSIONS – diminished as explained
Scale – 1 unit = $\frac{1}{1000}$ inch

EXTENSIONS – diminished as explained
Scale – 1 unit = $\frac{1}{1000}$ inch
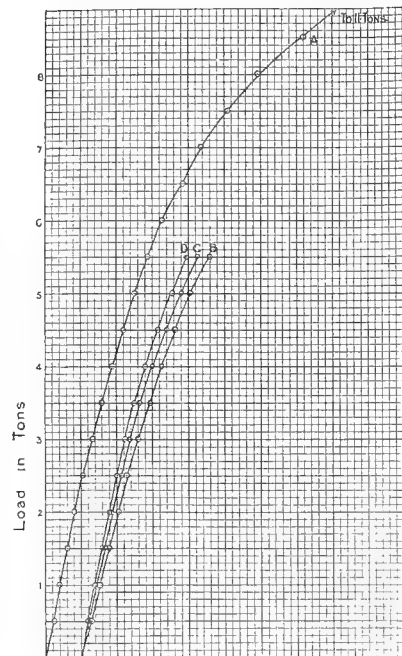
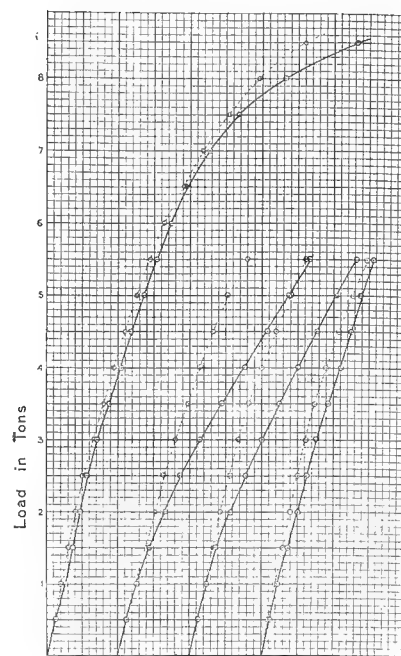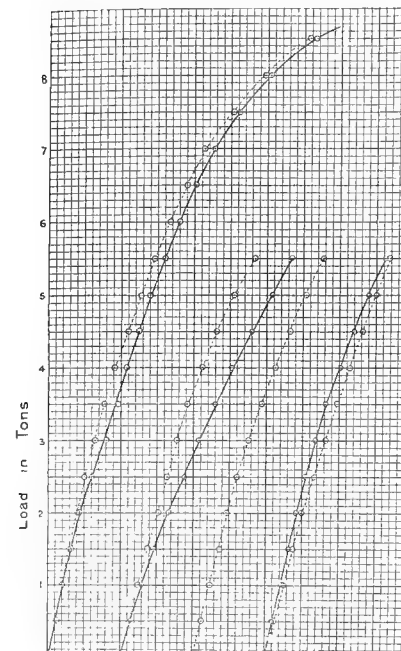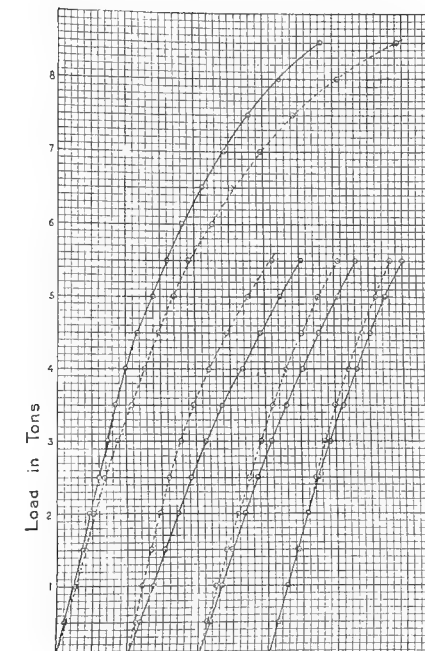EXTENSIONS – diminished as explained
Scale – 1 unit = $\frac{1}{1000}$ inch

EXTENSIONS – diminished as explained
Scale – 1 unit = $\frac{1}{1000}$ inch

EXTENSIONS – diminished as explained
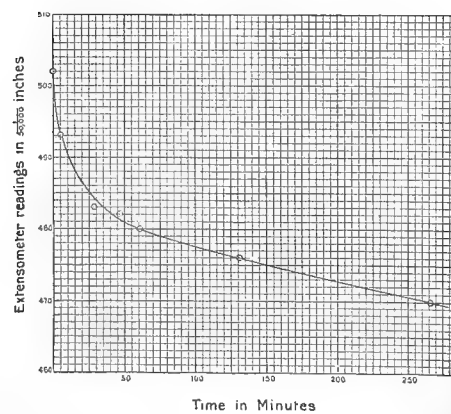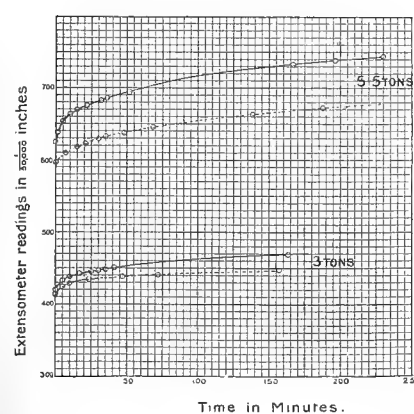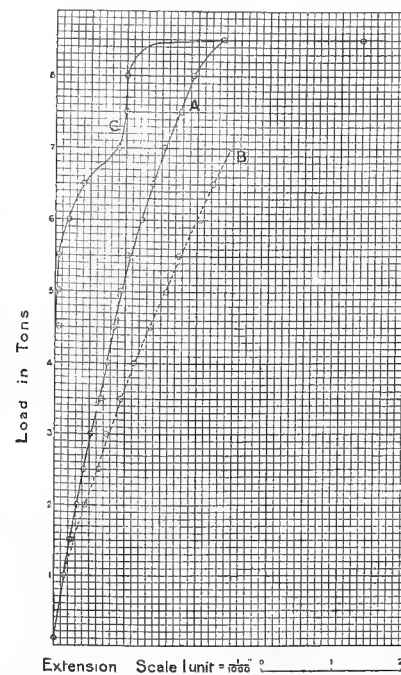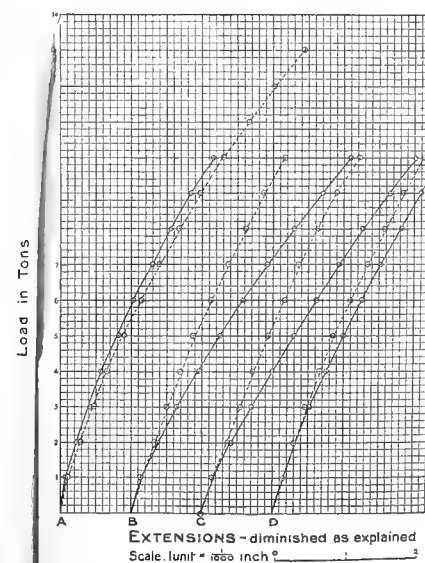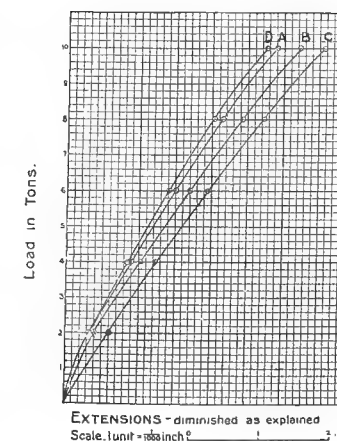Scale – 1 unit = $\frac{1}{1000}$ inch

FIG. 6.

FIG. 7.

FIG. 8.

FIG. 9.

FIG. 10.

Extensometer readings in $\frac{1}{1000}$ inches

Time in Minutes

Extensometer readings in $\frac{1}{1000}$ inches

5·5 TONS

3 TONS

Time in Minutes.

Load in Tons

Extension   Scale 1 unit = $\frac{1}{1000}$

Load in Tons

EXTENSIONS – diminished as explained
Scale. 1 unit = $\frac{1}{1000}$ inch

Load in Tons.

EXTENSIONS – diminished as explained
Scale. 1 unit = $\frac{1}{1000}$ inch

Fig. 2.
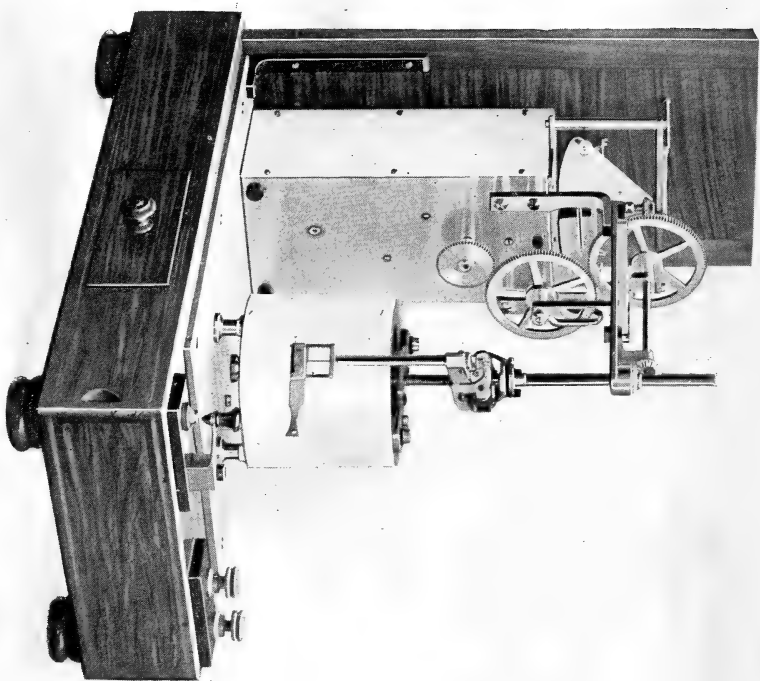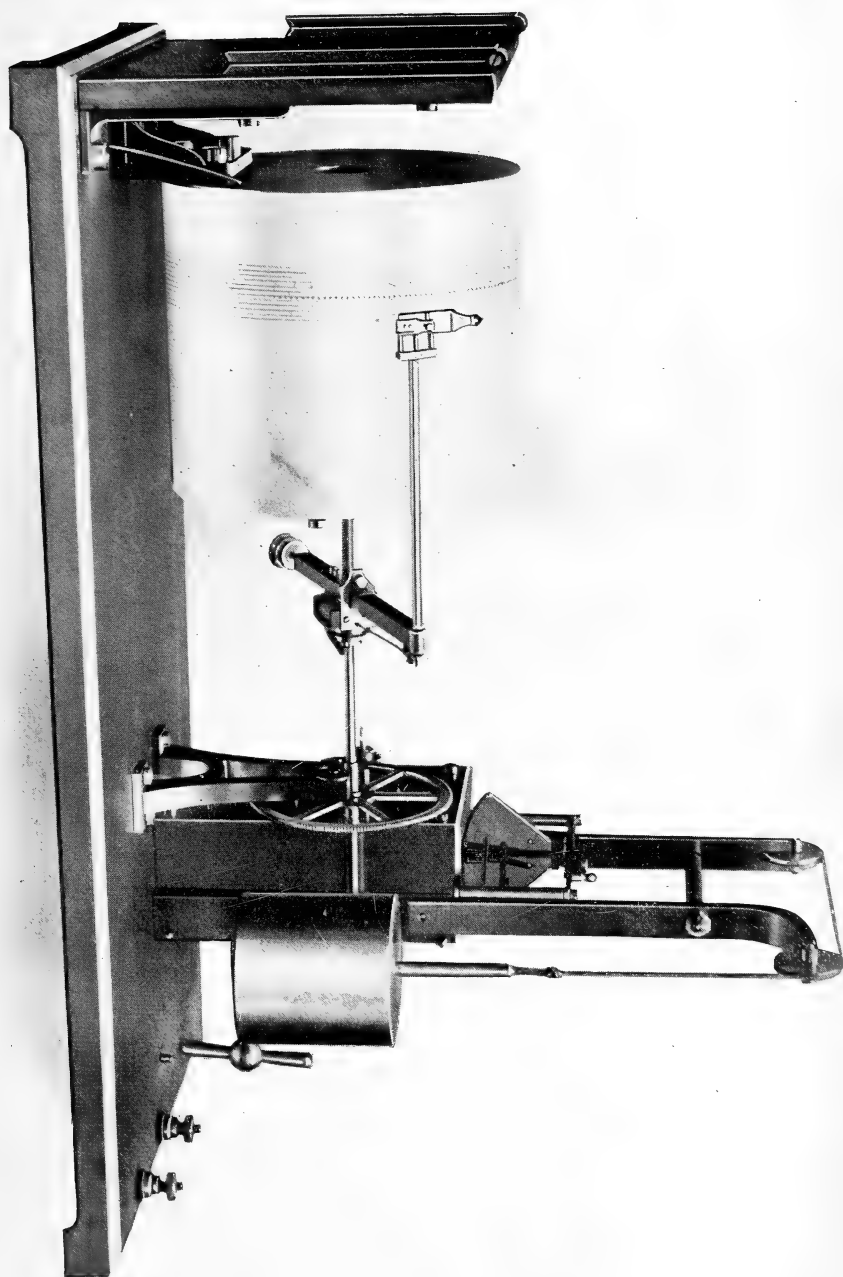
Fig. 1.

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[SIXTH SERIES.]

APRIL 1906.

XXXVIII. *On the Properties of Electrically Prepared Colloidal Solutions.* *By* E. F. BURTON, *B.A.,* 1851 *Exhibition Scholar of the University of Toronto, Research Exhibitioner of Emmanuel College, Cambridge**.

I. *Introduction.*

AS a result of experiments by Professor McLennan † and the writer, it was found that when a cylinder of any metal is enclosed within a second one of the same material, insulated from it, and surrounded by air or other gases, it gradually acquires a negative charge, and after a short time reaches a state of equilibrium at a definite potential below that of the enclosing cylinder. Several different metals were tried and each was found to attain a definite charge. These results led to the conclusion "that a process is going on at the surface of the metal, whereby an excess of positively charged particles is being continually emitted," thus leaving the metal itself with a negative charge.

Some time ago Bredig ‡ found that by means of electrical pulverization of metals it is possible to produce so-called colloidal solutions of metals in very pure water. Experiments on these solutions have hitherto pointed to the conclusion that they consist merely of suspensions of very finely divided metallic particles. The fact that these particles

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were in every case found to be negatively charged, suggested that they become so electrified by the emission of an excess of positively charged particles, as in the case of metals referred to above.

The experiments detailed in the present paper were carried out in order to give some indication as to how the particles in such solutions come to be charged. To this end colloidal solutions, or sols, of many different metals were formed in water and other liquids, and attempts made to determine the charge on each particle from the velocity with which they move in a known electric field and the size of the particles. The results of these measurements seem to preclude the idea that the charges arise from merely an emission of charged ions from the particles. A co-ordination of the results obtained with different liquid media indicates that the formation of colloidal solutions is due to a chemical reaction between the metal and the solvent.

II. *Preparation of Solutions in Water.*

The method of preparing the solutions was similar to that described by Bredig in the paper referred to above. The terminals (B, fig. 1) of a storage-battery capable of providing any required voltage up to 110 volts, were connected

Fig. 1.
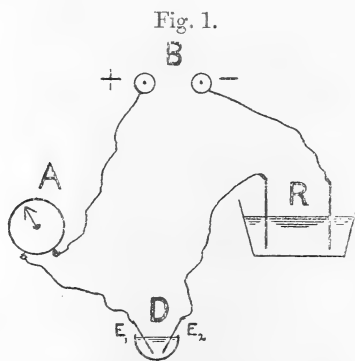
through a variable liquid resistance (R) and ammeter (A) to two electrodes (E₁, E₂) of the metal of which a sol was required. The voltage and current used depended on the metal of which the electrodes consisted; such details will be found in the tables given. The electrodes were held with their free ends beneath the surface of pure water contained in a clean porcelain evaporating-dish (D) of about 100 c.cs. capacity, and the current was allowed to spark between them.

Clouds of finely divided metal scatter from the cathode during the sparking and remain suspended in the water for a time depending on the nature of the metal. The dish (D) was surrounded by running tap-water in order to counteract the heating effect of the arc. The sparking was continued until the water became so clouded as to be quite opaque. The solutions, which usually contained a little sediment from the pulverization, were then carefully filtered. Using gold, silver, and platinum, the resulting filtrates retained nearly the whole of the metal sputtered from the electrodes and remained constant for months. When bismuth, lead, and iron were used, the filtrate still retained a large proportion of the metal, but these solutions were by no means as stable as the preceding ones. After sparking with electrodes of zinc and tin, the clouded solution began at once to clear, and the filtrate failed to retain any of the metal.

The water used in the experiments was first distilled through the large laboratory still, a trace of acid potassium sulphate added, and then redistilled through a silver spiral tube; the specific conductivity of this water was about 3×10^{-6}.

III. *Properties of these Colloidal Solutions.*

The properties of such solutions are quite analogous to those of the so-called colloidal solutions of arsenic sulphide, gold, silver, ferric hydroxide, and many organic substances, which have been prepared chemically and carefully investigated by Carey Lea[*], Linder and Picton[†], Barus and Schneider[‡], Zsigmondy[§], Hardy[||] and others. They diffuse in pure water very slowly, and may be separated by dialysis from impurities of various salts. They may be readily filtered through paper filters. When viewed with a microscope of the highest powers, no particles can be detected in the solution; yet when a beam of light is passed through a tube containing some of the solution, it is diffused and polarized. Bredig[¶] points out that these two latter properties fix the upper and lower limits to the size of particles which must be in the solution. The microscope can detect directly a particle of diameter as small as 14×10^{-5} cms.; while the fact that the particles diffuse light shows that their diameter is, at least, quite comparable with the wave-length

[*] Amer. Jour. of Science, xxxvii. p. 476 (1889).
[†] Jour. of Chem. Soc. vol. lxi. p. 148, vol. lxvii. p. 63, vol. lxxi. p. 568.
[‡] *Zeits. f. Phys. Chem.* vol. viii. p. 278.
[§] Liebig's *Ann.* ccci. p. 29 (1898).
[||] Proc. Roy. Soc. vol. lxvi. p. 110.
[¶] *Loc. cit.* and Drude's *Ann.* xi. p. 218 (1903).

of light, *i. e.* that the lower limit to the size is 10^{-5} cms. Such particles were first rendered visible by means of a special apparatus described by Zsigmondy and Siedentopf*, referred to later, and were then found to possess the Brownian movement common to all ordinary suspensions of fine particles in water. Again, in perfect analogy with the cases of ordinary suspensions, *e. g.* soap solutions, the metallic particles are completely precipitated by the addition of small quantities of any aqueous electrolytic solution. Of great interest, from chemical and physiological viewpoint, is the very strong catalytic action the platinum solution shows in bringing about the decomposition of hydrogen peroxide. Probably the most interesting of the properties of these solutions is the movement of the particles under the influence of an electric field ; in all the solutions which he prepared Bredig found that if two electrodes were introduced into a tube containing the solution, the particles would move towards the positive electrode, showing that they possessed a negative charge.

IV. *Determination of the Size of the Particles.*

The first method devised for rendering such ultra-microscopic particles visible is due to Zsigmondy and Siedentopf †. An intense, thin beam of light is focussed in the solution and viewed at right angles to its direction by a high-power microscope. The particles in the solution appear as bright points due to the light scattered by them into the microscope. This phenomenon is quite analogous to the very common means by which particles of dust floating in the atmosphere of a darkened room may be made visible by casting a bright beam of light into the room ; viewed at right angles to the direction of the beam, the particles of dust appear as bright moving specks. A rather simpler application of this principle, proposed by Mouton and Cotton ‡, was used in the present experiment. The apparatus is shown diagrammatically in fig. 2, which somewhat exaggerates the relative size of smaller parts. A drop of the solution to be examined is enclosed beneath a thin cover-glass (C) on a microscopic slide (S). The plate S is placed on a special block of glass A, good optical contact being made by an intervening thin layer of liquid whose refractive index does not differ much from that of the glass. The block A (for which a Fresnel rhomb serves very well) allows the direction of a beam of light to be easily adjusted so that, after being internally reflected at the lower surface of A, the succeeding internal reflexion at the upper surface may be made to take place at
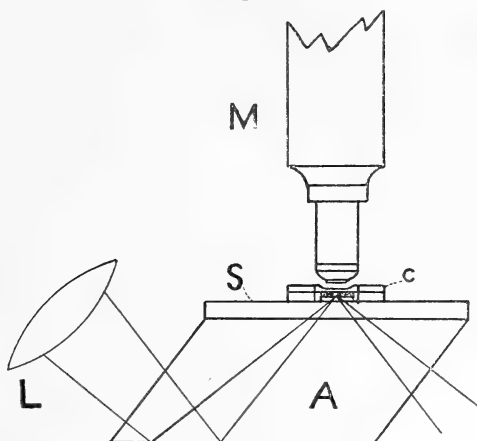
* Drude's *Ann.* x. p. 1 (1903).   † *Loc. cit.*
‡ *Revue Générale des Sciences*, xiv. 23, p. 1184 (1903).

the critical angle. The proper adjustment of the lens L and
block A allows the beam to come to a focus at C just at the

Fig. 2.

drop of liquid. As the particles in the solution come into
the path of the beam, the light striking them is now no longer
totally reflected, but is scattered into the microscope M. As
the field of view is otherwise dim, the appearance is that of
a sky filled with moving stars—the Brownian movement of
the particles being at once recognized. No means is afforded
of directly comparing the size or shape of the particles as,
in general, the better they are in focus the smaller they
appear.

An adaptation of the above apparatus was used to count
the number of particles in a given volume of a colloidal
solution. The difficulty of determining the volume of liquid

Fig. 3.
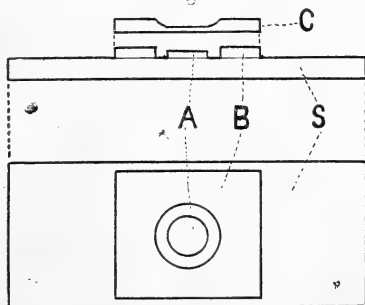
viewed was overcome by the use of the special Zeiss slide
used in counting the number of blood corpuscles in a
unit volume of diluted blood. Fig. 3 shows the con-

struction of the slide. At the centre of a circular piece of glass A an area of 1 sq. mm. is divided into small squares of $\frac{1}{20}$ mm. side by means of fine lines ruled with a diamond point. The plate B surrounds A so as to leave an annular trough about the central disk. The upper surface of B is exactly ·1 mm. above that of A, so that when the cover-glass C is placed on B, a layer $\frac{1}{10}$ mm. thick exists between A and C. The surfaces of A, B, and C are of course ground perfectly plane. When a drop of a sol is placed on A and covered with C, a volume of $\frac{1}{4000}$ cu. mm. can be discerned through the microscope. By raising and lowering the objective very slightly, it is possible to bring all the particles in a layer $\frac{1}{10}$ mm. thick into view, and so, for very dilute solutions, the number of particles per cu. mm. can be very approximately determined. A microscope which magnified about 350 times was used, and the slide was moved slightly so that the particles corresponding to 400 of the unit volumes (each $\frac{1}{4000}$ cu. mm.) were counted. The Brownian movement, at any rate in such a thin layer of liquid, did not show any constant drift of the particles, and was not sufficient to sensibly alter the disposition of the particles during the time (about one hour) taken to count the number in the volume ($\frac{1}{10}$ cu. mm.).

If the amount of metal in 1 c.c. of a given colloidal solution can be ascertained, then, assuming that the specific gravity of the metal in this state of fine subdivision retains its ordinary value, an idea of the size of each particle can be obtained from knowing the number of particles in unit volume. Such determinations have been made for the colloidal solutions of platinum, gold, and silver. The weight of metal in a given volume of the solution was obtained by evaporating the liquid in a previously weighed porcelain crucible, and finding the weight of the residue. A reference to the specific conductivities of the various solutions (Table II.) will show that the weight of dissolved impurity remaining after evaporation would be practically infinitesimal. The following is a sample of such determination, typical as regards method and the magnitude of the quantities involved :—

A silver solution containing 6·8 milligrams of metal per 100 c.cs. was diluted with distilled water to one hundred times its original volume. A drop of the dilute liquid showed the presence of 300 particles in the volume, $\frac{1}{10}$ cu. mm.

So that in the original solution, per c.c., there are 3×10^8 particles of total weight $6·8 \times 10^{-6}$ grms. If the specific gravity be taken as 10·5, the mean volume of the particles

in solution is $2 \cdot 2 \times 10^{-14}$ c.cs. Assuming that the particles are in the form of small spheres, the mean radius being ' a,'

$$\tfrac{4}{3} \pi a^3 = 2 \cdot 2 \times 10^{-15}$$
$$a = 1 \cdot 7 \times 10^{-5}.$$

The solutions of gold and platinum gave values for ' a ' of the same order as the above, but, as yet, the number of such determinations made permit only the statement that, for colloidal solutions of platinum, gold, and silver in water, the average diameter of the particles lies between the limits 2–6×10^{-5}. Such a result is quite in keeping with the limits assigned by Bredig from theoretical considerations (see page 427).

V. *Motion of Particles in an Electric Field.*

The investigation of this property of the so-called colloidal solutions dates from the work of Linder and Picton [*]. Prior to that [†], experiments on the motion of the particles of ordinary suspensions under an electric field had been carried out by various workers, and briefly the results obtained were as follows :—

The particles in suspension in water, of starch, platinum black, finely divided gold, copper, iron, graphite, quartz, feldspar, sulphur, shellac, silk, cotton, lycopodium, paper, porcelain earth, asbestos, move towards the positive pole.

When the above materials are suspended in a similar manner in turpentine oil, they all move towards the negative pole, with the sole exception of sulphur, which moves in the same direction in turpentine as in water.

Fine gas bubbles of hydrogen, oxygen, air, ethylene, carbon dioxide, and small liquid globules of turpentine and CS_2 when in water all move towards the positive pole.

Turpentine globules and small gas bubbles in ordinary alcohol move to the positive pole.

Quartz particles and air bubbles in carbon bisulphide move to the negative pole.

These results led to the conclusion that " in water all bodies appear, through contact, to become negatively charged, while, through rubbing against different bodies, the water becomes positively charged."

When Linder and Picton [‡] tested similar properties of the particles in chemically prepared solutions, they found that

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[*] *Loc. cit.*
[†] See Wiedemann's *Elektricität*, B. i. p. 1007 (1893).
[‡] Jour. Chem. Soc. lxxi. p. 568 (1897).

such a generalization was inexact.   They give the following results :—

Suspensions of aniline blue, arsenic sulphide, indigo, iodine, shellac, silicic acid, starch, and sulphur in water, and of bromine in alcohol show attraction to the positive electrode.

The following materials suspended in water move in the opposite direction : ferric hydrate, hæmoglobin, Hoffmann's violet, Magdala red, methyl violet, and rosaniline hydrochloride.

As a conclusion to their work, Linder and Picton make the significant statement that experiment seems to show that, if the solution is basic or tends to break up so as to leave a free base active, the motion is to the negative pole, *i. e.* the particle is positively charged ; if the solution is acidic, motion is to the positive pole, and consequently the particles are negatively charged.

To the above lists may now be added from the results of Bredig for electrically prepared solutions : suspensions of gold, platinum, and silver show convection to the positive pole.

The theory of this motion of finely divided particles in suspension in liquids was long since propounded by von Helmholtz[*] and later amplified by Lamb[†].   Without assuming, for the moment, anything with regard to the cause of the formation of the colloidal solutions, we may apply the same theoretical considerations to the movement of these particles.

The fundamental assumption is that when a particle suspended in a liquid becomes charged, there exists about it a double electric layer ; when the particle is negatively charged there is a layer of negative electricity on the surface of the solid particle, while in the liquid immediately surrounding it there is a corresponding layer of positive electricity.   "On the whole the algebraic sum of the two equals zero, and the centre of gravity of the complete system, solid particle and surrounding positively charged fluid layer taken together, cannot be moved by the electric forces which arise from the potential fall in the liquid through which the current passes. However, the electric force will tend to bring about a displacement, relatively to each other, of the positively charged fluid layer and the negatively charged particle, whereby the

* Wied. *Annalen,* vii. p. 337 (1879), and Memoirs Lond. Phys. Soc. 1888.

† Brit. Assoc. Rep. 1887, p. 495.

fluid layer follows the flow of positive electricity while the particle moves in the opposite direction. If the liquid were a perfect insulator the new position would still be a condition of equilibrium. Since, however, through the displacement of the layers the equilibrium of the galvanic tension between the solid particle and the liquid is disturbed, and, on account of the conductivity of the liquid, always seeks to restore itself, the original state of electrical distribution will tend to be continually reproduced, and so new displacements of the particle with respect to the surrounding liquid will continually occur " *.

This theory was put forward by Helmholtz in the course of his mathematical development of the explanation (suggested by Quincke) of the electric transport of conducting liquids through the walls of porous vessels or along capillary tubes ; Quincke assumed that there existed a contact difference of potential between the fluid and its solid boundaries. Throughout his treatment of the phenomenon, Helmholtz considers that there is no slipping of the fluid over the surface of the solids with which it is in contact. On this point Lamb disagrees with Helmholtz, holding that the solid offers a very great, but not an infinite, resistance to the sliding of the fluid over it, and that, while the effect of this slipping would be entirely insensible in such experiments as those of Poiseuille, it leads to appreciable results in the present case in consequence of the relatively enormous electrical forces acting on the superficial film of the liquid and dragging the fluid, as it were, by the skin through the tube. The practical difference between the views taken by Helmholtz and Lamb respectively may be shown in a simple case. By comparing with the numerical results found by Wiedemann, Helmholtz infers that for a certain solution of $CuSO_4$ in contact with the material of a porous clay vessel, the contact difference of potential E between the solution and the solid wall is given by

$$\frac{E}{D} = 1\cdot77,$$

where D is the E.M.F. of a Daniell's cell. The variation introduced by Lamb would change this equation into

$$\frac{E}{D} \cdot \frac{l}{d} = 1\cdot77, \quad \cdots \quad (1)$$

where $d=$ the distance between the plates of an air-condenser equivalent to that virtually formed by the opposed surfaces

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* Helmholtz, *loc. cit.*

of solid and liquid, and l is a linear magnitude. measuring the "facility of slipping" and equal to $\dfrac{\mu}{\beta}$;

μ being the coefficient of viscosity of the liquid,

β the coefficient of sliding friction of fluid in contact with the wall of the tube.

Lamb gives reasons for supposing that l and d are of the same order of magnitude (that of 10^{-8} cms.). Of course if $l = d$, Helmholtz's formulæ remain unchanged, and it is very probable that the ratio $\dfrac{l}{d}$ differs very little from unity.

Lamb deduces the following expression for the velocity (v) of a charged particle through a liquid under an electric force, when the motion has become steady :—

$$\mathrm{X}e = 4\pi a^2 \cdot \mu \cdot v \cdot \frac{1}{l}, \quad \cdots \quad (2)$$

where   X = gradient of electric potential in the liquid,

e = charge on the particle,

a = radius of the particle,

and μ and l as above.

We may look upon the particle with the double electric layer as a small condenser of two concentric spheres whose distance apart (d, same as before) is small compared with a. The capacity of such a condenser would then be given by

$$\mathrm{C} = \frac{a^2}{d}\mathrm{K}, \quad \cdots \quad (3)$$

where K = s.i.c. of the medium between the layers.

If V indicates the contact difference of potential between the solid and the liquid, we have

$$e = \mathrm{V}\frac{a^2}{d} \cdot \mathrm{K}; \quad \cdots \quad (4)$$

substituting this value of e in equation (2) and transposing we get

$$\mathrm{V} \cdot \left(\frac{l}{d}\right) = \frac{4\pi}{\mathrm{K}} \cdot \frac{\mu v}{\mathrm{X}}, \quad \cdots \quad (5)$$

all electrical measurements being made in electrostatic units.

This equation, which is similar to one given by Perrin[*],

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[*] *Jour. de Chim. Phys.* t. xi. no. 10, p. 607 (1904).

will enable us to find values of $\left(V \dfrac{l}{d} \right)$ for any solid and liquid if for known values of X and μ we can observe the corresponding values of v. The experiments described in the present paper will supply the corresponding values of v, X, μ for a number of cases.

VI. *Velocities of the Particles of Metal Solutions.*

The first systematic investigation of the velocities of these particles in an electric field is that afforded by the interesting experiments recently published by Whitney and Blake[*]. The colloidal solution was contained in tubes such as shown in fig. 4; they were 30-50 cms. long and 2·46–2·75 cms. diameter. The ends were closed with goldbeaters' skin and platinum electrodes placed in water-cells just outside these ends. When A was connected to the negative electrode and B to the positive electrode of a set of storage-cells, a migration of the particles set in down the tube. A reversal of the current in the tube caused the particles to rise again. In this way the authors have deduced the velocities of particles in solution of platinum and gold to be respectively 34×10^{-5} and 43×10^{-5}cm. per sec. per volt per cm. Various circumstances, unavoidable in the use of a tube of this construction, render the determination of the absolute values of the velocities of the particles under unit electric force somewhat unsatisfactory. The rate with which the surface of the particles in the solution moves down or up the tube varies in a non-uniform manner with the time. As the writers found, the specific resistance of the clear liquid above the surface of colloidal particles may be continually changing; this would naturally affect the potential gradient along the tube. Even when this is corrected for, as Whitney and Blake took precautions to do, still the specific resistance of the upper clear layer of liquid remains always different from that of the solution; a fact which must tend to affect the movement of the particles at the surface of separation, on account of the

Fig. 4.

[*] Jour. Amer. Chem. Soc. vol. xxvi. no. 10, p. 1339 (1904).

jump in the value of the electric force, just at the surface. In addition to this the method consists essentially in a compression of the colloid which may change the absolute velocity of the particles. Again, as the authors do not give the temperatures corresponding to these velocities of the gold and platinum particles, it is impossible to apply the formula (5), which involves the coefficient of viscosity of the liquid.

Fig. 5.

These difficulties are obviated if one uses a velocity tube (fig. 5) similar to that used by Hardy* in his work with colloidal solutions. That used in the present experiment consisted of a U-tube each limb of which was about 12 cms. long and about 1·5 cms. in diameter; the limbs were graduated in mms. throughout their length. Into the bottom of the U-tube is sealed a fine delivery-tube provided with a tap (T) and funnel (F); this tube is bent around so as to run up behind the limbs and to bring the funnel to the same height as the top of the U-tube.

The colloidal solution to be tried was poured into the funnel so as to fill the small tube and funnel around to the tap which was closed; water having a specific conductivity equal to that of the colloid was then poured into the U-tube so as to fill it to a height of about 3 cms. The whole tube was then placed in a large glass water-bath so as to be almost submerged; this water was kept at a constant temperature during the course of any experiment. At the end of a few minutes the tap (T) was very slightly opened, and the colloidal solution allowed to gently force the water up the limbs of the tube to any required height. If carefully manipulated, the surface of separation between the clear water and the solution was very distinct, and would remain so for hours. Two electrodes of coiled platinized platinum foil
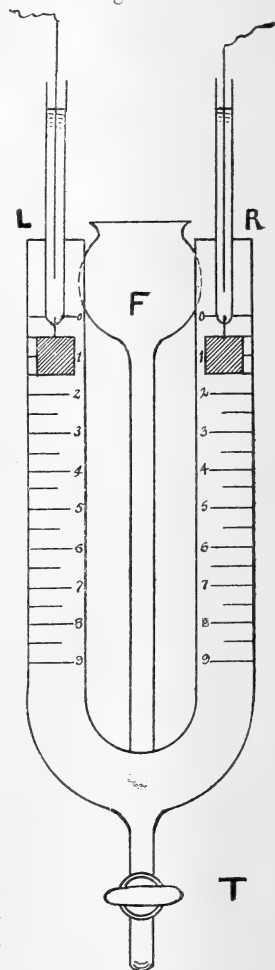
* Journ. of Physiology, vol. xxix. p. 26 (1903).

were supported at a convenient level in the two limbs of the tube
and the clear water allowed to rise well above them. The elec-
trodes were attached to the terminals of a set of storage-cells
of constant voltage, and when the circuit was completed, the
surface of separation in one limb would at once begin to rise
gradually, while that in the other sank. In practice the
connexions were made through a reversing key, and the
voltage, usually fixed at about 110 volts, was left on one way
for ten minutes and then reversed for twenty minutes. The
velocity was reckoned from the displacement of the surfaces
during this final twenty minutes; one half the sum of the
displacements in the two tubes was taken as the distance
travelled by a particle in the given time. A typical set of
observations is given in Table I.

<div align="center">TABLE I.</div>

Time.	Voltage. Sign of L.	Temp.	Height of Colloidal Surface.		Observed Velocity in cm./sec.
			Left.	Right.	
11·37	+118	11° C.	54 mm.	55 mm.	
11·47	+118		61	50	
Current off					
11·48	−118	11° C.	61	50	
11·58	−118		55	56	96×10^{-5}
12·08	−118	11° C.	50	62	

<div align="center">Electrodes at 15 mms. in each limb.</div>

It will be seen from the table that there has been an
apparent settling of the colloid in the tube while the current
was running. This was quite usual, but, as the reckoning
was made, it could not affect the rate, since this slight lowering
of the surface was uniform in both limbs, so that, while it is
added to the velocity in one limb, it is subtracted from the
velocity in the other.

In order to find the value of the electric force in the tube,
it is of course necessary to know the effective distance between
the electrodes A and B. In order to do this, the tube was
filled with a ·01 normal KCl solution, placed in the water-
bath, and the resistances taken with the electrodes placed at
the successive centimetre marks down the tube. In this way
it was found that the resistance of the curved part of the

tube from 90 in L around to 90 in R was 8·8 times the resistance of each cm. length of the single limbs. So that when, as in case cited in Table I., the electrodes were placed at 15 in each tube, the effective distance between the electrodes was 23·8 cms., and therefore the strength of the electric field was $\dfrac{118}{23\cdot 8} = 4\cdot 9$ volts per cm. Thus the absolute value of the velocity of the silver particles in water at a temperature of 11° C. would be $19\cdot 6 \times 10^{-5}$ cms. per sec. per volt per cm.

Such measurements required that the specific resistance of all the solutions should be carefully obtained. The resistance-cell used is shown in fig. 6. Two thick platinum disks (2·5 cms. in diameter) were fused by means of welded platinum wires into two fine glass tubes which were rigidly fastened into the ebonite lid (L) fitting into the glass vessel G. The disks were thus about ·75 cm. apart. A thermometer, T, was supported by the lid so that its bulb reached close to the upper disk. The constant of the vessel was taken by means of a ·01 normal KCl solution at suitable intervals during the course of the work. The resistances were determined by the ordinary Wheatstone's bridge method, the effects of polarization being annulled by means of a double commutator as used by Whetham[*].

Fig. 6.

Solutions of the metals platinum, gold, silver, bismuth, iron, and lead were formed, and the values of the velocities of the particles under unit electric force determined as given in Table II. As has been found by all experimenters, the particles of the platinum, gold, and silver solutions move toward the positive electrodes, showing that the particles themselves bear a negative charge. On the other hand, the particles of the other solutions named move in the opposite direction, corresponding to a positive charge on the particles. The velocities given have all been determined for a temperature of 18° C.

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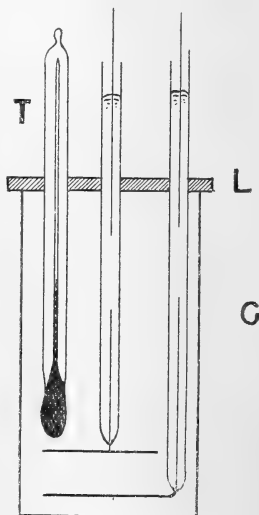
[*] Phil. Trans. vol. cxciv. A, p. 321 (1900).

TABLE II.

Metal in Sol.	Used for sparking in preparation.		Specific Conductivity of Sol.	Sign of Charge on Particles.	Velocity in cms. per sec. per volt per cm.
	Voltage.	Current.			
Platinum ...	40 v.	7·5 amp.	$21·3 \times 10^{-6}$	—	$20·3 \times 10^{-5}$
Gold	50 v.	7·5 ,,	$10·5 \times 10^{-6}$	—	$21·6 \times 10^{-5}$
Silver.........	60 v.	7·5 ,,	$19·0 \times 10^{-6}$	—	$23·6 \times 10^{-5}$
Bismuth......	30 v.	6·5 ,,	$7·0 \times 10^{-6}$	+	$11·0 \times 10^{-5}$
Lead	30 v.	6·5 ,,	$15·0 \times 10^{-6}$	+	$12·0 \times 10^{-5}$
Iron	50 v.	6·5 ,,	$9·4 \times 10^{-6}$	+	$19·0 \times 10^{-5}$

As has been mentioned above, the first three solutions of Table II. are extremely stable, the metals remaining apparently uniformly in solution for months, while the solutions of bismuth, lead, and iron would be precipitated in the course of a week or so. The bismuth solution is dark brown in colour, not unlike the silver solution in appearance. The lead solution is initially dark brown as well, but if left in contact with the air the particles rapidly turn white, doubtless on account of the formation of a carbonate; but whether in the brown or white state, the particles move to the negative electrode in an electric field. The iron solution always possesses a reddish-brown colour, which points to formation of a hydroxide; this quite agrees with the fact given by Linder and Picton that in chemically prepared colloidal solutions of ferric hydrate the particles move towards the negative electrode.

If there are particles of varying sizes present in any one solution, no indication of any difference in velocity of the various particles was evident; however, this does not justify the assumption that the particles are all of the same size, because, according to the theory given by Lamb, the velocity of similarly constituted particles is independent of the size and shape. If particles of different sizes do exist in these electrically prepared solutions the size would probably depend on the violence of the sparking during the preparation of the solution. Three silver solutions were prepared, using varying currents and voltages for producing the spark, all other conditions being the same except the time of sparking. As shown in Table III., the differences in the velocities, which were all determined at 11° C., are all within the limits of error in the experiment.

TABLE III.—Silver Solutions.

No.	Voltage.	Current.	Time of Sparking.	Velocity in cm./sec. per volt per cm.
1......	80 volts.	8·5 amperes.	10 minutes.	$19·7 \times 10^{-5}$
2......	60 ,,	7·5 ,,	20 ,,	$19·6 \times 10^{-5}$
3......	40 ,,	6·5 ,,	30 ,,	$19·3 \times 10^{-5}$

Since there is no *a priori* reason for assuming that all the particles are of a uniform size, these results would confirm Lamb's theory.

We are here dealing with the motion of small charged particles through a liquid, so that, other things being equal, the velocities of those particles should depend upon the viscosity of the liquid, and therefore to a large extent upon the temperature of the liquid. A neglect of this influence soon leads to bewildering results. That the dependence on the temperature is quite direct is shown by placing side by side the results given in Table I., and the velocity found for the same solution at a different temperature (Table IV.).

TABLE IV.

Metal.	Temp.	Coefficient of Viscosity of Water. (μ)	Velocity. (v)	$v\mu \times 10^{-5}$.
Silver...... {	11° C.	·012822	$19·6 \times 10^{-5}$	·251
	21° C.	·009922	$25·2 \times 10^{-5}$	·250

VII. *Remarks on Velocities in Table II.*

It will be noticed that while the velocities recorded above are distinctly lower than those given by Whitney and Blake, they are in close agreement with those found by Hardy [*] for solutions of a similar nature. Although they are of the same order as the velocity of the metallic ions, still the slowest moving ion travels more quickly than the fastest colloidal particle.

In considering the results given in Table II., one is struck with the fact that the particles of the more electro-positive, oxidizable metals are all positively charged, while the more

[*] Journ. of Physiol. vol. xxxiii. (Dec. 1905) p. 291.

electro-negative, non-oxidizable metals give negatively charged particles. When we recall that the iron in such solution appears to form the hydrate, and that they bear a charge of the same sign as the particles in chemically prepared colloidal solution of ferric hydrate, one is justified in suspecting that in the cases of iron, bismuth, and lead there is the formation of the hydrates, in greater or less degree, in the preparation of these electrically prepared solutions. Such an hypothesis at once suggests an analogous action on the part of gold, silver, and platinum, viz., an interaction between the metal and hydrogen—a view to which the already known existence of a hydride of platinum would lend some colour.

VIII. *Colloidal Solutions in other Liquids.*

The suggestions given by the results for the hydrosols are strengthened by the observations on colloidal solutions in the alcohols. Pure methyl alcohol and pure ethyl alcohol were used in the preparation instead of water.

Although repeated trials were made with the metals gold, silver, and platinum, the writer has never succeeded in getting them to remain suspended in either of these alcohols. On the other hand, the metals lead, tin, and zinc form solutions in both of the alcohols, while with methyl alcohol solutions were also obtained with bismuth, iron, and copper. In all these solutions the particles moved to the negative electrode in an electric field, *i. e.*, they are positively charged. The numbers are given in Tables V. and VI.

TABLE V.—Methyl Alcohol (free from acetone).

Metal.	Used in Preparation.		Specific Conductivity of Solution.	Sign of Charge on Particles.	Velocity in cms. per sec. per volt per cm.
	Voltage.	Current.			
Lead	30 volts.	6·5 amp.	10.0×10^{-6}	+	22×10^{-5}
Bismuth...	30 ,,	,,	14.3×10^{-6}	+	10.2×10^{-5}
Iron	40 ,,	,,	9.5×10^{-6}	+	⎫
Copper ...	40 ,,	,,	6.8×10^{-6}	+	motion very slight.
Tin	30 ,,	,,	12.5×10^{-6}	+	
Zinc	30 ,,	,,	7.0×10^{-6}	+	⎭

TABLE VI.—Ethyl Alcohol.

Metal.	Used in Preparation.		Specific Conductivity of Solution.	Sign of Charge on Particles.	Velocity in cms. per sec. per volt per cm.
	Voltage.	Current.			
Lead	30 volts.	6·5 amp.	$2·3 \times 10^{-6}$	+	$4·5 \times 10^{-5}$
Tin	,,	,,	$2·9 \times 10^{-6}$	+	$3·6 \times 10^{-5}$
Zinc	,,	,,	$1·7 \times 10^{-6}$	+	$2·8 \times 10^{-5}$

Viewing these results in the light of the chemical nature of the alcohols, the former suggestion as to the interaction between the liquid medium and the metals is strengthened. Although these alcohols have a neutral reaction, they act like weak bases in combining with acids to form salts, or, in other words, they have an easily replaceable OH^- group. Those easily oxidizable metals would thus be able to form at least a surface coat of hydrate, while the metals gold, platinum, silver, whose existence in a colloidal state we have been led to suspect to depend on a replaceable hydrogen in the liquid, cannot form in the alcohols.

A still further test of this hypothesis is afforded when one uses as the liquid medium a substance which has a replaceable H and not an OH. Anhydrous acids might be used, but there is great difficulty in keeping them free from water; and, as is well known, electrolytes containing acids have extremely strong power of coagulating solutions. Ethyl malonate is a liquid which fulfils the condition of having a replaceable H ; the samples used in the present experiment were pure from Kahlbaum. When platinum, gold, and silver were sparked

TABLE VII.—Ethyl Malonate.

Metal.	Used in Preparation.		Specific Conductivity of Solution.	Sign of Charge on Particles.	Velocity in cms. per sec. per volt per cm.
	Voltage.	Current.			
Platinum .	40 volts.	7·5 amp.	$·1 \times 10^{-6}$	—	$2·3 \times 10^{-5}$
Silver	60 ,,	,,	$·1 \times 10^{-6}$	—	$1·7 \times 10^{-5}$
Gold	60 ,,	,,	$·1 \times 10^{-6}$	—	$1·4 \times 10^{-5}$

underneath this liquid, very stable colloidal solutions were obtained; those of the first two metals named are apparently as stable as the corresponding hydrosols, while the gold solution coagulates at the end of a month or so. From the velocities which are given in Table VII., it will be seen that these particles all bear a negative charge, similar to those of platinum, silver, and gold in hydrosols. When the other metals, bismuth, lead, zinc, and iron were used, a colloidal solution could not be obtained.

IX. *Theoretical Considerations.*

There is practical unanimity in the opinion that these particles in colloidal solutions are enclosed by a double electric layer, the electricity of one sign on the surface of the particle being in equilibrium with an equal amount of electricity in the layer of liquid immediately surrounding the particle. It is a matter of doubt as to the means by which this double layer is formed.

Linder and Picton first suggested some interaction between the liquid and the particle:—" Experiment seems to show that if the solution is basic or tends to break up so as to leave a free base active, the motion is to the negative pole. If the solution is acidic (or tends to break up so as to leave a free ' H ' active) motion is to the positive pole."

This statement is remarkably confirmed by the determinations cited in the present paper for solutions in water, the alcohols, and ethyl malonate.

Following the conclusion given by Quincke regarding suspensions of microscopic particles in liquids, many writers have been contented to view the phenomenon as an effect expressed by the term " contact electrification "; the particles become charged by the rubbing of the moving particles of the liquid itself against the suspended particles.

The recent work of Perrin [*] has produced results which throw considerable light on the phenomena of electrification by contact between liquids and solids. By measurements of the electric osmose of liquids through diaphragms of various materials, he is led to announce these two laws :—

1. Electric osmose is only appreciable with ionizing liquids, or, in other words, ionizing liquids are the only ones which give strong electrification by contact.

2. In the absence of polyvalent radicals, all non-metallic substances become positive in liquids which are acidic, and negative in basic liquids.

* *Loc. cit.*

In explaining these results he suggests the hypothesis that a positive electrification of a wall bathed by an acidic liquid is formed by H^+ ions situated in the stationary liquid layer immediately contiguous to the wall. Opposed to it at a small distance there will be a corresponding excess of negative ions forming another layer. If the wall assumes a negative charge it is on account of similar action of the OH^- ions. It is found that H and OH ions move much more quickly than other ions; if, then, we explain this high velocity by assuming that they are smaller than other ions, we should expect them to penetrate nearer to the boundary of the liquid and so muster at the limiting layer of a liquid an excess of electric charges of one sign. Although the analogies between this phenomenon of electric osmose and that of the coagulation of colloidal solutions undoubtedly help in explaining the latter, the formation of these solutions can hardly be credited to merely physical diffusion of the H^+ and OH^- ions. As we have seen from the sign of charges borne by particles in solution, it is those which appear to depend on a replaceable H which are negative while those depending on a replaceable OH are positive.

The matter is summed up quite clearly in a recent statement of Noyes[*] :—

" In regard to the cause and character of the electrification two assumptions deserve consideration : one, that it is simply an example of contact electricity, the colloid particle assuming a charge of one sign and the surrounding water one of the other. This correlates the phenomena of migration with that of electric endosmose. It does not, however, give an obvious explanation of the facts that the basic colloidal particles become positively charged and the acidic and neutral ones negatively charged. The other assumption accounts for these facts. According to it the phenomenon is simply one of ionization. Thus each aggregate of ferric hydroxide molecules may dissociate into one or more ordinary hydroxyl ions and a residual positively-charged colloidal particle, and each aggregate of silicic or stannic acid molecules into one or more hydrogen ions and a residual negatively-charged colloidal particle. To explain the behaviour of neutral substances like gold and quartz by this hypothesis, it is necessary to supplement it by the assumption that in these cases it is the water or other electrolyte combined with or adsorbed by the colloidal particles which undergoes ionization. It seems not improbable that there may be truth in each of

these hypotheses, contact electrification occurring in the case of the coarser suspensions and ionization in the case of those which approximate more nearly to colloidal solutions."

Comparing the results given for the sign of the charges borne by the particles in different solutions, we have the following :—

1. Water ($H^+ . OH^-$) can form two classes of colloids, whose particles are respectively positively and negatively charged.

2. Replacing the mobile H^+ by the groups C_2H_5 and CH_3 so as to form the alcohols, seems to destroy the power of forming solutions with negatively-charged particles; while

3. Ethyl malonate, $CH_2(COOC_2H_5)_2$ which has the mobile H readily forms those solutions containing the negatively-charged particles, and those only.

It is thus evident that the formation of the solution depends on the chemical nature of the solvent.

This leads to the following theory of the constitution of the solution :—

1. In the case of gold, silver, and platinum in water and ethyl malonate, we have an incomplete chemical combination with the liquid : thus for platinum and water we have the equation :—

$$n . Pt + \overset{+}{H} . \overset{-}{OH} = (\overset{-}{Pt_n}\overset{+}{H}) + OH.$$
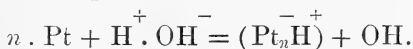
We may look upon the platinum-hydrogen aggregate as dissociating slightly so as to form an atmosphere of positively-charged hydrogen ions about the negatively-charged colloidal particle.

2. With the other metals in water and the alcohols, we have a corresponding formation of the hydroxides, thus :

$$n . Pb + \overset{+}{H} . \overset{-}{OH} = (\overset{+}{Pb_n} . \overset{-}{OH}) + H;$$
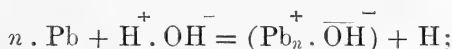
and by slight dissociation of the aggregate (Pb_nOH), we obtain a positively-charged colloidal particle, surrounded by a layer of OH^- ions in the liquid.

In accordance with Helmholtz's explanation, we may look upon the motion in an electric field as primarily due to electric endosmose. On this view, in formula (5) (p. 434) l will equal d and K will be the specific inductive capacity of the liquid medium. Values of V for different cases are

recorded in Table VIII. The specific inductive capacity of ethyl malonate was found by Nernst's [*] method for slightly conducting liquids.

TABLE VIII.—Values of V in volts.

Metal.	In Water. $K=80$.	In Ethyl Malonate $K=10.7$.	In Ethyl Alcohol. $K=25.8$.	In Methyl Alcohol. $K=33$.
Platinum..	−·031	−·054		
Gold	−·033	−·033		
Silver	−·036	−·040		
Lead	+·018	......	+·023	+·044
Bismuth...	+·017	......	......	+·022

This table shows a surprisingly close agreement among the differences of potential between the particles and the liquids. Taking into account the wide differences between the specific inductive capacity, say for water and ethyl malonate, we can deduce that the charge of electricity on the particle of a given metal must be much greater in water than in ethyl malonate ; in other words, the interaction between the particle and the solvent seems to be dependent on what may be defined as the ionizing power of the liquid. It is further interesting to note that these values for the differences of potential between the particles and the liquids are of the same order as the value found by Perrin for the difference of potential between chromium-chloride diaphragm and slightly acidulated water (·025 volt), and also agrees in the same way with Helmholtz's values for the difference of potential between very dilute aqueous solutions and the walls of glass tubes in which they were contained, if corrections are made by introducing the value for the specific inductive capacity of water.

X. *Conclusion.*

Summarizing the results here recorded, we have the following :—

1. The size of the diameters of the particles of gold, silver, and platinum in electrically prepared colloidal suspensions has been found to lie between the limits $(2-6) \times 10^{-5}$ cms.

2. The electro-negative, non-oxidizable metals, gold, silver,

* *Zeit. f. phys. Chemie,* vol. xiv. p. 622 (1894).

and platinum give solutions in water and ethyl malonate in which the particles are negatively charged.

3. The electro-positive, oxidizable metals give solutions in water, methyl alcohol, and ethyl alcohol in which the particles are always positively charged.

4. The velocities of the particles under a known electric force have been determined and the potential differences between the liquid and the particle have been deduced by using the formula

$$V = \frac{4\pi}{K} \cdot \frac{\mu v}{X}.$$

My best thanks are due to Professor J. J. Thomson for his kindly interest and advice during the course of the experiment. I also wish to acknowledge heartily the many helpful suggestions which Mr. W. B. Hardy has kindly given me.

Cavendish Laboratory,
 Sept. 1905.

XXXIX. *On the Decay of Torsional Stress in Solutions of Gelatine. By* A. O. Rankine, *B.Sc., Assistant in the Physics Department, University College, London* *.

THE results of the experiments on the time rate of change of stress in lead wires under constant strain, published in the Phil. Mag. Oct. 1904, suggested an investigation of the laws governing this variation in passing from purely viscous to perfectly elastic bodies. Lead exhibits both elastic and viscous properties to a marked extent, but does not obey the mathematically deduced laws of Maxwell for such bodies. The experimental curve of stress against time when the stress remains constant is not of the form $W = W_0 e^{-\lambda t}$, but is represented fairly accurately by the equation

$$W = W_0 - a \log (pt + 1).$$

The question arises, " Does a similar law hold when the viscous and elastic properties are combined in proportions differing from those in lead ; and, if not, how does the law change in passing from the perfectly elastic to the purely viscous cases ? " In order to find an answer experimentally, one must be provided with a series of substances of the same kind possessing elastic and viscous properties combined in different proportions and extending from the purely viscous on the one side to the perfectly elastic on the other. Now solutions of gelatine in water fulfil these conditions. Solutions of very low concentration possess no elastic properties,

and at the same time it is possible with high concentrations to make specimens which are perfectly elastic for fairly large stresses.   Intermediate concentrations give elastic and viscous properties in varied proportions.   Gelatine was therefore chosen for the investigation of the question proposed above.

The consistency of solutions of gelatine in water which have been allowed to congeal is very varied in character.   All of these, however, seem to possess, to a greater or less degree, elastic properties.   It also appears that any particular specimen has a definite elastic limit provided that none of the conditions which determine its consistency vary.   It may be subjected to a stress of less than a certain limiting value, and the resulting strain will be constant with regard to time and it will recover on the removal of the stress to its original state.   If, however, this limiting stress is exceeded, the strain is no longer constant but increases with time.   The removal of the stress does not now produce complete recovery, although it should be noted that if the stress is only applied momentarily the recovery is very nearly complete.

Now suppose a specimen of jelly stressed beyond its elastic limit.   There will result an immediate strain followed by a gradual increase with time.   This increase, however, can be prevented by the gradual removal of the stress, and in order to keep the strain constant this removal must be continued until the remaining stress is equal to the elastic limit of stress for the particular specimen.

It was to investigate this change in the special case of the rate of decrease of couple in solutions of gelatine twisted through a constant amount, that the following experiments were carried out.

I. *Method of Experiment.*

The method used has been to apply equal torsional stresses to gelatine solutions of different concentrations, to maintain the strain always equal to the initial immediate strain, and to record the rate of falling off of the stress.   It was arranged that the removal of the stress should be effected automatically by a method very similar to that used by Professor Trouton and myself in our experiments on lead wires (Phil. Mag. October 1904).

The interspace between two concentric glass cylinders is filled with a prepared solution of gelatine which is allowed to set.   A torsional stress can then be produced in the jelly by twisting the inner cylinder and retaining the outer one fixed. This inner cylinder forms the lower part A B of figure 1, and is rigidly fixed to the brass piece C D which is itself supported

vertically by means of two steel points at C and E. The brass projection F G is perpendicular to the direction of the cylinder and has a platinum point at its end G which, when the cylinder is twisted by means of threads passing round the wheel H, can be brought into contact with a fixed platinum plate, thus completing an electrical circuit through a Post-Office relay. The whole is supported on a stand (figure 2). The threads producing the couple pass over two pulleys K K and are attached to one arm of a balance. From this same arm hangs a tin vessel L, containing water with a layer of oil on the top to prevent evaporation. The weight of this vessel and a certain amount of the water it contains is counter-balanced by weights placed on the other pan of the balance so

Fig. 1.
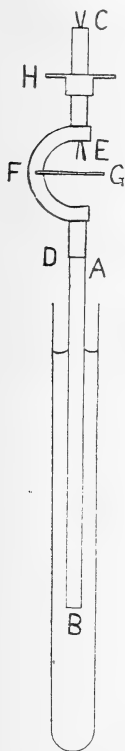
that the tension in the threads at any time is equal to the difference between the two weights. If necessary, therefore, all the tension may be removed by drawing off sufficient water from the vessel L. The water is run off by means of a siphon through the electrically controlled valve M into a second vessel N, which is sus-pended from a spring, and the weight of water in the vessel N at any time is recorded on the revolving drum O by the pen P.

The current passing through the platinum point and plate before mentioned is far too small to itself actuate the water valve M, and it is necessary to introduce two relays R_1 and R_2 to effect a sufficient increase of current.

Contact between the platinum point and plate causes the water valve to open and a small quantity of water leaves L. This, however, reduces the torsional stress in the jelly, a small recovery takes place, contact is broken and the valve closes. By means of this intermittent action the inner glass cylinder which is imbedded in the gelatine is always kept in the same posi-tion, *i. e.* the strain remains constant; and the curve drawn on the revolving drum gives the weight removed at any time, and since this weight is proportional to the torsional stress its value may be taken as a measure of the latter.

Dimensions :—The total length of the outer cylinder was about 50 cms., and the end of the inner cylinder was about 20 cms. from the bottom of the former. In all cases the length of the inner cylinder imbedded in jelly was the same, viz. about 26 cms. The diameter of

the outer cylinder was about 3 cms. and that of the inner 1 cm. The diameter of the wheel about which the couple was applied was about 4·5 cms.

Experiments were made with 3·4, 4, 4·2, 4·4, and 4·5 per cent. solutions, and in each case the couple was produced by 500 gr. wt. acting round the wheel H. The platinum plate was then adjusted so as to come into contact with the point G, and the action of the valve began.

Fig. 2.

The friction of the points C, E and the pulleys K, K was not large, being only equivalent to a couple produced by ·5 gr. wt. acting round the wheel H, but in order to obtain a smooth curve it was necessary to eliminate it. If it is not allowed for, the pen draws a curve consisting of a series of alternate vertical and horizontal lines, each upward step being equivalent to the frictional couple. This difficulty was overcome in the following way :—An iron plate S (fig. 2) suspended from the second arm of the balance is adjusted to such a distance from the electromagnet T that when a given current flows in T, S is pulled downwards with a force equal to the frictional force. This causes the tension in the strings to be relieved to that extent. The current flows in T whenever the water valve opens. Hence the platinum

point is free to recover without having first to overcome friction. In this way a series of smooth curves was obtained.

Another precaution had to be taken. Preliminary experiments sufficed to show that the consistency of any specimen of jelly depended largely on the temperature. For purposes of comparison between solutions of different concentrations, therefore, it was necessary to maintain them throughout the experiment as far as possible at equal temperatures. To effect this the outer cylinder containing the jelly was surrounded by a water-jacket which was itself wrapped round and round with felt. It was then possible to keep the temperature of the jelly fairly constant for about three hours.

The following table shows the variation of temperature in the actual experiments :—

Experiment.	Percentage of Gelatine.	Maximum & Minimum Temperatures. Centigrade degrees.	Mean Temperature for two hours.
1	3·4	16·32 − 16·44	16·38
2	4·0	16·23 − 16·59	16·41
3	4·5	16·10 − 16·52	16·31
4	4·2	16·15 − 16·63	16·39
5	4·4	16·18 − 16·76	16·47

II. *Preparation of Solutions.*

The consistency of any specimen of jelly does not solely depend on the percentage of gelatine present and the temperature. It depends also on the way in which the solution is prepared and the time allowed for setting. Thus, if one solution of gelatine be boiled for some time and another of equal concentration be made simply by dissolving the gelatine in hot water without boiling, then, even if the times of setting are the same, the two have different consistencies. A gradual hardening, too, occurs in any jelly even after the temperature of the atmosphere has been reached.

In order to compare specimens varying only in concentration, it was essential to ensure that all the other conditions determining the consistency remained constant.

The temperature was kept constant in the way before described. All the solutions were prepared by simply dissolving the gelatine in water that had just been boiled and making up to the required concentration by the addition of hot water. The liquid (equal volumes in each case) was then poured into the interspace between the two glass cylinders and allowed to set for a fixed time—22½ hours.

III. *Results.*

The curves shown in figure 3 are copied from the experimental curves. The break in the curve, which probably indicates the point at which the elastic limit is reached, will

Fig. 3.
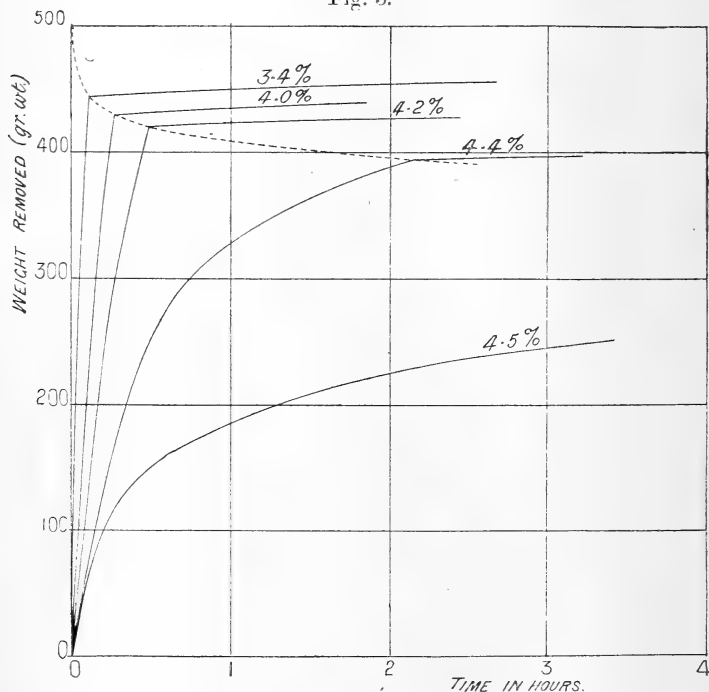
be noticed in the cases of the 3·4, 4·0, 4·2, and 4·4 per cent. specimens. It was not found practicable to maintain the temperature constant for a sufficient time to obtain such a break in the 4·5 per cent. curve. It will be seen, too, that these points lie on a curve which, from theoretical considerations, must pass through the point $t=0$ $W=500$. For in the limiting case when the substance possesses no elasticity, it follows that it has no tendency to recover when stress is removed; and hence the whole weight would be removed instantaneously, and the curve representing the removal of weight with time would simply consist of the two straight lines at right angles, $t=0$ and $W=500$. If we suppose that all the curves do break when they meet this dotted curve, then it is obvious from the diagram that the breaking point of the 4·5 per cent. curve must be at a considerably greater time and a considerably less weight than that of the 4·4 per cent. curve. This supposition is to some extent justified by

the curve shown in figure 4, which represents final stress against concentration. This curve in the region 4·4 per cent. concentration is very steep, and a small change of concentration makes a comparatively large change in the elastic limit.

Returning now to figure 3, it will be seen that the curves are not quite horizontal after the break has been reached; that is to say, the elasticity still gradually diminishes. Possibly this may be accounted for by the fact that during each experiment the temperature was slowly rising, which condition would tend to decrease the elastic limit of the specimen.

No type of equation has been found which will fit all the curves, but an equation of the form $W = a \log (pt + 1)$ fits the 4·5 per cent. curve with considerable exactness. This particular curve is of the same type as those for lead wire treated in a similar manner (" On the stretching and torsion of lead wire beyond its elastic limit," Phil. Mag. Oct. 1904), for not only do equations of the same form approximately represent the curves, but both curves show similar departures from those equations. The above equation also satisfactorily represents all curves corresponding to higher concentrations than 4·5 per cent.; these are not shown in the diagram.

It is noticeable also that the dotted curve passing through the points of breakage of the four upper curves is of like

Fig. 4.
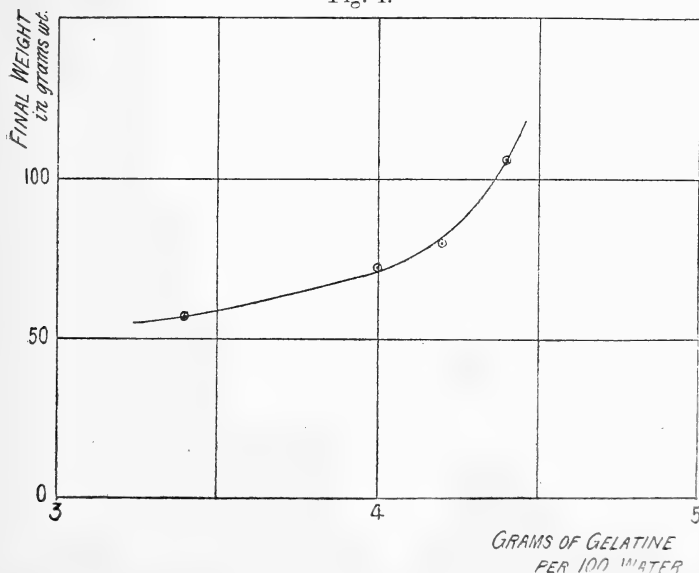
shape, being fairly represented by an equation of the form $W = W_0 - a \log (pt + 1)$.

Figure 4 shows the final weight, which is of course

proportional to the couple, plotted against concentration, and, if we regard each specimen as being elastic to the extent shown, it indicates that as the concentration increases the elastic limit increases more and more rapidly.

The elastic limits and the times at which they are reached are shown in the following table :—

Concentration.	Elastic Limit.	Time of reaching limit.
3·4	57 gr. wt.	0·114 hour.
4·0	72 ,,	0·274 ,,
4·2	80 ,,	0·496 ,,
4·4	106 ,,	2·160 hours.

IV. *Conclusions.*

The following conclusions may be drawn from the results of these experiments :—

1. That any specimen, the concentration of which exceeds some fixed limit, is capable of permanently supporting a certain stress.

2. That the nature of the variation of stress for constant strain depends on the amount by which the limiting stress is exceeded. Thus we see that the curves of the 3·4, 4·0, and 4·2 per cent. concentrations differ in kind from that representing the 4·5 per cent. concentration; that is to say, they are not represented by equations of the type $W = a \log (pt + 1)$. Owing to the smallness of the time scale it is impossible to reasonably fit equations to them, and it is intended to repeat these determinations using a more quickly moving drum, in order to obtain a more open scale. But the 4·4 per cent. curve is not of the same type as the 4·5 per cent. No equation of the form suitable to the latter fits it. It appears to partake of the shape of both upper and lower curves. We conclude, therefore, that for specimens of greater concentration, *i. e.* those in which the elastic preponderates over the viscous, the variation of stress under constant strain follows the logarithm law $W = W_0 - a \log (pt + 1)$ just as in the case of lead ; on the other hand, if the concentration is low and viscosity is the more prominent property, some other law is followed, and solutions of intermediate strength are governed by a combination of these laws.

3. That, in some cases at least, the whole of the excess

stress is dissipated within a finite time, contrary to what we should expect from theoretical considerations. At any rate, whether or not it is admitted that the breaks in the curves indicate that the elastic limit has been reached, it is certain that at these points a discontinuity occurs, the law on the one side being entirely different from that on the other side.

In conclusion I wish to thank Professor Trouton and Professor Porter for the kind interest they have shown in the work, and for the suggestions which have enabled me to surmount many of the difficulties which have arisen during the investigation.

XL. *The H Theorem and* Professor J. H. Jeans's *Dynamical Theory of Gases.* By S. H. BURBURY, *F.R.S.*[*]

1. BOLTZMANN'S H Theorem professes to prove that whenever the state of a gas is other than the normal state, $\frac{d\mathrm{H}}{dt}$ is negative, and H is a minimum in the normal state.

H is the function $\iiint\limits_{-\infty}^{+\infty} (f \log f - f) \, du \, dv \, dw$; $f \, du \, dv \, dw$ being the number per unit of volume of molecules whose component velocities lie between u and $u + du$, v and $v + dv$, w and $w + dw$. In the normal state as usually understood, $f = \mathrm{A} e^{-hm(u^2 + v^2 + w^2)}$. As a consequence of the theorem, H, once minimum, remains minimum for ever, making the process irreversible. Then the objection was made that the Theorem, proving $\frac{d\mathrm{H}}{dt}$ to be necessarily negative, if not zero, proves too much, because we have only to reverse all the velocities simultaneously, and the system will retrace its course with H increasing. The question is how to explain the paradox.

2. Professor J. H. Jeans, a very strong mathematician, has dealt with this question in his recent work on the Dynamical Theory of Gases. He gives (art. 11) a definition of the density, ν, of a gas at a point P : namely, ν is the number of molecules in a volume at P, which volume is very great compared with the mean molecular distance, but very small compared with the scale of variation of density of the gas. The definition is free from ambiguity. But it ignores variations of density on the scale of the intermolecular distances.

There seems to be no reason why the density should vary at all on a scale infinitely larger than that of the intermolecular distances, unless in a field of external force such as gravity. Jeans's definition virtually assumes ν to be constant, except as varied by external forces.

3. He then gives the usual proof of the H Theorem on the hypothesis of " molecular chaos " (art. 15). This is the hypothesis which I call condition A, namely, that the chance of a molecule having velocities within assigned limits is independent of the positions of all the other molecules for the time being, and also independent of the velocities of all the other molecules for the time being, subject only to the constancy of $\frac{1}{2}\Sigma m(u^2 + v^2 + w^2)$, the total kinetic energy. The legitimacy of this assumption is, Jeans says, not self-evident, because it is conceivable, for instance, that molecules having nearly equal velocities should tend to flock together. He reserves the consideration of this question to later chapters.

4. In Chapter II. he introduces a new notation. Given a system of N molecules, he calls the 3N position coordinates $x_1 y_1 \ldots z_N$, and the 3N component velocities $u_1 v_1 \ldots w_N$, the " coordinates of a point in a generalized space of 6N dimensions." Every state of the system is represented by a point in that space, and the series of states which follow each other in natural motion are represented by a line in that space. This is a change in notation, advantageous it may be, but still only in notation. And as no new hypothesis is expressly introduced, the physical relations between the molecules must be understood to remain unaffected. Therefore every proposition concerning these relations, true in the usual notation, is true in the new notation, and *vice versa*. Also every proposition which can be proved by the use of the new notation, can be proved—I do not say proved equally well, but can be proved—by the use of the old notation. If the new method leads, or appears to lead, to any physical results not attainable by the ordinary method, that can only be because along with the new method we have introduced some new hypotheses unawares.

5. Jeans then discusses, arts. 39–54, the " Partition of the generalized space among the position coordinates." We are to suppose the three-dimension space Ω in which the molecules are moving to be divided into n equal cells $\omega_1 \omega_2 \ldots \omega_n$, each of volume ω, so that $\Omega = n\omega$. The number of ways in which N molecules can be assigned to the n different cells, so that there shall be a_1 in ω_1, a_2 in ω_2, &c. with $a_1 + a_2 + \ldots a_n = N$ is

$$\theta = \frac{N!}{a_1! \ a_2! \ldots \ a_n!} \text{ (equation 60, p. 39).}$$

This means that the molecules are scattered at random through the space Ω. The most probable distribution is found by making θ maximum, and is when

$$\sum_{s=1}^{s=n} a_s \log a_s \text{ is minimum, that is when } a_1 = a_2 = \ldots = a_n.$$

Subsequently (art. 55) he discusses the Partition of the generalized space as regards the velocities. This he says is to be effected in the same way as in the case of the position coordinates. That is, we are to suppose another three-dimension space, or diagram of velocities, and u, v, w are the coordinates in that diagram of points scattered at random through it, but they are to be subject to one new condition, namely, that $\Sigma m (u^2 + v^2 + w^2) = $ constant (equation 98). This new space, let us call it Ω', may be divided into n equal cells, $\omega_1' \, \omega_2' \, . . \, \omega_n'$, each of volume ω', and the number of ways in which N molecules may be distributed through Ω', so that there shall be f_1 in ω_1', f_2 in ω_2', &c., with $\Sigma f = n$, is

$$\theta' = \frac{N\,!}{f_1!\ \ f_2!\ .\,.\,.\,.\,f_n\,!}.$$

It is shown that θ' is maximum when

$$\iiint f \log f \, du \, dv \, dw \text{ is minimum.}$$

So the solution is given by making

$$\iiint f \log f \, du \, dv \, dw \text{ minimum,}$$

given

(1) $\iiint f \, du \, dv \, dw = 1$ (97), expressing only the fact that the number of molecules is constant;

(2) $\iiint m (u^2 + v^2 + w^2) \, du \, dv \, dw = $ constant (98).

6. The usual method leads to

$$f = \mathrm{A}e^{-hm\,(u^2+v^2+w^2)}, \ i.\ e. \text{ Maxwell's law.}$$

No mention has been made of any possible relation between velocities and space coordinates, such as that referred to by Jeans's art. 15. If any such existed it would manifestly vitiate the reasoning. It is therefore assumed by necessary implication (a) that no such relation exists, and therefore that the chance of a molecule having velocities within assigned limits is independent *of the positions* of all the other molecules for the time being. Further, it is expressly stated that the only relation existing among the

velocities is (98). It is therefore assumed (*b*) that the chance of a molecule having velocities within assigned limits is, subject to (98), independent of *the velocities* of all the other molecules for the time being. Assumptions (*a*) and (*b*) together amount to Condition A, or "molecular chaos." Jeans has therefore by implication assumed Condition A. Maxwell's law distinctly asserts Condition A. If therefore you could prove Maxwell's law without expressly or by implication assuming Condition A, you would prove Condition A to be a physical fact, self-existent I suppose. But Jeans has proved Maxwell's law only on the implied assumption of Condition A. He has proved it, that is, on precisely the same assumption as all former writers have done. He has in no way altered the logical position of the Kinetic Theory of Gases.

7. With regard to the proof contained in arts. 39–54, Jeans proves, as I said, that θ is maximum when $a_1 = a_2 \ldots = a_n$. And this he shows is equivalent to making ν constant on average. And further that, the a's being very great, θ becomes infinitely less than its maximum when ν differs considerably from its constant or mean value, so that ν may be regarded as everywhere constant. I think that with infinitely small molecules and no forces acting, this follows from the definition of ν. Also, is not the result that all the a's are equal when θ is maximum evident from the mere inspection of θ? And then may we not write $a_s = \omega_s \nu_s$. And, therefore, $\nu_1 = \nu_2 \ldots = \nu_n$?

8. The question occurred to me whether the number of possible arrangements of the molecules, which Jeans calculates, has any effect on the state of any physical system. Consider, for instance, a vertical column of gas under the earth's attraction. The number of ways in which the molecules can be assigned to equal elements of the column is immensely greater, according to Jeans's calculation of θ, when they are distributed with uniform density than when they are distributed according to any other law. Has this fact any influence on the actual distribution? I think none whatever, if the molecules are, as Jeans assumes (p. 39), infinitely small. I do not think Jeans would say that it has any. In Jeans's distribution the chance of a given molecule being in the cell ω, is $\dfrac{\omega}{\Omega}$, and is independent of the position of the cell. We might say that equal cells have equal value for the purposes of the distribution. In the vertical column the chance of a given molecule being in the elementary cell ω is $\dfrac{\omega}{\Omega} e^{-2hgs}$, s being the height of the

cell above the base of the column. We might say that the value of a cell for the purposes of this distribution is proportional to e^{-2hgs}. On this principle we find that the density at height s is proportional to e^{-2hgs} as Boltzmann teaches, and is not affected in any way by Jeans's law of distribution.

9. In arts. 62–65 Jeans discusses the assumption of molecular chaos. The number of molecules whose space coordinates lie between x and $x + dx$, &c., and their velocities between u and $u + du$, &c., whatever be the positions and velocities of the other molecules, is denoted by v_p. It can evidently be written

$$v_p = \mathrm{N} f(u\,v\,w)\,du\,dv\,dw\,dx\,dy\,dz \iiint dx_b\,dx_c \ldots dy_b\,dy_c \ldots ,$$

in which $x_b\,y_b\,z_b$, x_c, &c. denote the space coordinates of the molecules not included in v_p. Similarly the number whose position coordinates lie between $x' \ldots x' + dx'$, &c., and their velocities between $u' \ldots u' + du'$, &c., is given by

$$v_p = \mathrm{N} f(u'\,v'\,w')\,du'\,dv'\,dw'\,dx'\,dy'\,dz' \iiint dx_c \ldots dy_c \ldots dz_c,$$

$x_c\,y_c$, &c. now denoting the position coordinates of all the molecules except those which are included in v_p or v_q.

If $v = \iiint dx\,dy\,dz \iiint du\,dv\,dw$, over all the molecules, v includes all possible states of the system and means the same thing as the volume of Jeans's generalized space. And $\dfrac{v_p}{v}$, $\dfrac{v_q}{v}$ represent respectively the chances of a molecule being included in v_p or in v_q.

The question is whether these chances are independent of each other. Now the motion being continuous, the velocities and positions of all the molecules at any instant are determinate functions of their velocities and positions at the initial instant t seconds ago. Consider two molecules m and m'; let u_0 and u_0' denote their initial velocities. If m and m' exert no forces on each other during the time t, and no third body sensibly influences both during the time t, and if $u\,u'$ were initially independent, then at the given instant u cannot be a function of u'. If $\phi(u)\,du$ be the chance of u lying between assigned limits, $\phi(u')\,du'$ the same for u', it is certain that $\dfrac{d\phi(u)}{du'} = 0$. Similarly $\dfrac{d\phi(u')}{du} = 0$. $\phi(u)$ and $\phi(u')$ are then independent chances.

Clearly, then, if there be no intermolecular forces at all, the chances are independent. And they may be independent if t be small enough and the initial distance between m and

m' great enough.   But if, during the time t, m and m' exert sensible forces on each other, it is certain that in general the two chances $\phi(u)$ and $\phi(u')$ are not independent.   The question depends on the forces, and on nothing else whatever. It cannot be affected by any change in notation.   There is indeed an attempted justification of the assumption of independence in the latter part of art. 65, p. 57.   But it is only by reference to the analysis of Chapter III., the Partition of the generalized space, in which was assumed by necessary implication that molecular chaos or Condition A exists.   It is impossible to prove a physical fact by merely using a new notation.

10. Jeans then, pp. 59, 60, gives an "Analysis of the H Theorem."   He reminds us that to obtain strictly accurate results, the quantities he deals with must not be regarded as applying to a single system, but to the average of all systems satisfying certain conditions.   These conditions I suppose are that each system consists of the same number N of similar molecules, and has the same mean kinetic energy. Then he assumes (art. 68) that the function f (art. 1 above) consists of two numerically equal parts, one half containing those molecules whose velocities are u, v, w, and the other half those whose velocities are $-u$, $-v$, $-w$.   If one of these equal parts corresponds to a direct motion, the other corresponds to the reversed motion (art. 1 above).   Whence it follows that, taking both motions into account, $\dfrac{d\mathrm{H}}{dt}$ is on average as often positive as negative, is in fact on average zero.

11. This argument, it should be noted, does not affect the mathematical correctness of the H Theorem.   The theorem is founded on a certain assumption, namely, that the number per unit of time of collisions between pairs of molecules, one from the class F, and other from the class f, the two molecules being so situated that after collision they are in the classes F′ and f' respectively, is proportional to Ff, that is to the number of such pairs which exist—and is not generally proportional to F′f'.   This assumption, though I deny its legitimacy, justifies, if true, the H Theorem.   But if it applies to the direct, it cannot apply to the reversed motion, because the pairs of molecules which in the reversed motion pass from the classes F and f to the classes F′ and f', are the identical pairs which in the direct motion passed from F′ and f' to F and f, and their number is therefore by the assumption proportional, not to Ff, but to F′f'.   Therefore the H Theorem is inapplicable to the reversed motion by

failure of the assumption on which it is based. And the mathematical correctness of Boltzmann's theorem, granted his assumption, is preserved. The doubt is whether the assumption itself is valid.

12. It is proved *only in the normal state* that u, v, w are as likely to occur as $-u, -v, -w$. We have then, it may be said, no right to assume it in an analysis of the H Theorem, because that theorem deals only with systems in abnormal states. I understand, however, Jeans's meaning to be this. According to the H Theorem the system, given in an abnormal state, passes into the normal state. But *when it is normal,* u, v, w are as probable as $-u, -v, -w$, and therefore, the *motion being continuous,* the passage from the abnormal to the normal is just as frequent, and no more so, than the reverse. In this form the argument is good against Boltzmann as an *argumentum ad hominem.* But it makes the continuity of the motion a vital point. For if there be any the least discontinuity in the original motion, caused, say, by some external disturbance, it ceases to be true that the system, starting with velocities $-u, -v, -w$, will retrace the course of a system which ended with u, v, w. And therefore we cannot prove that on average of both motions $\dfrac{d\mathrm{H}}{dt}$ is zero. The discontinuity would be fatal to Jeans's argument, but not to the H Theorem.

13. I agree with Jeans that as a fact on average of all systems and of all time, $\dfrac{d\mathrm{H}}{dt} = 0$, provided that the motion be continuous and protected from external disturbance. I agree with him also that, as a fact, on the average of all systems and all time, the states u, v, w, and $-u, -v, -w$ occur with equal frequency. I agree further with him that on average of all systems and all time H is nearly minimum. I expressed that myself in the form that when H differs little from its minimum, $\dfrac{d\mathrm{H}}{dt}$ and $\dfrac{d^2\mathrm{H}}{dt^2}$ are very small, whereas if H differs much from its minimum $\dfrac{d\mathrm{H}}{dt}$ may be very great. For by consideration of the direct and reverse motions, we see that the number per unit of time of collisions by which two molecules pass from the classes F and f to the classes F$'$ and f'', is proportional, not indeed necessarily to Ff as Boltzmann assumes, but either to Ff or else to F$'f''$, and as often to one as the other. But when H is nearly minimum F$'f'' -$Ff' is in

general very small, and $(F'f' - Ff) \log \dfrac{Ff'}{F'f'}$ of the second order of small quantities. But $\dfrac{dH}{dt}$ consists of terms either of this form or of the form $(Ff - F'f') \log \dfrac{Ff}{F'f'}$ and is therefore, whatever its sign, very small, and so is $\dfrac{d^2H}{dt^2}$.

It follows that any given system will remain for very long periods of time with H nearly minimum, though H may occasionally for short periods deviate into higher values. As for each system on average of time, so at each time on average of all systems, H is nearly minimum, which agrees with Jeans's conclusion.

Let us suppose there are N states in which $H = H_0$, and is minimum. Of these N states there are say n_1 states, each of which is the end of a course in which H has fallen from H_1, its last maximum, to H_0, and n_2 in which it has fallen from H_2, its last maximum, to H_0. And if $H_2 < H_1$, $n_2 > n_1$. Similarly for each of the N states there is a last maximum of H, and $N = n_1 + n_2 + \ldots + n_N$. Then I should say $\dfrac{n_1}{N}$ measures the chance of H having the value H_1, $\dfrac{n_2}{N}$ the chance of its having the value H_2, and so on; and it may be that $\dfrac{n}{N}$ is infinitesimal for all values of H not infinitely near to H_0. That gives Jeans's result.

14. If, however, there are in fact as many courses in which H passes from H_0 to H_1, as in which it passes from H_1 to H_0, it is not true that the variations of H constitute an irreversible process. It is not necessary to assume that the fall from a maximum H_1 to H_0 is always followed by a rise to precisely the same value H_1 as the next maximum, so that the system oscillates. It is sufficient to say, as we have said, that, on average of all systems and all time, motions in one direction happen as frequently as in the other direction.

15. Boltzmann's own explanation of our paradox was that the reversed motion, in which H increases, although possible, is in a very high degree less probable than the direct motion in which H diminishes, and on this he founded his doctrine of the connexion between reversibility and probability. I think this explanation cannot be reconciled with the theorem; for if Maxwell's law holds when H is minimum, which the theorem is supposed to prove, the state in which the velocities are typically $+u$, $+v$, and $+w$, and the state in which they are $-u$, $-v$, and $-w$ are equally probable

states. To assert that one is in a high degree more probable than the other, would be contrary to Maxwell's law.

16. The real defect in the H Theorem, and in every proof yet given of Maxwell's law, consists in my opinion in this, that it ignores the continuity of the motion altogether. The H Theorem is based wholly on the assumption mentioned in art. (11) above. Now let us suppose that at a given initial instant the state of the system is formed as follows :— One person assigns component velocities u, v, w to each molecule according to any law he pleases, subject only to the condition that $\Sigma m(u^2 + v^2 + w^2) = E$. And another person, to whom these assigned velocities are wholly unknown, scatters the molecules through the space Ω. In that case Condition A is satisfied at the initial instant, Boltzmann's assumption art. (11) above is satisfied, and if H is not minimum, $\dfrac{d\mathrm{H}}{dt}$ is at that instant negative. But in the supposed distribution, the velocities are just as likely to have all the opposite values $-u$ $-v$ $-w$ as u, v, w. In either case at the initial instant $\dfrac{d\mathrm{H}}{dt}$ is negative. If, however, the system were left to itself for a finite time, and then the final velocities were reversed, the system would retrace its course back to the initial state. And immediately before it arrives at that state $\dfrac{d\mathrm{H}}{dt}$ is evidently positive. But when it has arrived at that state $\dfrac{d\mathrm{H}}{dt}$ becomes discontinuously negative, because then the original velocities u, v, w would have become $-u$ $-v$ $-w$. And, as we have seen, $\dfrac{d\mathrm{H}}{dt}$ is negative in that state as well as in the original state. $\dfrac{d\mathrm{H}}{dt}$, if not zero, must be discontinuous when the system in its reversed course passes through its initial position.

17. That is the history of the initial state formed as I have supposed it to be formed. In dealing with the subsequent course of the system, to prove the H Theorem, or Maxwell's law, we assume, and if we are to prove the proposition we must assume, that the independence of the molecular velocities, or as I call it Condition A, exists at every instant during the motion. But if the motion be continuous, the state of the system at any instant is a determinate function of the initial state and of the time elapsed, and of

nothing else whatever. Now that the velocities of a set of molecules, being functions of the same set of initial variables and of the same time, should be always independent of each other, is, if it is to be taken as an exact statement, a mathematical impossibility. There must arise correlation between the velocities, at least of neighbouring molecules, as Jeans suggests in his art. 15.

18. In my unorthodox opinion therefore, neither Maxwell's law nor the H Theorem are true accurately, if the motion is continuous. As regards Maxwell's law, it may be for some important purposes accurate enough. If, for instance, we wish to calculate the theoretical value of the coefficient of diffusion between rare gases for comparison with experiment, approximate numerical results are all we require, and all we can have. And it may well be that Maxwell's law gives a sufficiently near approximation. This I have never denied But in considering the question of reversibility, the method of approximation is ineffectual. For suppose two states of a gas, state A and state B. In state A the position and velocities of the molecules are denoted respectively by x, y, z and u, v, w. In state B they have nearly the same values as in A, and the differences are for some purposes negligible. Nevertheless these differences, however small, may in time cause the two systems to diverge widely from each other. One may be asymptotic, H continually approaching its minimum, the other may be cyclic. The asymptotic character cannot be established by any approximate reasoning.

19. If I am right in saying that Maxwell's law, founded as it is on the assumption of Condition A, and asserting as it does the truth of Condition A, cannot in continuous motion be accurately true, nevertheless the system started as I have supposed in art. 16 above and left to itself to move in continuous motion, must ultimately settle down into stationary motion of some kind, and must then have a normal state of some kind. If it be not Maxwell's law, what is it? I should answer the question thus—Firstly, in that normal state Condition A must not completely prevail. Secondly, the state in which the velocities of the N molecules are typically u, v, w, and the state in which they are typically $-u$, $-v$, $-w$, must in the stationary motion be equally probable. Both conditions are satisfied if " The chance of " the N molecules having velocities within assigned limits is " $A\epsilon^{-hQ} du_1 \ldots dw_N$, and

" $Q = \Sigma m(u^2 + v^2 + w^2) + \Sigma\Sigma b(uu' + vv' + ww')$,

" where b is a function of r, the distance between the molecules

" whose velocities are $u\,v\,w$ and $u'\,v'\,w'$, which is evanescent
" except for small values of r." This does not satisfy Con-
dition A. Also the second condition is evidently satisfied
because Q is a quadratic function.

20. I suggested myself in 'Nature,' February 1895, in
aid of the H Theorem, that no material system ever does
remain for any considerable time in exactly continuous
motion, free from external disturbances. Such disturbances
are always happening, and their effect, if they come at hap-
hazard without regard to the state of the system for the time
being, is *pro tanto* to renew or to maintain the independence
of the molecular motions, *i.e.* Condition A, and so to make H
diminish. I think that is true. We may assume that
if the disturbances are frequent enough and great enough,
$\frac{d\mathrm{H}}{dt}$ will be in general negative or zero. But then we must
remember that, as regards this disturbed system, it is not
true that were all the velocities at any instant reversed the
system would retrace its course, because the disturbances are
not included in the reversal. I think therefore that the true
explanation of the paradox of art. 1 above is as follows :—

The system either is or is not isolated, that is protected
throughout the motion from all external disturbances. If we
take the first hypothesis, that it is isolated, it is not true that
the condition of independence (art. 17) exists. Therefore it is
not true that $\frac{d\mathrm{H}}{dt}$ is necessarily negative. But it is true that
on reversal of the velocities the system would retrace its
course. There is a reversed motion, but there is no H Theorem.
If we take the second hypothesis, that disturbances are con-
tinually happening, it may be true that $\frac{d\mathrm{H}}{dt}$ is generally
negative. But it is not true that on reversal of the velocities
the system would retrace its course. There is an H Theorem,
but there is no reversed motion. The reason why we obtain
for the same system first one and then the other of two incon-
sistent results, first that $\frac{d\mathrm{H}}{dt}$ is negative, secondly that $\frac{d\mathrm{H}}{dt}$
is positive, is because we apply to the same system first one
and then the other of two inconsistent hypotheses.

XLI. *On the Recombination of Ions in Air and other Gases.
By* W. H. Bragg, *M.A., Elder Professor of Mathematics
and Physics in the University of Adelaide; and* R. D.
Kleeman, *B.Sc., Demonstrator* *.

I T is well known that when positive and negative ions are
distributed through a given space, a process of com-
bination goes on until ions of one sign only are left. Let
there be p positive ions, and n negative ions in each cubic
centimetre at any instant, and suppose that the relations of
any ion to all those of opposite sign are of the same character.
Then the chance that an ion, say a positive one, will enter
into combination before the end of a short time δt is propor-
tional to $n\delta t$; and generally the number of combinations
taking place in that time may be denoted by $\alpha pn\delta t$, where α
is the "co-efficient of recombination." This has been clearly
established by the experiments of Rutherford, Townsend,
McClung, Langevin, and others.

As a consequence, the current passing between two elec-
trodes in a gas in which ions are being formed by external
agents depends on the magnitude of the potential gradient
or electric force. The relations between current and force
have been carefully studied by many workers, and the
observed facts have been compared with the results of cal-
culation based on theory. The comparison is partly, but not
completely, satisfactory.

Certain experimental results which we propose to describe
in this paper seem to throw light on the reason of the dis-
crepancy. They point to the existence of another cause,
distinct from that represented by the expression αnp, which
prevents ions from reaching the electrodes in the gas in which
they are formed. This cause appears to be a process of re-
combination of newly-formed ions with the atoms from which
they have just been separated. The effects of it are propor-
tional to the number of ions formed in a c.cm. in unit time,
not to the product of the existing numbers of positives and
negatives. They are independent of the shape of the ioni-
zation chamber, and in this they differ from those of general
recombination. They depend directly on pressure, and vary
greatly from gas to gas.

In order to bring these effects into relief, it is only necessary
to reduce the number of ions in a c.cm. until the number of
those that are lost by general recombination is negligible
compared with the number of those that are formed. When

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* Communicated by the Authors. From the 'Transactions of the Royal
Society of South Australia,' vol. xxix. 1905.

this is done, it is found that it is still necessary to apply a high potential in order to extract all the ions from the gas. For example, in air at atmospheric pressure an electric force of 25 volts to the cm. will only extract about 80 per cent. of the ions which are obtained when the force is increased to 1000. The following example will serve as an illustration:— The width of the ionization chamber is 4 mm., the upper electrode being a metal plate, the lower a sheet of gauze. A thin layer of radium is placed 6·2 cm. below the sheet, and α particles emitted from Ra C cross the chamber and ionize the air, which is at atmospheric pressure. The area of the plate on which the rays fall is about 18 sq. cm. The capacity of the electrometer to which the upper plate is connected is about 150 cm., and a potential of ·125 volt applied direct to the electrometer causes a deflexion of 722 divisions on the scale; ten divisions = 1 mm. When the lower plate is raised to 400 volts positive, so that the electric force is 1000 volts per cm., there is a deflexion of 982 divisions in 10 seconds, under the influence of the α rays. When a potential of 10 volts is applied, giving a force of 25 volts per cm., there is a deflexion of 772 in 10 seconds.

In the latter case the charge Q received per sq. cm. of electrode in one second, measured in electrostatic units, is—

$$\frac{772 \times 150}{10 \times 722 \times 8 \times 300 \times 18} = 3\cdot5 \times 10^{-4}.$$

The number of ions falling on each sq.cm. of electrode per second is therefore $1\cdot2 \times 10^{6}$ nearly.

The velocity of ions at this potential gradient is nearly $25 \times 1\cdot5$, or 37 cm. per second.

Thus, if n be the number of ions in a cubic centimetre, $37n = 1\cdot2 \times 10^{6}$, and therefore $n = 3\cdot2 \times 10^{4}$. Hence, the number of recombinations taking place in a second in the space between two opposing square centimetres of the electrodes is equal to $\alpha \times \cdot4 \times (3\cdot2 \times 10^{4})^{2}$. If we take the value of α to be $3400 \times 3 \times 10^{-10}$, we find this number to be nearly 420. Finally, therefore, the number of ions recombining in each second is 420, whilst the number received is $1\cdot2 \times 10^{6}$, and thus only 1/3000 of the ions are lost in this way.

But the current at 25 volts is only 772/982, or about 80 per cent. of the current at 1000 volts.

It is clear from this example that there is some cause which prevents the current attaining its full value other than general recombination between positive and negative ions.

Now, it is possible that ions newly formed might be specially liable to recombine with each other. Such a possi-

bility has been already suggested by Rutherford ('Radio-activity,' p. 33). An electron, which has just been ejected from an atom by a passing α particle, does not go far before encountering a neighbouring atom. The encounter, perhaps, results in a temporary attachment, for we know that ion-clusters are formed in this way. In any case, it is probable that the electron loses much of its velocity of projection. Now, it is still under the attraction of the atom from which it has come. Supposing this atom to have only lost one electron, the strength of the electric force which it exerts at the distance of the mean free path is equal to

$$e/r^2 = 3 \times 10^{-10}/10^{-10} = 3 \text{ E.S.U.,}$$

or 900 volts per cm. This is large compared with the usual impressed electric forces of experiment. It is by no means improbable, therefore, that the electron may finally slip back into its old place. Such a possibility is not considered in the equations as usually formed. For all writers begin their arguments by the statement:—" Let p be the number of positive ions in a cubic centimetre, and n the number of negative." In doing so they tacitly assume that the relations of any one ion to all others of opposite sign are of the same character. But if a pair of newly-formed ions ran a special chance of recombination until they got away from each other, then the relations of either of these two to the other would be quite different to its relation to all other ions.

Let us, then, for the moment suppose that there is a special form of recombination, which we may call " initial," as distinguished from general recombination, and let us consider the nature of its effects, in order that we may find means of testing the correctness of the supposition.

Now, it is clear that the effects of initial recombination do not depend upon the shape of the ionization chamber, and this at once differentiates them from those of general recombination. For the special or initial recombination concerns only the ion and its parent atom. But general recombination depends on the chance of an ion meeting others of the opposite sign, which chance depends on the number in a c.cm., and this, again, on the shape of the chamber. If, for example, α particles cross a chamber 3 mm. wide, and a sufficient potential gradient is applied, most of the ions will be carried to the electrodes. If the width of the chamber is increased to 6 mm. the magnitude of the stream of ions is doubled, each positive meets twice as many negatives as before, and therefore the chance that any one ion enters into recombination is twice as great. Suppose the saturation current for a 3 mm. chamber were 100, using any arbitrary

system of units, and the actual current for a moderate potential were 90, then for the 6 mm. chamber, under an equal potential gradient, the current would be 160, not 180; the saturation current being 200. This is recognized in the usual formulæ. For example, Langevin finds that—

$$\frac{\epsilon Q}{\sigma} = \log\left(1 + \frac{\epsilon Q_0}{\sigma}\right)$$

where Q_0 is the saturation current per sq.cm. of electrode, and Q is the current when such a potential is applied that σ is the density thereby caused to exist on each sq.cm. of the electrode. When Q and Q_0 are both small compared to σ, it follows that :—

$$\frac{\epsilon Q}{\sigma} = \frac{\epsilon Q_0}{\sigma} - \frac{\epsilon^2 Q_0{}^2}{2\sigma^2}.$$

$$\therefore \quad \frac{Q_0 - Q}{Q_0} = \frac{\epsilon Q_0}{2\sigma}.$$

Thus, the relative lack of saturation, viz. $(Q_0 - Q)/Q_0$, is proportional to Q_0, which itself depends on the depth of the chamber. Other formulæ show the same dependence.

But experiment shows that when the density of the ions is small the depth of the ionization chamber has very little effect on the degree of saturation. This may be illustrated by the following experiments :—

Five mmg. of radium bromide were so placed that the α rays passed upwards through an aperture in a lead plate and crossed the gauze of the ionization chamber. The rays formed a cone whose vertical angle was about 20°. The apparatus used was the same as that of the previous experiment described, but the currents were so strong that a capacity of 1070 cms. had to be put in parallel with the electrometer. Determinations were then made of the strengths of the current at various potentials :—(1) When the ionization chamber was 3 mm. wide; (2) when 6 mm. wide ; and (3) when 9 mm. wide. The values obtained were then reduced so that the saturation current in each case was set at the same value. Comparison then showed that the curves were almost identical except at low potentials, and this was in agreement with the hypothesis now put forward. For at all but low potentials αn^2 was so small as to be negligible. When the potential was very low, one or two volts per centimetre, then the ions moved so slowly that n was larger, and αn^2 was not negligible, and under those circumstances the curve showed a difference of the right kind. That is to say, the 9 mm. curve was further from being saturated than the others. The currents were specially made not too weak in order to

bring out this contrast between the effects at low and at high potentials. The figures are given in the following table :—

Relation of current to potential gradient for different widths of the ionization chamber, the currents being small ; potential gradients in volts per centimetre : currents in arbitrary units, reduced to common maximum.

Width of Chamber.

3 mm.		6 mm.		9 mm.	
Potential Gradient.	Current.	Potential Gradient.	Current.	Potential Gradient.	Current.
1000	400	1000	400	1000	400
34·8	341	35·7	335	36·0	333
11·9	308	12·7	302	16·4	300
8·65	294	9·47	292	9·83	290
6·6	285	7·47	283	7·81	283
4·66	274	5·50	272	5·83	271
2·75	252	2·75	247	3·10	244
1·92	235	1·89	224	2·23	220
1·50	221	·98	169	1·36	174
1·06	196	·51	108	·84	134
·59	147	·02	4	·04	68
·385	106				
·20	47				

These figures are plotted, as far as 36 volts per cm., in curves A, B, and C of fig. 1. An open scale is chosen so as to show the separation of the curves at low values of the field, when n is not very small.

These figures and curves show that the ratio of the current at any particular strength of field to the saturation current is almost independent of the shape of the ionization chamber, when the current is small. As this seems an important point, we have made many experimental tests of it. We give below the details of one such test, in order to illustrate the methods employed and their degree of accuracy.

The arrangements were the same as those just described ; and the special object of the experiment was the determination of the degree of saturation under a certain moderate potential gradient in the case of chambers of two different depths. The depths were reckoned in turns of the screw, which raised the upper from the lower plate of the chamber ; eleven turns = 1 cm. The currents were allowed to run into the electrometer for 10 seconds. The electrometer was not dead-beat, and therefore the first and second resting-places on the scale were observed, and the mean taken. For example, the second line, marked †, of the subjoined table,

Fig. 1.

Curves showing relation between potential gradient, and current in air.

Curve			A	B	C	D	E	F
Width of ionisation chamber in mm.			3	6	9	3	5	10
Order of saturation current in amperes per sq. cm. of electrode			10^{-12}			10^{-9}		

D, E, and F are calculated from the results of experiments made by Retschinsky.

Saturation current = 4.00

Volts per cm. →

Current →
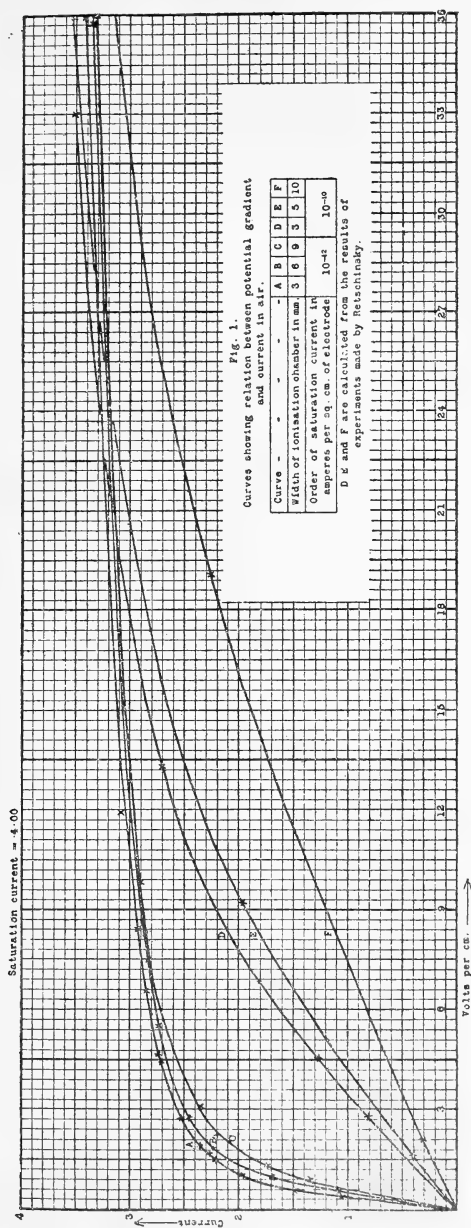
shows that the first deflexion was to 57·47 cm., and then back to 54·92, zero being 4700. The experiment repeated gave 57·42 to 54·92, and again 57·47 to 54·93. The leak was also measured with a metal plate over the radium, and the difference taken as the proper value of the leak for that experiment, a small proportion only being due to β rays. The first measurements relate to a chamber of depth 6 turns— (i.) under a potential gradient of 600 volts for the 6 turns ; (ii.) a gradient of 20 volts for the same distance ; (iii.) under 600 volts again. The difference between (i.) and (iii.), as shown in the table, was due to the variation in sensitiveness of the electrometer. In almost all our experiments this variation has been negligible ; in this special case it was not so, because so large an amount of radium was used, viz., 5 mmg. The γ rays penetrated all the metal casings, and caused a leak in the charge of the needle. The leak had an exaggerated influence on the readings because the capacity of the electrometer was increased by the addition, in parallel, of a plate-condenser of 1000 cm. capacity. This disturbed the usual balance of the electrometer, in which leakage of the needle's charge had little effect on the deflexion for a given quantity of electricity. To obviate any error from variation of sensibility the results of (i.) and (iii.) were averaged, and compared with the result of (ii.). It will be observed that successive determinations of the same leak were very consistent with each other. This implies that almost all the observed effect was due to the radium ; extraneous influences were very small.

6 TURNS.   Zero = 4700.

600 volts (Metal over Ra).	47 47	75 69 63 59 } Mean leak = 67	} Net leak = 852 = I_6 (say).
† 600 volts.	57 54	47 42 47 92 92 93 } Mean = 919	
20 volts (Metal over Ra).	47 47	50 54 40 46 } Mean = 48	} Net leak = 695 = I'_6.
20 volts.	55 53	49 39 49 43 36 42 } Mean = 743	
600 volts (Metal over Ra).	47 47	70 69 61 55 } Mean = 94	} Net leak = 829 = I_6.
600 volts.	57 54	19 16 13 73 70 66 } Mean = 893	

$\therefore$ Mean value of I_6 = 840.
Value of I'_6 = 695.

3 TURNS.

300 volts (Metal over Ra).	47 47	53 53 44 44 } Mean = 49	Net leak = 416 = I_3 (say).
300 volts.	52 50	27 30 31 99 00 03 } Mean = 465	
10 volts (Metal over Ra).	47 47	41 42 34 36 } Mean = 38	Net leak = 340 = I'_3.
10 volts.	51 50	26 31 31 23 28 27 } Mean = 378	
300 volts (Metal over Ra).	47 47	52 51 45 42 } Mean = 48	Net leak = 396 = I_3.
300 volts.	52 50	96 07 04 83 83 83 } Mean = 445	

∴ Mean value of I_3 = 406.
Value of I'_3 = 340.

Hence, $I_6/I'_6 = 1·208$, and $I_3/I'_3 = 1·193$. A repetition of the experiment, in different order, gave $I_6/I_3 = 1·947$, and $I'_6/I'_3 = 1·897$. These agree well with each other, for we find from the first set that $I_6 I'_3/I'_6 I_3 = 1·013$, and from the second that the same fraction = $1·025$.

The fraction I_6/I'_6 is the ratio of the saturation current in a chamber about 6 mm. wide to the current when the potential gradient is about 35 volts per cm.; and I_3/I'_3 is the ratio when the chamber is 3 mm. wide, all other conditions being exactly the same. It ought, perhaps, to be mentioned that the current for the chamber of double width was not quite twice that for the other, because the widening was effected by raising the top plate, and so adding to the chamber a layer of air which was about 3 mm. further away from the radium than the original layer. As a little heap of radium bromide was used, the curve was of the form shown in Plate xviii., Philosophical Magazine, December 1904, so that ionization decreased as distance from the radium increased. These results show clearly the existence of at least one effect which we should expect to find as a result of initial recombination.

Again, we ought to find that variation in current strength, caused by altering the power of the ionizing agent, makes little difference to the form of the curve when the current is small. We have made several experiments in this direction also. In fig. 2, curves A and B show the results of experiments with currents which were of an order ten times smaller

Fig. 2.

Curves showing relation between potential gradient and current in air

Curve	A	B	C
Width of ionisation chamber in mm.	8	4	3
Pressure in cm. of mercury	78	76	50
Order of saturation-current in amperes per sq. cm. of electrode		10^{-12}	
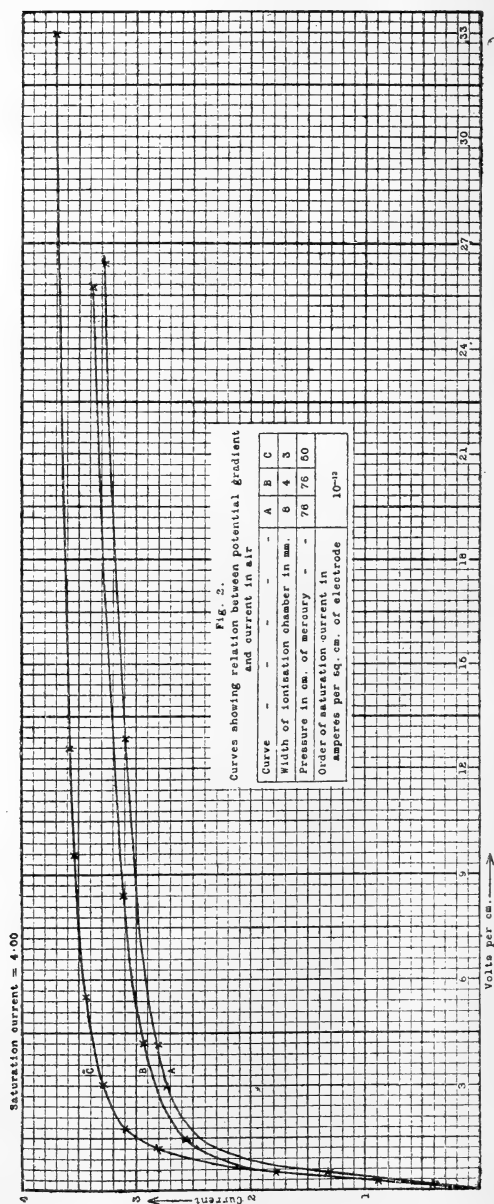
than those already described ; yet their form is very similar. The curve A shows results with an 8 mm. ionization chamber ; curve B, 4 mm. The ionization was due to a thin layer of radium, surmounted by a set of vertical tubes, as described by us in the Philosophical Magazine for September 1905. Other experimental results may be expressed in terms of I_4, the saturation current for 4 turns of the screw (rather less than 4 mm.), and I'_4 the current for a potential gradient of 25 volts per cm., and the same depth of chamber. On one occasion it was found that $I_4/I'_4 = 1\cdot17 : I_8/I'_8 = 1\cdot21$; and on another $I_4/I'_4 = 1\cdot18 : I_8/I'_8 = 1\cdot23$. In these experiments the radium was $5\cdot05$ cm. from the gauze. When the distance was $6\cdot25$ it was found that $I_4/I'_4 = 1\cdot27 ; I_8/I'_8 = 1\cdot29$; and again $I_4/I'_4 = 1\cdot30, I_8/I'_8 = 1\cdot30$. In the latter cases the α rays did not all get across the chamber : possibly the small variation of the ratios with distance may, in some way, be due to this fact.

It might be argued that we ought not to find much variation in the lack of saturation when the current is increased by shooting a greater number of α particles across the chamber in one second, on the following grounds :—Each particle as it flies across makes something like 10^5 ions in a centimetre of its path. If there are only about 10^4 or 10^5 ions in a c.cm. at any one time, it is clear that these must be all the work of one particle, and that all the ions it produces are cleared away before the next one comes. Thus, the ions made by one α particle have no chance of combining with those made by another, and recombination cannot be proportional to the square of the number per c.c. But this consideration, though no doubt true, cannot furnish an explanation of the fact that the curves are little altered when the chamber is altered in depth. It was, indeed, in view of this argument that we made the experiments with the varying depths of the chamber.

It is very instructive to compare these figures with the results obtained by Retschinsky, and described by him in a paper contained in Drude's *Annalen*, No. 8, 1905. Very careful measurements have been made by this observer of the relation between current and potential gradient in the case when the currents are of an order 100 to 1000 times greater than those of the experiments described above. Curves D, E, and F, in fig. 1, are plotted from the table on page 531 in his paper, being reduced to a saturation value 400, so as to be comparable with the other curves in the same figure. It will be seen that in this case the curves for different widths of the ionization chamber differ very widely at low potential

gradients, and this is in accordance with the present hypo-
thesis. For, when the currents are so large, the value of αn^2
is great, and the effects of general recombination must be
considerable, unless the potential gradient is much increased.
In fact, the general characteristic of these curves is that the
larger the current the higher the potential gradient must be
at the point where the effects of altering the depth of the
chamber cease to be visible.

Several observers have determined the form of the curve
connecting current and potential gradient, and have calcu-
lated therefrom the recombination coefficient. Let us now
consider the result of neglecting the effects of initial recom-
bination in these calculations.

If the currents are very great, the effects of initial recom-
bination may be small as compared with those of general
recombination. But they must always be there, and their
effect will be of greater relative importance when the current
is made smaller, either by using a weaker source of ionization
or by lessening the width of the ionization chamber. If both
effects are ascribed to one cause, whose influence is measured
by α, then α must be given a value which is fictitiously
large. The smaller the chamber, the greater the apparent
value of α must be; and this is actually the case, as found
and remarked upon by Retschinsky. For when the chamber
is very small the effects of general recombination ought to be
small; and if, as is the case, there is still a considerable lack
of saturation at moderate voltages, the whole of which is
ascribed to general recombination, the value found for α
must be very great. It is possible to find any desired value
of α in this way, if only the currents are made small enough.
This is especially true if we use the first formula employed
by Retschinsky, and ascribed by him to Riecke. In this the
determination of α depends on the difference between two
current-values taken from the upper part of the curve, where
the slope is due rather to initial than to general recombination.
To make this point clear consider the following determinations
of α :—

Retschinsky gives the following form of Riecke's equation
where the quantities are expressed in electrostatic units :—

$$\alpha = 5 \cdot 52 \times 10^{-4} \frac{(C-c)F_0^2}{c^2 l} \left(1 - \cdot 2 \times \frac{C-c}{c}\right),$$

where C = saturation current per sq.cm. of electrode,
   c = current for a potential gradient F_0,
   l = depth of chamber.

He has found by experiment that when l is 1 cm., and F_0 is 151 volts per cm., then (in amperes)

$$C = 8\cdot03 \times 10^{-8}/200 \text{ (area of electrode} = 200 \text{ sq.cm.)}$$
$$= 4\cdot01 \times 10^{-10},$$
$$c = 3\cdot94 \times 10^{-10}.$$

Therefore,

$$\frac{C-c}{c^2} = 4\cdot5 \times 10^7,$$

and by substitution in the equation it can be found that
$$\alpha/e = 4434 \text{ (}loc\ cit.\text{ p. 530).}$$

Now, in a similar experiment, with far smaller currents, we find that when l is 1 cm. and F_0 is 150 volts per cm.,

$$C = 4\cdot23 \times 10^{-13},$$
$$c = 3\cdot90 \times 10^{-13}.$$

Therefore

$$\frac{C-c}{c^2} = 2\cdot2 \times 10^{11},$$

and substitution in the equation gives a value of α/e about 5000 times greater than Retschinsky's, or about 2×10^7.

In the second formula (Stark's) the values of α are more correct, because the current values used are taken, one from the lower part of the curve, and one from the saturation values, so that their difference depends less on the effects of initial recombination. Retschinsky draws attention to these anomalies in his results, but ascribes them to absorption of ions by the electrodes. He argues that in a shallow ionization chamber this effect must be greater than in a deeper one; and so he accounts for the lack of saturation in the small chamber, a lack which is excessive if attempt is made to explain it as wholly due to general recombination. But we think that a more reasonable explanation is to be found in the hypothesis and results described in this paper, in connection with which Retschinsky's results fall naturally into place.

When, therefore, the ionizing agent is feeble, the only part of the curve which can be altered by varying the current is that where the potential gradient is small; the feebler the agent, the smaller the gradient. Let us now consider whether our hypothesis makes it probable that we can alter the shape of the rest of the curve by any variation of the conditions of the experiment.

Now, if initial recombination takes place because the

ejected electron does not get far enough away from its parent
atom before it is stopped by encountering another atom, then
diminution of pressure ought to make it much easier to
saturate.   But this is a well-known fact (Rutherford,
Philosophical Magazine, vol. xlvii. p. 160).   In order to
obtain results comparable with those we had already obtained
at ordinary pressures, we made several experiments in which
all the conditions were the same, except that the pressure
was less than that of the atmosphere.   Curve C in fig. 2
shows the results of such an experiment.   If this is compared
with the other curves in the same figure, it will be clear that
alteration has taken place in the very portion of the curve
where we should have expected it, and where change in the
strength of the current has small influence, viz., all along the
upper part of the curve up to the high potential end.   The
saturation current per sq. cm. was about 10^{-13} amp.   In
further support of our hypothesis it may be pointed out that
it gives a ready explanation of an experiment due to
Rutherford, and described by him in the Philosophical
Magazine, vol. xlvii. p. 158 *.   He found that the satu-
ration value of the current through a gas could be obtained for
a much lower potential gradient when the gas was drawn
away from the uranium which ionized it, and treated in a
separate vessel.   This is to be expected when it is considered
that under the circumstances of the experiment initial
recombination was wholly absent.

It is now convenient to consider these phenomena as they
are manifested in other gases than air.   It is well known
that the relations between current and potential in carbon
dioxide are in some way abnormal.   But the peculiarities of
this gas are even intensified in ethyl chloride (C_2H_5Cl).   The
fact is that this effect, which makes it difficult to draw all the
ions to the electrodes in the case of air, is far greater in more
complex gases, and thus it is extremely difficult to obtain the
saturation current unless very high potentials are employed.
We find it necessary to use a potential gradient of two to
three thousand volts per cm. in the case of ethyl chloride at
60 cm. pressure.   In the investigations which were made by
us (Philosophical Magazine, September 1905), with regard
to the ionization curves in different gases, we found the
currents to be unexpectedly small in the case of some gases.
We suggested that possibly some of the ions made by the α
particles did not get away from their parent atoms.   We
proposed to make a special investigation of the point, and it

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* This statement may need amendment.   The matter is discussed in a
later paper.

was with this purpose that the work described in this paper was undertaken.

It now appears that our suggestion was justified, but it is also clear that we should have obtained larger currents if we had used a higher potential gradient ; 500 volts per cm. was insufficient.

Consider the curves in fig. 3. In A is shown the relation between current and potential gradient up to 3000 volts per cm. for ethyl chloride at 56 cm. pressure, the saturation current per sq.cm. being about 3×10^{-13}. B shows the same relation in the case of air at atmospheric pressure, the saturation current being rather smaller. Comparison of these two shows how much more difficult it is to obtain the full current in the more complex gas. Again, C shows the results of experiments in which the depth of the ionization chamber was varied. The crosses refer to a 2 mm. chamber, the dots in circles to a 4 mm. chamber. The currents were of the 10^{-13} order. The two sets of observations lie on practically the same curve. This shows that general recombination is not responsible for the lack of saturation, and that the cause is probably similar to that whose effects in the case of air have been described above. Curves D and E refer to experiments in which the chamber was maintained at the same depth, 2 mm., but the currents were altered by varying the distance of the radium. In the former curve the saturation current is about 10^{-13}, in the latter six times as much. In the case of the results shown in C, D, and E, the gas contained a certain proportion of air.

These results all go to show that the form of the curve for ethyl chloride is almost independent, as in the case of air, of strength of current and depth of ionization chamber, when the ionization is small. But also, as in the case of air, it depends greatly on the density of the gas. F represents the results of experiments at a pressure of 36 cm., and is to be compared with A. All the conditions, except as regards pressure, were the same for the two curves.

We have also carried out experiments, similar to some of those just described, for a mixture of carbon tetrachloride and air, and obtained similar results. Although there was only 5 per cent. (by pressure) of the denser gas in the mixture, yet the current at a potential gradient of 330 volts per cm. was only 82 per cent. of the saturation value, whils in air under similar conditions it was 93 per cent.

It is hardly surprising that initial recombination should be more effective in a complex gas than in air. For the molecule contains many atoms, each one of which is just as

Fig. 3.

Curves showing relation between potential and ionisation current in ethyl chloride and air.

A. Ethyl chloride: Width of chamber, 3 mm., pressure, 56 cm.; saturation current = 3 x 10⁻¹⁰ amp.

B. Air: Width of chamber, 10 mm.; pressure, 76 cm.; saturation current = 2 x 10⁻¹⁰

C. Ethyl chloride and air: Pressure, 76 cm.; results for 2 mm. chamber denoted by crosses, for 4 mm. chamber by circles.

D and E. Ethyl chloride and air: Saturation current in E = six times saturation current in D

F. Ethyl chloride: Width of chamber, 3 mm., pressure, 36 cm.; saturation current = 2 x 10⁻¹⁰ amp.

likely to lose an electron as if it were not associated with other atoms. Perhaps, therefore, the molecule as a whole loses two or three electrons, and its electric field is the more intense. Recombination of this kind must also be easier, the shorter the free path.

It will be clear from the foregoing that certain effects occur which are characteristic of a process of initial recombination, a process which is *a priori* not improbable. The question now arises as to whether any other cause could produce the same effects.

When we consider the great increase of current in a complex gas which is caused by an increase in the electric force applied, we cannot but ask whether any of it is due to the production of other ions by those actually made by the α particle. Could the electric force aid the freed electron to gather speed sufficient to ionize by collision? A process of this kind would be similar in its results to initial recombination, in that it would be independent of strength of current and depth of ionization chamber. It seems probable, however, that its effectiveness would rather be increased than diminished by lowering the pressure ; and also it would be difficult to account for the existence of a saturation value of the current. Nevertheless, it does not seem safe as yet to say that no such process occurs. Probably further light could be thrown on the subject by an investigation into the total number of ions produced in different gases under varying conditions. Some initial experiments of this kind will be described presently.

Rutherford has recently shown that the α particle of Ra C has only lost 40 per cent. of its velocity when it ceases to ionize. If this fact is considered in conjunction with our investigations into the form of the ionization curves for gaseous mixtures, it is at once clear that the α particle stops ionizing in every gas when its speed has fallen to precisely the same value. For, if not, the ionization curve for a mixture would show a superposition of simple curves, of which effect there is no trace. This and other considerations seem to show, as we have already said (Philosophical Magazine, September 1905), that the α particle performs the same number of acts of ionization in every gas. If, then, we find the total saturation current to be different in different gases, we must come to the conclusion that either the ions in the gases of higher conductivity produce others by the help of the electric field, or that in the gases of lower conductivity some of the ions made by the α particle do not get free, even under conditions of saturation, from their parent atoms, or

that both these effects take place. With the object of helping to a decision on this point we have begun a set of experiments, of which those now described are the first examples.

The method used is to measure the co-ordinates of some standard point on the ionization curve of the gas investigated, under different pressures. The point chosen is that where the side of the Ra C curve is struck by the top of the curve which belongs to the α particles of next velocity to those of Ra C. This point in air, at 760 mm. and 20° C., is at a height of nearly 4·83 cm. It is a convenient point to choose, for the following reasons:—Being on a part of the Ra C curve where no great change in the ionization takes place for a considerable alteration in range, the measurements there are usually pretty consonant with each other, even though they are taken quickly, and if several be taken on the Ra C curve they check each other. The ordinate of this point can also be determined with great precision by measuring two or three points along the top slope of the curve of Ra A (or emanation, whichever it finally proves to be). Thus, a few readings can be quickly taken in succession which determine the point accurately, and very little leakage of air into the apparatus takes place while the experiments go on. This is a desirable thing, because our apparatus leaks slowly when the pressure within is much reduced, on account of the large number of connecting tubes and mechanical arrangements. We find that this method is very satisfactory. We may mention also that to save time it is not well, in the case of gases like ethyl chloride, which are at first in the liquid form, to admit any of the liquid into the apparatus, as it takes so long to evaporate completely. It is better to let the liquid evaporate in another chamber, which can be quite small, and then to take over gas only.

Fig. 4 shows the results of some experiments with C_2H_5Cl. The curves shown are portions of the ionization curves in this gas at different pressures. In all cases the apparatus was exhausted of air to about 10 mm. pressure, then partly filled with gas, re-exhausted, and filled again to the desired pressure. The observations were made at once, those in the neighbourhood of the standard point being made first, so that the gas might be as pure as possible whilst the important readings were being taken. A potential of 900 volts was used for the three greater pressures, and of 300 for the low pressure. The chamber was 3 mm. wide, and therefore these potential gradients were, respectively, 3000 and 1000 volts per cm.

The results for ethyl chloride and for air are contained

Fig. 4.

Ionisation curves of ethyl chloride at different pressures.

Position of standard point	A	E	C	D
Pressure in cm. of mercury	22·4	32·5	41·0	53·8
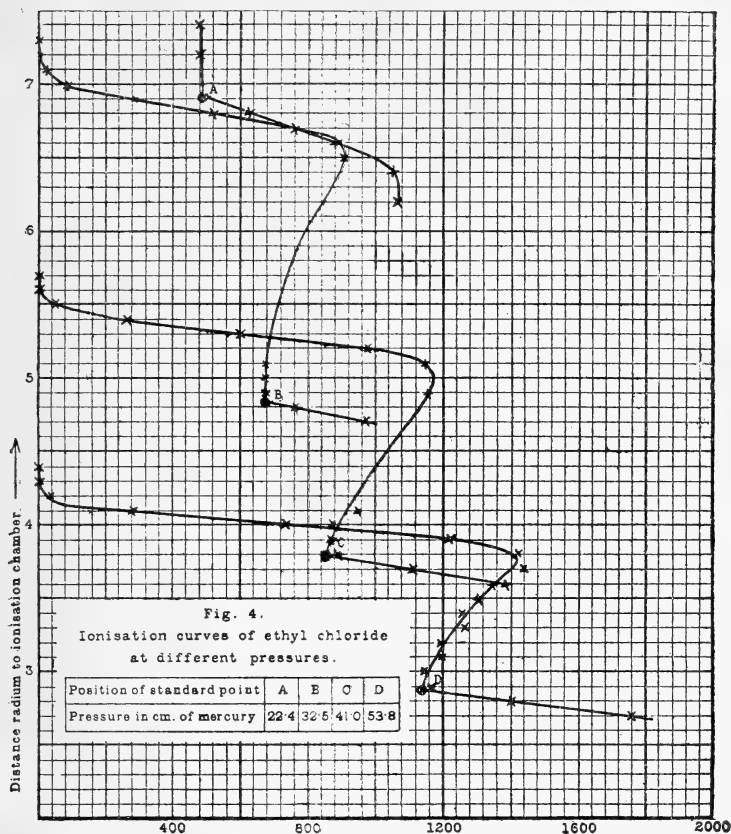
in the following table, where P denotes the pressure, R the range, and I the ionization on an arbitrary scale :—

Ethyl Chloride.

P.	R.	I.	R × I.	R × P.
53·8	2·87	1140	3260	154
41·0	3·78	860	3260	155
32·5	4·83	666	3220	157
22·4	6·92	476	3300	155

Air.

75·3	4·72	546	2570	355
57·2	6·08	432	2620	352
46·9	7·42	340	2520	348
38·8	9·00	283	2540	349

These results relate to two gases only ; but so far as they go they show :—(i.) That the range varies inversely as the pressure, which result might have been anticipated ; (ii.) that the total number of ions set free in a gas is independent of the pressure, but is different in different gases. The total ionization is greater in ethyl chloride than in air. This is a contrary result to that which we obtained during our experiments on absorption. We were unaware at that time of the enormous force required to saturate the complex gas.

Finally, the following experiments may be briefly described:—

We have tried the effect of reversing the field on the relation between current and potential, and found a result which was practically negative. We have found a similar result when the α particles were not shot straight across the ionization chamber in the direction of the lines of force, but in a slanting direction. These experiments were made in the endeavour to find whether there was any relation between the direction in which electrons were projected and the direction of the applied field. We have also tried to alter the range in air by using different potential gradients, with the idea that it might be possible to obtain ions from an atom traversed by a slower α particle, if only enough electric force were applied. But the result was the same, no matter whether the force was 20 volts to the cm. or 2000 ; and a variation of ·2 mm. could hardly have escaped detection.

During the progress of this work, one of us (R. D. Kleeman) left Australia for England. We wish to acknowledge with gratitude the assistance of Mr. H. J. Priest, B.Sc., in completing the observations.

XLII. *On the Electrical Conductivity of Flames containing Salt Vapours for Rapidly Alternating Currents. By* H. A. WILSON, *M.A., D.Sc., M.Sc., Professor of Physics, King's College, London, Fellow of Trinity College, Cambridge, and* E. GOLD, *B.A., Hutchinson Student, St. John's College, Cambridge* [*].

THE following paper contains an account of a series of experiments on the electrical conductivity of a Bunsen flame containing various alkali salt vapours, the currents used being alternating ones with frequencies varying from $7·14 \times 10^4$ to $6·2 \times 10^6$ per second.

The conductivity was measured between two platinum electrodes immersed in the flame, and the variation of the

conductivity with the amount of salt present in the flame and with the nature of the salt was investigated. The variation of the conductivity with the frequency of alternation, the maximum electromotive force, and the distance between the electrodes was also examined. The results obtained enable a comparison to be made between the conductivities of the various alkali salt vapours for alternating currents and their conductivities for steady currents as previously determined [*].

It appears that the relative conductivities for rapidly alternating currents are nearly proportional to the square roots of the corresponding conductivities for steady currents.

The flame is found to behave for very rapidly alternating currents more like a dielectric of high specific inductive capacity than like a conducting medium ; and it is shown that this result is in accordance with the ionic theory.

The rest of the paper is divided into the following sections :—

(1) Description of apparatus used.
(2) Variation of the conductivity with the concentration and nature of the salt vapour.
(3) Variation of the conductivity with the maximum P.D., the frequency, and the distance between the electrodes.
(4) Theory of the conductivity for rapidly alternating currents.
(5) Summary of results.

(1) *Description of Apparatus used.*

To produce a steady flame containing a definite amount of salt vapour, an apparatus similar to those described in the two papers referred to above was used. The principal parts of the apparatus are shown in fig. 1.

A mixture of coal-gas and air containing spray of a salt solution was burnt from a glass tube TT tipped with a short thin copper tube 1 cm. in diameter.

The mixture was formed in a glass bulb B, from which it passed through a wide tube AA to the burner. In the bulb and AA the coarser spray settled, and was allowed to escape through the tubes D and D'. The spray was produced by a

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[*] " The Electrical Conductivity and Luminosity of Flames containing Vaporised Salts," by A. Smithells, H. M. Dawson, and H. A. Wilson, Phil. Trans. A 193. 1899. " On the Electrical Conductivity of Flames containing Salt Vapours," by H. A. Wilson, Phil. Trans. A 192. 1899.

Gouy sprayer K, worked by compressed air, which was supplied by a rotary blower worked by a ½-H.P. motor. The compressed air entered at H, and the salt solution at N.

Fig. 1.
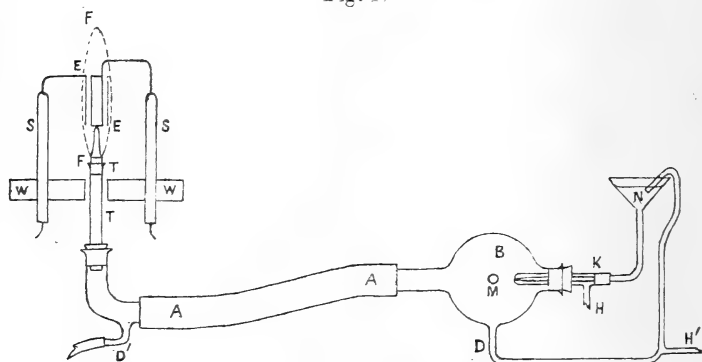
FF. Flame.	B. Sprayer-bulb.
EE. Electrodes.	K. Sprayer.
SS. Glass tubes.	N. Solution-tube and funnel.
WW. Wood block.	HH'. Compressed-air tubes.
TT. Flame-tube.	M. Gas tube.
AA. Wide indiarubber tube.	DD'. Solution-overflow tubes.

The pressure of the air-supply was indicated by a water-manometer, and was always kept at the same value, about 80 cms. of water. The salt solution was contained in a large funnel, and its level always kept a constant height above the sprayer-nozzle. The amount of gas entering the bulb was measured by a water-meter, and its pressure kept constant by means of a regulator; it did not vary appreciably.

The gas and air supplies were so adjusted that a "non-luminous" flame having a sharply-defined inner cone was obtained. This flame was very steady in appearance, and could be maintained constant for any length of time required.

The electrodes used in most of the experiments consisted of two concentric cylinders made of thin platinum, of the following dimensions :—

Diameter of outside cylinder . . 2·4 cms.
Height   ,,   ,,   . . 5·0 ,,
Diameter of inside cylinder . . 1·2 ..
Height   ,,   ,,   . . 5·2 ,,

They were supported symmetrically about the axis of the flame at such a height that the inner cone of the flame just reached up to the level of the lower ends of the cylinders.

The conductivity between the electrodes was determined by means of a Wheatstone-bridge arrangement, of which the electrodes formed one arm, and the other three arms consisted of small air-condensers, the capacity of one of which was adjustable. The arrangement is shown in fig. 2.

Fig. 2.
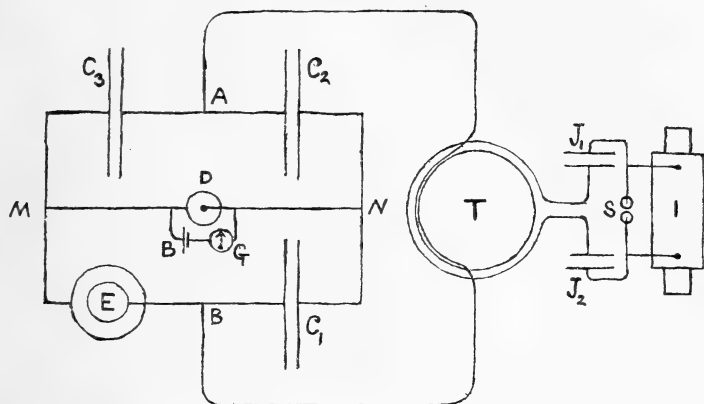
E. Flame-Electrodes.
$C_1 C_2$. Air-condensers.
C_3. Adjustable Air-condenser.
D. Detector.
B. Cell.
G. Galvanometer.
T. Tesla Coil.
$J_1 J_2$. Leyden jars.
S. Spark-gap.
I. Induction-coil.

An induction-coil, I, charged up two Leyden-jars, $J_1 J_2$, and these discharged at a spark-gap S. The outside coatings of the jars were connected through the primary of a Tesla coil, T. The primary of this coil consisted of 33 turns wound into a spiral, 29 cms. long and 19·2 cms. in diameter, on a large glass cylinder. The secondary coil had three turns, and was placed inside the glass cylinder, halfway up it. It was connected to the bridge arrangement at A and B, as shown.

An "electrolytic detector," D, was connected to the points M and N of the bridge. This detector consisted of two platinum electrodes dipping into 20 per cent. sulphuric acid. One electrode was a platinum cylinder 3 cms. in diameter and 4 cms. high, while the other was a platinum wire $\frac{1}{1000}$ inch in diameter sealed into a glass tube and cut off close to the surface of the glass. The large electrode was contained in a test-tube, which it just fitted, and the glass tube carrying the small electrode was supported by a cork fitting the test-tube. The small electrode was just below the surface of the acid at the middle of the tube. The two electrodes were connected to a silver-chloride cell B and a

moving-coil galvanometer G. The cell B gave about one
volt and served to polarize the electrodes. When an alter-
nating P.D. was produced between M and N, the detector was
depolarized and a current passed through the galvanometer.
The deflexion of the galvanometer-coil was read by means of
an incandescent lamp and a scale, a current of 10^{-9} ampere
giving a deflexion of one scale-division.

The condensers C_1, C_2 each consisted of two parallel
circular disks 10 cms. in diameter, supported on ebonite
rods. The distance between the disks of C_1 was 0·15 cm.
and of C_2 0·75 cm., in most of the experiments. The con-
denser C_3 consisted of two brass disks 10 cms. in diameter,
whose distance apart could be adjusted and measured by
means of an accurate micrometer-screw.

To determine the conductivity between the flame-electrodes,
the rapidly alternating P.D. produced by the Tesla coil was
applied at A and B (fig. 2), and the condenser C_3 was ad-
justed until the deflexion of the galvanometer was a minimum.
On starting the alternating current, the galvanometer de-
flexion increased to a maximum value and then fell off
slowly. The alternating current was always kept on for
15 secs. and the maximum deflexion noted, and then after an
interval of 15 secs. the current was turned on again and so
on. During the intervals the condenser was adjusted, and
the deflexions corresponding to a series of positions of the
adjustable condenser-disk were thus obtained. A curve was
then drawn on squared paper showing the relation between
the galvanometer deflexion and the condenser-screw reading,
and so the screw reading for which the deflexion was a
minimum was obtained. The observations at each position
of the condenser-disk were repeated several times and the
mean taken ; the series of observations was also repeated
first in one direction and then in the other, while various
intermediate positions were also tried so as to make as certain
as possible that the correct relation between the deflexion
and the condenser distance was obtained. These precautions
were very necessary because the apparatus was sometimes
irregular in its action. No great difficulty was experienced
in keeping the flame sufficiently constant during the experi-
ments, but it was not easy to keep the alternating current
constant. This was due partly to variations taking place at
the spark-gap and partly to irregularity in the action of the
induction-coil interrupter. The spark-gap finally adopted
consisted of two platinum spheres kept in an atmosphere of
hydrogen. In most of the experiments the length of the
gap was about 0·2 cm.

Several different kinds of interrupters were tried, but finally the ordinary App's platinum contact-breaker was used. The platinum contacts were always carefully filed smooth before starting an experiment and a 10-inch coil was used with a four-volt battery, the contact-breaker being adjusted so that the coil gave only a short spark when not connected to the Leyden jars. In this way the apparatus was made to work sufficiently steadily to obtain fairly satisfactory observations. The distance between the condenser-plates corresponding to the minimum galvanometer-deflexion could be obtained within about 5 per cent. of its value. It was only after a long series of attempts extending over nearly a year that the apparatus was got to work well enough to obtain reliable results, and numerous modifications were tried before the form above described was finally adopted.

To compare the conductivities due to different salts, the increase in the apparent capacity of the flame-electrodes consequent on introducing each salt was calculated in terms of the capacities of the three condensers.

With a bridge arrangement each arm of which is a capacity with negligible self-induction, the condition for a balance is $C_1 C_3 = C_2 C_4$, a condition independent of the frequency. It was found that when the ratio $\dfrac{C_2}{C_1}$ was altered, then C_3 changed approximately proportionally, so that it appeared justifiable to apply the equation $C_1 C_3 = C_2 C_4$. Let d_1 be the distance between the plates of the adjustable condenser at the minimum for the flame free from salt, and d_2 for the flame containing salt. The capacity of the adjustable condenser in the first case is $\dfrac{A}{4\pi d_1} + D$, where D is a quantity nearly independent of d_1 and A the area of each condenser-plate [Clerk Maxwell, 'Electricity and Magnetism,' art. 202].

We have therefore

$$C_1 \left[\frac{A}{4\pi d_1} + D \right] = C_2 C_4,$$

where C_4 is the apparent capacity of the flame-electrodes with the flame free from salt. Also if C_4' is the apparent capacity with salt in the flame we have

$$C_1 \left(\frac{A}{4\pi d_2} + D \right) = C_2 C_4'.$$

Hence
$$\frac{C_1 A}{4\pi}\left(\frac{1}{d_2} - \frac{1}{d_1}\right) = C_2 \left(C_4' - C_4\right) ;$$

so that the change, in the apparent capacity, due to the introduction of the salt is given by the equation

$$C_4' - C_4 = \frac{C_1}{C_2}\frac{A}{4\pi}\left(\frac{1}{d_2} - \frac{1}{d_1}\right).$$

In the experiments we have made, the self-inductions of the arms of the bridge could be neglected in comparison with the capacities without appreciable error.

It was found possible with the bridge arrangement de-scribed to obtain an approximate balance. That is, the minimum galvanometer-current was always small compared with the currents when the adjustable condenser-plate was far from the position which gave the minimum deflexion. This showed that the arm of the bridge containing the flame behaved like a capacity simply, or like a capacity and self-induction in series. If the flame had behaved like a capacity and resistance in parallel, then a balance could not have been obtained. The current through the flame with a given maximum P.D. and frequency is proportional to the apparent capacity, so that it is reasonable to regard the apparent capacity as a measure of the conductivity of the flame for the rapidly alternating currents employed.

To obtain relative values of the apparent capacities, it is therefore sufficient to calculate $\dfrac{1}{d_2} - \dfrac{1}{d_1}$ for each salt solution sprayed. Since the absolute amount of salt in the flame could be only roughly estimated, it was useless to attempt to obtain exact values for the absolute apparent capacities; so that it was unnecessary to know the ratio $\dfrac{C_1}{C_2}$ exactly.

The value of d_1, the minimum position for the flame free from salt, was about half the value obtained with no flame. Consequently $\dfrac{1}{d_1}$ corresponds to twice the capacity of the flame-electrodes with air as dielectric. The capacity in this case was 3·6 cms., and $\dfrac{C_1}{C_2}$ was very nearly $\frac{1}{5}$ in most of the experiments described below.

(2) *Variations of the apparent Capacity with the Concentration and Nature of the Salt Vapour.*

The salt solutions sprayed were made up with distilled water and pure salts carefully dried. We shall first give a few examples of the curves obtained, showing the relation between the distance apart of the condenser-plates and the galvanometer deflexion, from which curves the position of the minima were deduced. Fig. 3 shows several such curves. The sharpness of the minimum was usually greater

Fig. 3.

Distance between plates of Adjustable Condenser.
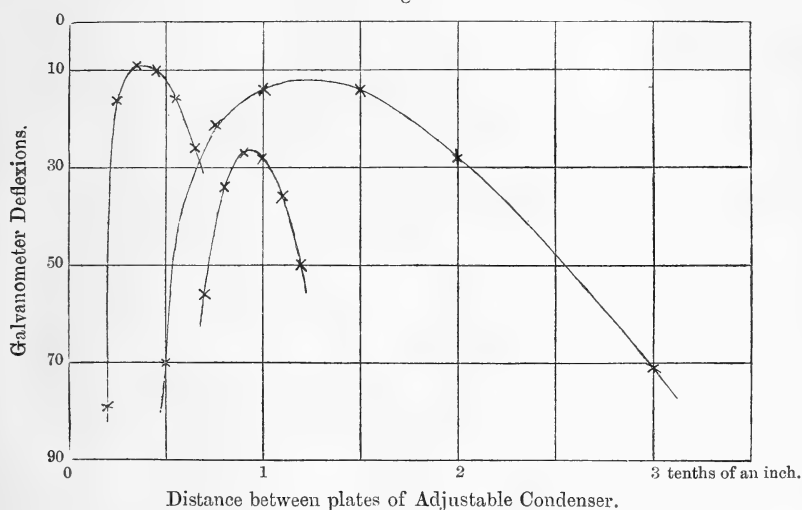
when the conductivity was large and the distance therefore small, than when the converse was the case. The accuracy with which the minimum could be found was consequently nearly the same for large as for small conductivities.

During the course of the experiments the ratio of the two condensers, C_1, C_2, was changed on one or two occasions. When this had been done, some of the observations made before the change were repeated, and the factors required to reduce all the results to the same standard so determined. The following table contains the results obtained, the number of alternations per second being $3\cdot2 \times 10^5$, and $d_i = 3\cdot33$ tenths of an inch.

2 K 2

Salt.	Grams per litre.	d_2 (in tenths of an inch).	$\dfrac{1}{d_2} - \dfrac{1}{d_1}$.
CsCl	50	0·08	12·2
,, ...	10	0·18	5·26
,, ...	1	0·34	2·64
,, ...	0·333	0·48	1·78
,, ...	0·1	1·10	0·61
Cs_2CO_3	25·5	0·13	7·40
,, ...	2·55	0·30	3·03
,, ...	0·26	0·59	1·40
,, ...	0·026	1·90	0·23
RbCl	48·1	0·22	4·24
,, ...	9·6	0·29	3·15
,, ...	1·92	0·41	2·14
Rb_2CO_3	50·4	0·19	4·96
,, ...	10	0·25	3·70
,, ...	1	0·50	1·70
K_2CO_3	100	0·062	15·8
,, ...	10	0·297	3·07
KCl	100	0·20	4·70
,, ...	20·8	0·30	3·03
,, ...	1	0·50	1·70
NaCl	100	0·47	1·83
LiCl	102·5	1·3	0·47

Fig. 4.
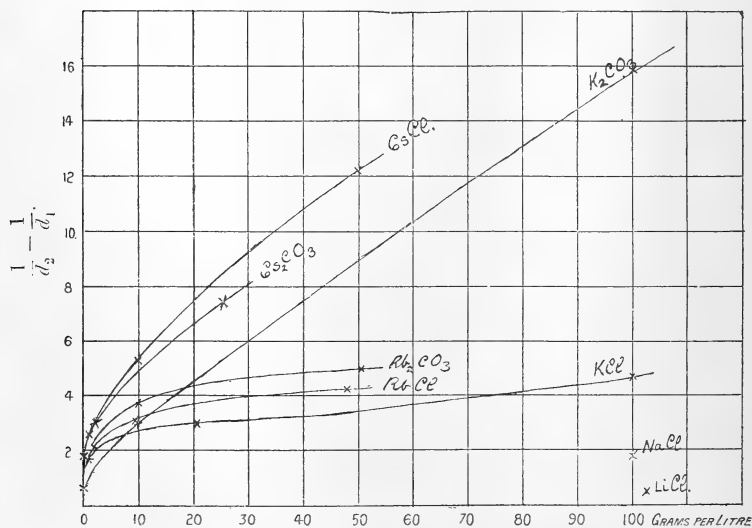
In fig. 4 the above values of $\frac{1}{d_2} - \frac{1}{d_1}$, and the corresponding strengths of the solutions, are shown graphically, for each salt. Fig. 5 shows the steady currents due to an E.M.F. of 0·227 volt, taken from the papers referred to above.

Fig. 5.
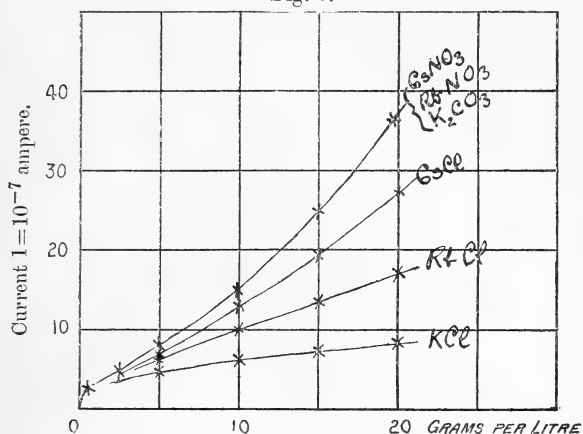
The amount of salt entering the flame in the present experiments was nearly the same as in the older experiments.

The following table gives the values of $\frac{1}{d_2} - \frac{1}{d_1}$ for decinormal solutions obtained from the curves in fig. 4, and also the steady currents due to 0·227 volt.

Salt.	$\frac{1}{d_2} - \frac{1}{d_1}$ (k)	Salt.	Steady Current $(1=10^{-7}$ ampere). (k')	Ratios.	$\frac{\sqrt{k'}}{k}$.
CsCl.........	6·7	CsCl.........	22·2	3·3	0·70
$\frac{1}{2}$Cs$_2$CO$_3$ ...	5·9	CsNO$_3$......	36·6	6·2	1·03
$\frac{1}{2}$Rb$_2$CO$_3$ ...	3·7	RbNO$_3$......	25·9	7	1·37
RbCl	3·2	RbCl	11·3	3·5	1·05
$\frac{1}{2}$K$_2$CO$_3$......	2·9	$\frac{1}{2}$K$_2$CO$_3$	11·2	3·9	1·15
KCl	2·6	KCl	5·75	2·2	0·92

The fifth column contains the ratios of the numbers expressing the conductivities for steady currents to the values of $\frac{1}{d_2} - \frac{1}{d_1}$. In the previous work the conductivities of cæsium and rubidium carbonates were not measured; so the values for nitrates are given, since the conductivities of all oxysalts of the same metal were found to be nearly equal for

steady currents.   It will be seen that, roughly speaking, the relative conductivity for steady currents varies in a similar way to the conductivity for rapidly alternating currents as represented by $\dfrac{1}{d_2} - \dfrac{1}{d_1}$.

The last column contains the square root of the conductivity for steady currents divided by $\dfrac{1}{d_2} - \dfrac{1}{d_1}$.   The numbers in this column do not vary much, showing that the conductivity for rapidly alternating currents varies, roughly speaking, as the square root of the conductivity for steady currents.

The conductivities of KCl and RbCl were found to vary nearly as the square root of the concentration in the case of steady currents ; so that we should expect them to vary as the fourth root of the concentration for rapidly alternating currents.   The following table shows that this is nearly the case.

Salt.	Grams per litre. (C)	$\dfrac{1}{d_2} - \dfrac{1}{d_1}$. (k)	$k/\sqrt[4]{C}$.
RbCl.........	48·1	4·24	1·6
,, ...	9·6	3·15	1·79
.. ..	1·92	2·14	1·81
KCl	100	4·70	1·49
,, ...	20·8	3·03	1·42
,, ...	1·0	1·70	1·70

Considering the large variation in C the values of $k/\sqrt[4]{C}$ are surprisingly constant for these two salts.

The following table shows the relative variation of the conductivity CsCl for steady and alternating currents.   The numbers are taken from figs. 4 and 5.

	Grams per litre.	$\dfrac{1}{d_2} - \dfrac{1}{d_1}$. (k)	Steady Current. (k')	$\sqrt{k'}/k$.
CsCl	20	7·3	26	0·70
,, ...	15	6·4	19·5	0·69
,, ...	10	5·3	13	0·68
,, ...	5	4·1	7·4	0·66
,, ...	1	2·6	3·0	0·67

It is clear from these results that $\dfrac{1}{d_2} - \dfrac{1}{d_1}$ varies nearly as the square root of the steady current due to a small P.D.

(3) *The Variation of the Conductivity with the Potential-Difference and Number of Alternations per Second.*

The variation of the conductivity of the flame as measured by $\dfrac{1}{d_2} - \dfrac{1}{d_1}$, with the maximum P.D. applied to the bridge arrangement, was effected by varying the length of the spark-gap in the primary circuit of the Tesla coil. The spark passed between two platinum spheres in air at the ordinary pressure. The following table gives the results obtained when spraying a solution of CsCl containing one gram per litre :—

Spark-length.	P.D. (E.S. Units at gap.)	d_2.	$\dfrac{1}{d_2} - \dfrac{1}{d_1}$. (k).	$k\sqrt{\overline{\mathrm{P.D.}}}$
0·0055 cms.	2·60	0·23	4·05	6·55
0·011 ,,	3·6	0·42	2·08	3·95
0·017 ,,	4·7	0·43	2·03	4·4
0·028 ,,	6·4	0·48	1·78	4·5
0·044 ,,	8·8	0·50	1·70	5·0
0·10 ,,	16	0·53	1·59	6·3
0·20 ,,	27·8	0·60	1·37	7·2

The last column contains the products of $\dfrac{1}{d_2} - \dfrac{1}{d_1}$ and the square root of the corresponding potential-difference as estimated from the spark-length in the primary circuit of the Tesla coil. The numbers in this column vary between 4 and 7, while the P.D. varies by a factor of 11. Taking into account the roughness of the method used to estimate the P.D. and the large change made in it, we may conclude that $\dfrac{1}{d_2} - \dfrac{1}{d_1}$ probably varies approximately inversely as the square root of the maximum P.D. applied.

The effect of varying the number of alternations per second was tried by altering the capacity in the primary circuit of the Tesla coil. A solution containing one gram of CsCl per litre was sprayed. The following table gives the results obtained :—

Capacity.	Self-Induction.	Alternations per second.	d_2.	$\frac{1}{d_2} - \frac{1}{d_1}$.
33000 cms.	136500 cms.	$7\cdot14\times10^4$	0·4	2·1
6400 „	..	$1\cdot62\times10^5$	0·4	2·1
1600 „	„	$3\cdot24\times10^5$	0·34	2·6
150 „	„	$10\cdot8 \times10^5$	0·30	3·0
150 „	4100 cms.	$6\cdot2 \times10^6$	0·30	3·0

It will be seen that the value of $\frac{1}{d_2} - \frac{1}{d_1}$ varies as the number of alternations per second is changed. Unfortunately it was not found possible to obtain the variation of d_2 with the frequency very exactly; and all that can be said is that changing the frequency from $7\cdot14\times10^4$ to $6\cdot2\times10^6$ per second does not change $\frac{1}{d_2}-\frac{1}{d_1}$ by more than 25 per cent. of its mean value.

If we suppose that $\frac{1}{d_2}-\frac{1}{d_1}$ varies as n^x, where n is the frequency, then the results just given show that x lies between $+0\cdot05$ and $-0\cdot05$.

For a pure capacity $x=0$, and for a pure self-induction $x=-1$; so that it appears that the flame behaves nearly like a pure capacity. That it does not include much conductivity in parallel with the capacity is shown by the fact that it could be nearly balanced by three condensers. In the section below, on the theory of conductivity, it is shown that these conclusions from the experiments might have been anticipated.

To obtain the apparent specific inductive capacity of a salt vapour, it is only necessary to multiply the numbers for $\frac{1}{d_2}-\frac{1}{d_1}$ given above by $2d_1=6\cdot66$ and add on unity. The apparent specific inductive capacities so obtained vary from about 100 for the strongest solution of K_2CO_3 sprayed to about 4 for the solution of LiCl *.

Some experiments were made to find out how the conductivity varied with the distance between the electrodes.

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* These values of the apparent specific inductive capacity have of course no relation to the true specific inductive capacity, which must be nearly unity.

Two parallel vertical platinum disks each 1·5 cms. in diameter were used, and it was found that $\frac{1}{d_2} - \frac{1}{d_1}$ was independent of the distance between them when this was less than 3 or 4 millimetres.

When they were 10 millimetres apart $\frac{1}{d_2} - \frac{1}{d_1}$ was about double its value between 0 and 4 mms. This increase in the apparent capacity is no doubt due to the fact that the cross-section of flame acted on by the P.D. is greater when the distance between the electrodes is comparable with their diameter.

We may therefore conclude that $\frac{1}{d_2} - \frac{1}{d_1}$ would be independent of the distance between the electrodes if they were very large. This is in agreement with the results for constant P.D.'s, for which the current is independent of the distance between the electrodes when they are near together.

(4) *Theory of the Conductivity of Ionized Gases for rapidly alternating Currents.*

It will be convenient now to describe an approximate theory of the conductivity of ionized gases for rapidly alternating currents—a theory which affords an explanation of the experimental results obtained in this investigation.

Suppose a large parallel plate-condenser filled with a uniformly ionized gas, and let the distance between the plates be D cms. Let the potential-difference between the plates be given by the formula $V = V_0 \sin pt$, and let the number of positive or negative ions per c.c. be n, each ion carrying a charge $\pm e$. In a Bunsen flame the velocity of the negative ions is about 1000 $\frac{\text{cms.}}{\text{sec.}}$ for one volt per cm., while the velocity of the positive ions under the same potential gradient is only about 60 $\frac{\text{cms.}}{\text{sec.}}$ Moreover, the mass of the positive ions is probably very large compared with that of the negative ions. Consequently, in a rapidly alternating electric field the amplitude of vibration of the negative ions will be large compared with that of the positive ions. As a first approximation, therefore, we shall assume that the positive ions do not move at all, so that all the current is carried by the negative ions. We shall also suppose that all the negative ions move in the same way with the same velocity, so that the number of negative per c.c.

remains n except within a distance d of each electrode, d being twice the amplitude of vibration of the negative ions. It is easy to see that on these assumptions the negative ions will practically all be contained in a slab of thickness $D-d$, which will vibrate between the plates so as just not to touch either of them. For if a new negative ion is formed outside the slab, it will almost immediately strike the electrode near it ; whereas a new negative ion formed in the slab cannot reach either electrode except by diffusion, which we shall neglect. Thus we may regard the whole space between the plates as filled with positive electricity of density $+ne$, and the vibrating slab of thickness $D-d$ as containing also negative electricity of density $-ne$. Thus inside the slab the total density is zero and outside is $+ne$.

Let X denote the electric intensity between the plates at a distance x from one of them. Then, inside the slab $\frac{dX}{dx}=0$, and outside $\frac{dX}{dx}=4\pi ne$.

Let A and B be the two plates, and let the slab be represented by the space between the two dotted lines E, F. Let $AB=D$, $AE=t_1$, $FB=t_2$, and suppose the potential of A kept zero while that of $B=V$. Let the rise of potential in AE be V_1, in EF be V_3, and in FB be V_2. In EF $\frac{dX}{dx}=0$ so that X is constant. Let its value be X_0. Then
$$V_3 = -X_0(D-d),$$
where $d=t_1+t_2$. In AE we have
$$\frac{d^2V}{dx^2} = -4\pi\rho,$$

where $\rho=ne$. From this we get
$$\frac{dV}{dx} = -4\pi\rho x + C \quad \text{and} \quad V = -2\pi\rho x^2 + Cx + D,$$

where C and D are constants to be determined. When $x=0$
$$V = D = 0,$$
and when $x=t_1$
$$\frac{dV}{dx} = -4\pi\rho t_1 + C = X_0.$$

Hence
$$V_1 = 2\pi\rho t_1^2 - X_0 t .$$
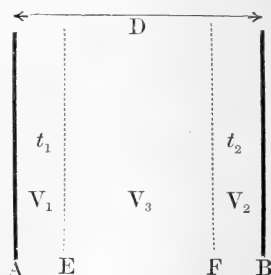
In the same way in FB we have

$$V = -2\pi\rho x^2 + C'x + D'.$$

When $x = D - t_2$

$$\frac{dV}{dx} = -X_0 \quad \text{and} \quad V = V_1 + V_3 \, ;$$

so that we get for V at $x = D$,

$$V = -X_0 D + 2\pi\rho(t_1^2 - t_2^2).$$

Now $t_1 + t_2 = d$, so that

$$V = -X_0 D + 2\pi\rho d(2t_1 - d). \quad \cdots \quad (1)$$

The force acting on a negative ion is $-X_0 e - A\dfrac{dt_1}{dt}$, where A is a constant representing the viscous resistance to motion with unit velocity. Let m be the mass of a negative ion; then its equation of motion is

$$-X_0 e = m\frac{d^2 t_1}{dt^2} + A\frac{dt_1}{dt}. \quad \cdots \quad (2)$$

The current-density inside the slab is given by the equation

$$i = -\rho\frac{dt_1}{dt} + \frac{K}{4\pi}\frac{dX_0}{dt},$$

where K is the specific inductive capacity of the medium between the plates in the absence of ions. Thus K is unity, and

$$i = -\rho\frac{dt_1}{dt} + \frac{1}{4\pi}\frac{dX_0}{dt}. \quad \cdots \quad (3)$$

Now in a flame containing a salt vapour the fall of potential nearly all takes place near the electrodes, so that X_0 is probably very small, even when rapidly alternating currents are used. Consequently, since ρ is large, $\dfrac{1}{4\pi}\dfrac{dX_0}{dt}$ may be neglected in comparison with $-\rho\dfrac{dt_1}{dt}$. Hence (3) becomes $i = -\rho\dfrac{dt_1}{dt}$ approximately. Substituting in (1) the value of X_0 got from (2) we get

$$V = V_0 \sin pt = D\left(\frac{m}{e}\frac{d^2 t_1}{dt^2} + \frac{A}{e}\frac{dt_1}{dt}\right) + 2\pi\rho d(2t_1 - d).$$

This gives

$$V_0 p \cos pt = \frac{mD}{e}\frac{d^3 t_1}{dt^3} + \frac{AD}{e}\frac{d^2 t_1}{dt^2} + 4\pi\rho d\frac{dt_1}{dt}.$$

But $\dfrac{dt_1}{dt} = -\dfrac{i}{\rho}$. Hence

$$-V_0 p \cos pt = \frac{mD}{e\rho}\frac{d^2i}{dt^2} + \frac{AD}{e}\frac{di}{dt} + 4\pi\,di. \quad . \quad . \quad (4)$$

The solution of this equation is

$$i = \frac{-\dfrac{e\rho}{mD}V_0 \sin(pt-\alpha)}{\left\{\left(1 - \dfrac{4\pi de\rho}{p^2 mD}\right)^2 p^2 + \dfrac{A^2}{m^2}\right\}^{\frac{1}{2}}}, \quad . \quad . \quad (5)$$

where

$$\tan\alpha = \frac{-\left(1 - \dfrac{4\pi de\rho}{p^2 mD}\right)pm}{A}.$$

If a P.D. $V = V_0 \sin pt$ is applied to a condenser of capacity C, the current is given by the equation $i = CV_0 p \cos pt$. For the flame, if A and m are both negligible (5) becomes

$$i = -\frac{V_0 p \cos pt}{4\pi d},$$

so that the apparent capacity is $\dfrac{1}{4\pi d}$ per unit area. Now $\dfrac{d}{2}$ is the amplitude of vibration of the negative ions so that $d\rho$ must be the amount of electricity flowing during a half-vibration.

Let $\qquad i = -\dfrac{dQ}{dt} \qquad$ so that $\qquad Q = \dfrac{V_0}{4\pi d}\sin pt.$

Then we have, integrating from 0 to π,

$$d\rho = \frac{V_0}{2\pi d} \quad \text{or} \quad d = \sqrt{\frac{V_0}{2\pi\rho}},$$

so that the apparent capacity per unit area is

$$\frac{1}{4\pi d} = \sqrt{\frac{\rho}{8\pi V_0}}.$$

If $\rho = 0$ this makes the capacity zero, whereas it should be $\dfrac{1}{4\pi D}$. This is due to the omission of $\dfrac{1}{4\pi}\dfrac{dX}{dt}$, which would no be negligible if ρ were very small. If, however, we take $\sqrt{\rho/8\pi V_0}$ to be not the apparent capacity but the increase in the apparent capacity due to the presence of the ions, then no error will be made even if ρ be small. The quantity $\dfrac{1}{d_2} - \dfrac{1}{d_1}$ which has been determined is proportional, as we have seen, to the increase in the apparent

capacity when salt is added to the flame.   In what follows we shall speak of $\dfrac{1}{d_2} - \dfrac{1}{d_1}$ as the apparent capacity.

Thus we should expect the apparent capacity per unit area to vary as the square root of the number of ions per c.c., and inversely as the square root of the maximum P.D. applied, but to be independent of the distance between the electrodes. The experimental results are in surprisingly good agreement with these conclusions.   Thus we have seen that the apparent capacity is nearly independent of the number of alternations per sec., and varies as the square root of the steady current conductivity.   For small E.M.F.'s the conductivity for steady currents is proportional to the number of ions per c.c. and to the velocity of the ions.   But it has been shown ** that all alkali-salts in flames give ions having the same velocity, so that the conductivity for steady currents should vary nearly as the number of ions per c.c. present.   Hence the observed apparent capacity varies nearly as the square root of the number of ions present.   The apparent capacity was also independent of the distance between the electrodes, provided this was small compared with their diameters. Further, we have seen that the apparent capacity varies roughly inversely as the square root of the maximum P.D., measured by the length of the spark-gap in the primary coil of the Tesla transformer.   It appears, therefore, that the expression $\sqrt{\rho/8\pi V_0}$ does represent approximately the observed variations of the apparent capacity of the electrodes in the flame containing salt vapour.

This expression has been obtained by neglecting the mass of the ions and the resistance to their motion through the flame-gases, so that it appears that the amount of alternating current through the flame is determined merely by the density of the layer of positive charge left in the gas near the electrodes when the negative ions move under the action of the applied field.

If a steady P.D. V is applied to two electrodes immersed in an ionized gas, and if the positive ions cannot move, it is easy to see that a current will only pass for the short time required for the accumulation of positive charge near the negative electrode to become sufficient to make the electric force near the positive electrode zero.   Thus the two electrodes will behave like a condenser when the P.D. is applied.   When a rapidly alternating P.D. is applied it is easy to see that even if the positive ions can move, provided their velocity is small

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* H. A. Wilson, Phil. Trans. A. 1899.

compared with that of the negative ions, the arrangement will behave like a condenser if the number of ions per c.c. is very large and the mass of the negative ions very small.

Denoting the apparent capacity per unit area by C, we have

$$C = \sqrt{\frac{\rho}{8\pi V_0}} = \frac{K}{4\pi D},$$

where K is the apparent specific inductive capacity. Consequently $\rho = \frac{V_0 K^2}{2\pi D^2}$. The length of the spark-gap used in the experiments on the variation of the apparent capacity with the concentration and nature of the salt was about 2 mms. in hydrogen gas at atmospheric pressure. The Tesla coil had 33 turns in its primary coil and 3 turns in its secondary, so that V_0 was about 400 volts or 1·2 E.S. units. D was 0·6 cms., so that for the strongest K_2CO_3 solution sprayed, for which K = 100 (p. 496), we have

$$\rho = \frac{1·2 \times 10000}{2 \times 3·1 \times 0·6^2} = 5400 \text{ E.S. units.}$$

The charge on one ion is 3×10^{-10} E.S. units, so that the number of ions per c.c. was

$$\frac{5400}{3 \times 10^{-10}} = 18 \times 10^{12}.$$

The amount of salt entering the flame was determined by finding the loss of weight of a bead of sodium-chloride placed in an equal Bunsen flame, so that the light emitted by this flame was equal to that emitted by the flame when a solution of NaCl containing 10 grams per litre was sprayed. In this way it was found that 0·53 milligram entered the flame per minute. Consequently the amount of K_2CO_3 entering the flame per minute with the strongest solution was 5·3 milligrams. The velocity of the flame-gases was about 200 cms. per second, and the diameter of the flame about 3 cms., so that the amount of salt per c.c. in the flame was about

$$\frac{5·3}{200 \times \pi (1·5)^2 \times 60} = 7 \times 10^{-5} \text{ milligram.}$$

Hence, taking the mass of an atom of hydrogen as 10^{-24} gram, the number of salt molecules per c.c. was about

$$\frac{7 \times 10^{-8}}{138 \times 10^{-24}} = 5 \times 10^{14}.$$

It thus appears that about one molecule in 30 molecules of K_2CO_3 was ionized in the flame. For most of the other salt solutions sprayed the proportion of the molecules ionized in the flame comes out less than one in 30.

In the paper on the conductivity of flames for steady currents, referred to above, numbers are given which are proportional to the molecular conductivity of all the salts tried, so that it is not necessary here to discuss further the variation of the conductivity with the nature and concentration of the salt vapour.

Let q be the number of ions produced per c.c. per sec. in the flame and n the number present per c.c. Then $q = \alpha n^2$, where α is a constant which has been shown by Langevin to be equal to $f \cdot 4\pi e(k_1 + k_2)$, where f is a proper fraction, and k_1, k_2 are the ionic velocities due to unit electric intensity. For the flame we have $k_2 = 1000 \dfrac{\text{cms.}}{\text{sec.}}$ for one volt per cm., or $3 \times 10^5 \dfrac{\text{cms.}}{\text{sec.}}$ for one E.S. unit of electric intensity. Hence

$$\alpha = 4\pi \cdot f \cdot 3 \times 10^{-10} \times 3 \times 10^5 = 1 \cdot 1 \times 10^{-3} f.$$

Now for the strongest K_2CO_3 solution sprayed $n = 1 \cdot 8 \times 10^{13}$, hence

$$q = f\, 1 \cdot 1 \times 10^{-3} \times 1 \cdot 8^2 \times 10^{26} = 4 \times 10^{23} f.$$

The number of salt molecules per c.c. is about 5×10^{14}; so that it appears that each salt molecule is ionized and recombines about $10^9 f$ times per second. The value of f under ordinary conditions is about $0 \cdot 2$. Probably in the flame it is less, say $0 \cdot 1$. Taking the temperature of the flame-gases to be $2000°$ C., we get for the number of collisions made by a salt molecule per second in the flame rather less than 5×10^8. So it appears that the K_2CO_3 molecules are ionized once for every 5 collisions with another molecule. It is therefore probable that the cause of the ionization of salt vapours in flames is the shock of molecular collision.

The current which could be carried by the 5×10^{14} ions produced per c.c. is about 60 amperes, which is enormously greater than the observed currents per c.c. of flame between the electrodes. This result agrees with the conclusion [*] that the observed steady currents through flames containing salt vapours are very far from the saturation value. Since each salt molecule is ionized many times per second, the salt would all be carried to the electrodes as ions if the current were sufficiently great. If we suppose that the electrodes absorb the ions which reach them, then the maximum possible current would be equal to that required to electrolyse the same amount of salt in a solution. This has been previously found to be the case for alkali salts vaporised in a current of air [†].

[*] H. A. Wilson, Phil. Mag. October 1905.
[†] H. A. Wilson, Phil. Mag. August 1902.

Equation (5) may be written

$$i = \frac{- pe\rho V_0 \sin (pt - \alpha)}{\{(mDp^2 - 4\pi de\rho)^2 + A^2 D^2\}^{\frac{1}{2}}}.$$

It is easy to show that the term $A^2 D^2$ is negligible compared with $(mDp^2 - 4\pi de\rho)^2$. We have $D = 0.6$ cm. and $Xe = Av$, where $X =$ electric intensity and $v =$ velocity of negative ions due to X. If $X = 1$ E.S. unit, $v = 3 \times 10^5 \frac{\text{cms.}}{\text{sec.}}$; so that $D^2 . A^2$ is equal to $\dfrac{0.6 \times 9 \times 10^{-20}}{9 \times 10^{10}} = 3 \times 10^{-29}$.

In $(mDp^2 - 4\pi de\rho)^2$ the term $4\pi de\rho$ is about 6×10^{-9} for $e = 3 \times 10^{-10}$, and $4\pi d\rho$ is about 20. Also when p is say 10^6, mDp^2 must be small compared with $4\pi de\rho$, because, as we have seen, changing p does not much affect the apparent capacity. Hence $(mDp^2 - 4\pi de\rho)^2$ is of the order 10^{-18}.

The expression for i becomes therefore on putting $\alpha = -90°$,

$$i = \frac{pe\rho V_0 \cos pt}{4\pi de\rho - mDp^2}.$$

Hence the apparent capacity per unit area is

$$C = \frac{e\rho}{4\pi de\rho - mDp^2}.$$

Therefore if C_1 and C_2 are values of C corresponding to values p_1, p_2 of p, we have

$$4\pi d\rho - \frac{m}{e} Dp_1{}^2 = \frac{\rho}{C_1},$$

$$4\pi d\rho - \frac{m}{e} Dp_2{}^2 = \frac{\rho}{C_2};$$

$$\therefore \ D . \frac{m}{e} . (p_2{}^2 - p_1{}^2) = \rho \left(\frac{1}{C_1} - \frac{1}{C_2} \right).$$

Also approximately $\rho = 8\pi V_0 C^2$ when p is small enough for mDp^2 to be small compared with $4\pi de\rho$, so that

$$\frac{e}{m} = \frac{D(p_2{}^2 - p_1{}^2)}{8\pi V_0 \left(\dfrac{1}{C_2} - \dfrac{1}{C_1} \right) \times C^2}.$$

If C_1 nearly $= C_2$ we can put $C^2 = C_1 C_2$, and get

$$\frac{e}{m} = \frac{D(p_2{}^2 - p_1{}^2)}{8\pi V_0 (C_2 - C_1)}.$$

Thus if the variation of C with p were known with sufficient accuracy, $\dfrac{e}{m}$ for the negative ions could be calculated. It is hoped that further experiments will enable this to be done for the negative ions of different salts.

Summary of Results.

(1) For rapidly alternating currents a flame containing an alkali-salt vapour behaves like an insulating medium having a high specific inductive capacity.

(2) The conductivity of different alkali-salt vapours in a flame for rapidly alternating currents as measured by the apparent capacity of platinum electrodes immersed in the flame varies as the square root of the conductivity of the same salt vapours for steady currents. This result confirms the view that the negative ions from all salts have the same velocity.

(3) The apparent capacity varies nearly inversely as the square root of the maximum applied P.D.

(4) The apparent capacity is nearly independent of the number of alternations per second.

(5) The apparent capacity is nearly independent of the distance between the electrodes.

(6) The results (1) to (5) are in agreement with the ionic theory of the conductivity of the flame for rapidly alternating currents when the velocity of the positive ions and the inertia and viscous resistance to the motion of the negative ions are neglected in comparison with the effects due to the number of ions per c.c.

(7) The apparent capacity per sq. cm. area of the electrodes is equal, according to the theory just mentioned, to $\sqrt{ne/8\pi V_0}$, where n is the number of positive ions per c.c., e the charge on one ion, and V_0 the maximum applied P.D.

(8) Not more than one molecule in 30 salt molecules is ionized at any instant in the flame, but each molecule is probably ionized and recombines several million times per second.

(9) The steady currents observed through salt vapours in flames are very far from the maximum possible currents corresponding to the number of ions produced per second.

XLIII. *The Electrical Conductivity of Metallic Oxides.* By
F. Horton, *D.Sc., B.A., Fellow of St. John's College, Cambridge* *.

THE theory of electrolytic dissociation has furnished a simple explanation of conductivity, in electrolytes, by assuming that the current is carried as a stream of electric charges by the ions into which a certain proportion of the molecules of an electrolyte are dissociated. Conduction in

metals is not due to this cause, for we cannot conceive that a molecule of an elementary substance could be split up into two dissimilar atoms, or groups of atoms, having opposite charges ; and experiments have shown that the conduction of electricity through metals is not accompanied by any detectable transportation of matter [*].

A theory of metallic conduction not involving the transportation of matter in the atomic state has been put forward by Professor Thomson [†]. On this theory metallic conductors owe their conductivity to the dissociation of the *atoms* of the metal, the products of dissociation arising from a single atom being an extremely small negatively charged corpuscle and the remaining larger portion of the atom which bears an equal positive charge. These negative corpuscles are identically the same in mass and charge from whatever atom they are produced, and they can move about freely in the mass of the metal. When an electric force acts upon them, they travel in a direction opposite to that of the force, and this convection of charges constitutes the electric current.

The conductivity of compound bodies may belong to either the metallic or the electrolytic class. The distinction between these two classes is generally based on the sign of the temperature coefficient of the conductivity of the conductor in question. If the conductivity decreases with rise of temperature, the conduction is taken to be metallic ; if it increases, the conduction is considered to be electrolytic. This, however, cannot be taken as an infallible rule, for, although the conductivity of liquid electrolytes generally increases with the temperature, a series of electrolytic liquids is known which behave in the opposite manner. Such acids as sulphuric and phosphoric, the hydrolytic dissociation of which takes place exothermically, belong to this series. In these cases an increase of temperature decreases the electrolytic dissociation, causing a diminution in the conductivity which more than compensates for the increased mobility of the ions. On the other hand, the conductivity of all the metalloids, which cannot be electrolytic on account of the elementary nature of the conductors, increases with rise of temperature, the increase being probably due to changes in the crystalline structure when heated [‡].

It has long been known that many substances which, at the ordinary temperature of the laboratory, may be regarded

* E. Riecke, *Phys. Zeitschr.* ii. p. 639 (1901).

† J. J. Thomson, *Rapports présentés au Congrès International de Physique*, vol. iii. p. 138 (Paris, 1900).

‡ Siemens, Pogg. *Ann.* clix. p. 117 (1876).

as non-conductors, become conducting when their temperature is raised. The conductivity of such substances was studied by Faraday[*], who found that silver sulphide, lead fluoride, corrosive sublimate, and periodide of mercury, all increase very rapidly in conductivity with rise of temperature. Since the time of Faraday this subject has been investigated by many experimenters. It has recently received special attention from Nernst[†], who has given it practical application in the Nernst filaments for incandescent lamps. The conduction of electricity through these filaments is considered by Nernst to be electrolytic, but the recent experiments of Streintz[‡] and of Guinchant[§] have shown that a large number of substances which become conducting at high temperatures give no trace of electrolysis. The conduction in these cases must therefore be metallic, and on Professor Thomson's theory due to the negatively charged corpuscles liberated from the atoms. If this supposition be true, it follows that either the number of free corpuscles within these substances, or the mean free path of these corpuscles, increases very rapidly with rise of temperature. Now Wehnelt[||] has studied the emission of corpuscles from the alkaline earths when they are raised to high temperatures, and has found that the rate of emission increases very rapidly with the temperature, and at 1500° C. is quite enormous. At these high temperatures, therefore, there must be a large number of free corpuscles inside the substance, which probably owes its conductivity to their presence. Such an explanation would account for the great increase in conductivity with rise of temperature observed by Faraday, Streintz, and others, and explain the loss of insulating power of quartz and other non-conductors when raised to high temperatures.

The object of the present research was to investigate the variation of electrical conductivity of solid metallic compounds with change of temperature, and to ascertain whether the conduction is carried on electrolytically or metallically. For this purpose the metallic oxides seemed to be the most

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[*] Faraday, Exp. Res. vol. i. § 432 (1833), § 1340 (1838).
[†] Nernst, *Zeitschr. Elektrochem.* vi. p. 41 (1899); *Gessel. Wiss. Göttingen, Nachr. Math.-Phys.* p. 328 (1900).
[‡] Streintz, *Ann. d. Physik*, iii. p. 1 (1900); *Akad. Wiss. Wien. Sitz.-Ber.* iii. 2 a, p. 345 (1902).
[§] J. Guinchant, *Comptes Rendus*, cxxxiv. p. 1224 (1902).
[||] Wehnelt, *Ann. d. Physik*, xiv. p. 425 (1904); Phil. Mag. [6] vol. x. p. 80 (1905).

suitable compounds, especially as the emission of corpuscles from a large number of metallic oxides had been studied by Wehnelt.

The oxides were used in the form of slabs about 1 sq. cm. in area, and from 1 to 2 mms. in thickness. A slab of oxide was placed between two stout platinum plates which were pressed firmly together by means of two porcelain rods 1 cm. in diameter. These rods had a small hole along their axis, and platinum wires from the electrodes passed through them and served to connect to the Wheatstone bridge used for measuring the resistance. The surfaces of the electrodes and of the oxide in contact with them were ground or filed plane so as to make good electrical contact all over.

Two types of electric heating-furnace were used in these experiments. With the more easily fusible oxides it was necessary to watch the oxide to see that the temperature was not raised above the melting-point. The furnace used to heat these oxides consisted of an iron box lined with asbestos cardboard. The porcelain rods passed vertically through the box, the slab of oxide being in the middle. A strip of platinum foil, 5 cms. wide, passed above and below the oxide, and this was heated by an alternating current from a transformer. The oxide could be seen by opening a small door in the side of the furnace. Oxides such as lime, magnesia, and quartz, of high melting-point, were heated in an electric furnace consisting of a coil of No. 20 B.W.G. platinum wire wrapped round a porcelain tube 2·5 cms. in diameter and 13 cms. long. This was placed inside a larger earthenware tube and the space between filled with magnesia. This furnace is shown in section in fig. 1. With both forms of furnace the temperature of the oxide was obtained by means of a thermocouple of wires of platinum and an alloy of platinum with 10 per cent. of rhodium, the junction being close to the oxide.

In order to standardize the thermocouple the melting-point of potassium sulphate was taken as the fixed point, and the galvanometer deflexion corresponding to this temperature (1066° C.) was found. The platinum temperatures given by the galvanometer-readings were corrected to degrees centigrade by means of the curve given by Professor Callendar [*]. The galvanometer deflexion corresponding to the melting-point of platinum was also found and used for standardizing readings up to 1700° C.

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[*] Callendar, Phil. Mag. vol. xlviii. p. 519.

In some cases the resistance of the oxide was obtained by measuring the current passing through it under a known

Fig. 1.—Electric Furnace.
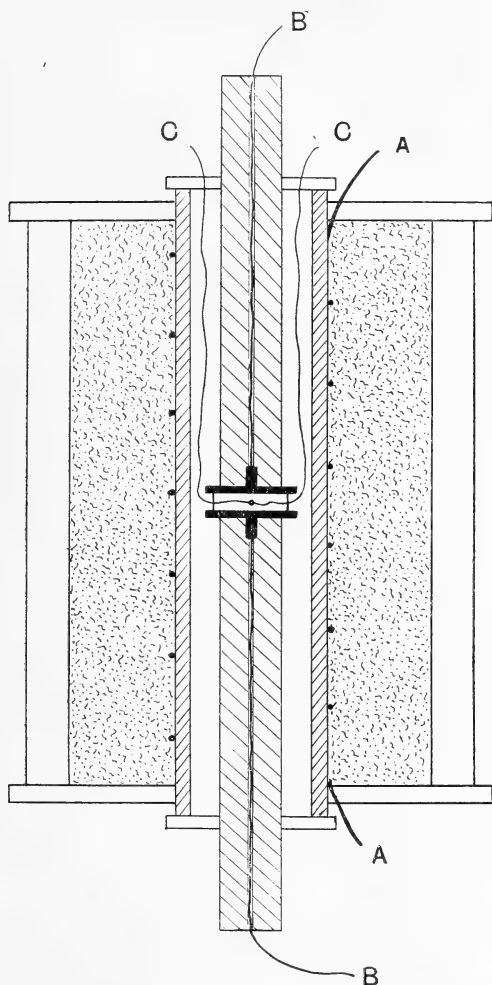
A A are the leads to the heating-coil; B B are the platinum wires from the electrodes; C C the thermocouple wires.

E.M.F. by means of a delicate d'Arsonval galvanometer; but more generally the resistance was measured directly by a

Wheatstone bridge. With some of the oxides electrolysis was found to accompany the conduction, and with these another method had to be used for obtaining the resistance. Kohlrausch's method of using an alternating current and a telephone indicator does not work well for resistances over 10,000 ohms, and the resistance to be measured was usually greater than this value. The method of using a rotating commutator to reverse the battery and galvanometer circuits together many times a second requires an exceedingly well-made commutator. Unfortunately the one made for the researches which Mr. Fitzpatrick carried out in this laboratory was in use at the time and not available for this work. Other commutators were tried, but none worked satisfactorily.

The method finally adopted was not of great accuracy, but was sufficiently accurate for the object of the present research. It consisted in having the galvanometer permanently connected up to the Wheatstone bridge without a key in circuit, and noticing the direction of its first kick on momentarily closing the battery circuit. The resistance was connected to a reversing key which was reversed after each "make" of the battery circuit, so that successive momentary currents were sent in opposite directions through the electrolyte. This method worked better than I had expected, and, by taking care not to let a continuous current pass through the oxide, results in very good agreement were obtained. There was sometimes a small thermoelectric force developed on heating the oxide, and consequently the resistance in one direction was slightly greater than in the opposite direction. The mean of these resistances was always taken, and the circuit through the oxide was kept open except just when observations were being made. The voltage used on the battery circuit varied, according to the resistance to be measured, from 1 to 500 volts.

The oxides experimented on were lime, baryta, magnesia, bismuth trioxide, plumbic oxide, cupric oxide, sodium peroxide, and quartz. The results obtained will now be given.

Lime.

The piece of lime used was ·85 cm. square by ·70 mm. thick. Its resistance at the temperature of the laboratory was greater than 100 megohms. The following table contains the resistance, and the conductivity calculated therefrom, at different temperatures :—

Temperature Centigrade.	Resistance in ohms.	Conductivity in mhos.
763	70,000,000	1.38×10^{-9}
879	9,125,000	1.06×10^{-8}
930	4,175,000	2.33×10^{-8}
1011	2,000,000	4.85×10^{-8}
1105	600,000	1.62×10^{-7}
1175	245,500	3.96×10^{-7}
1235	104,100	9.33×10^{-7}
1304	35,100	2.77×10^{-6}
1370	2,045	4.75×10^{-5}
1448	122.7	7.90×10^{-4}
1466	91.0	1.035×10^{-3}

Diagram I.—Showing increase of conductivity of *Lime* with temperature.
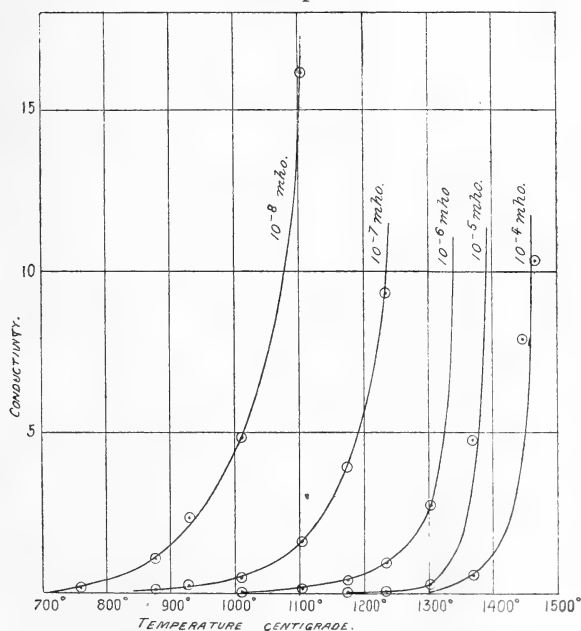
The relation between the conductivity and the temperature is shown graphically in diagram I. The ordinates give the value of the conductivity, the abscissæ being temperatures in degrees centigrade. In the first curve, starting from the left, each unit of the ordinates represents 10^{-8} mho; in the

succeeding curves the value of the ordinate is successively multiplied by ten, so that in the second curve each unit is equal to 10^{-7} mho, in the third 10^{-6} mho, in the fourth 10^{-5} mho, and in the last 10^{-4} mho.

These curves are exactly similar to those obtained by Wehnelt [*] for the relation between the number of corpuscles given out by lime (measured by the saturation current) and the temperature. Wehnelt found that the number of corpuscles emitted from glowing lime obeyed the law found by O. W. Richardson [†] and by H. A. Wilson [‡] to hold for the negative corpuscles from hot platinum, viz., $n = A\theta^{\frac{1}{2}} e^{-\frac{b}{\theta}}$ where n is the number of corpuscles, θ the absolute temperature, and A and b are constants. If the conduction is carried on by the corpuscles set free in the interior of the oxide at high temperatures, we might expect a similar relation to hold between the conductivity and the temperature.

Diagram II.
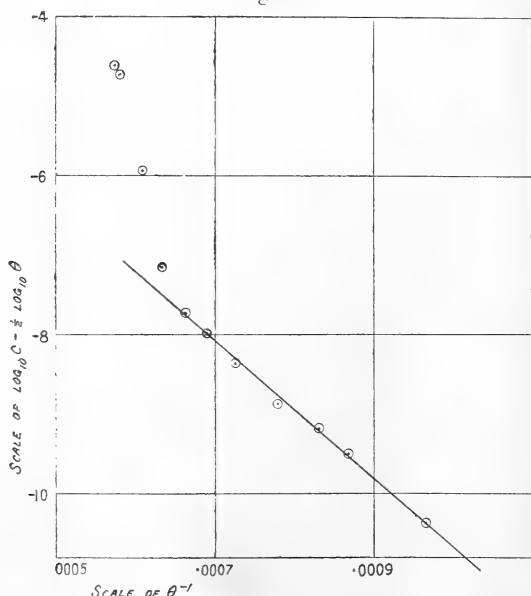
In order to test this we may denote the conductivity by c, and write the equation in the form

$$\log_{10} c - \tfrac{1}{2} \log_{10} \theta = \log_{10} A - \cdot 434 b \, \theta^{-1},$$

so that on plotting the values of $\log_{10} c - \tfrac{1}{2} \log_{10} \theta$ against

[*] Wehnelt, *Ann. d. Phys.* xiv. p. 425 (1904).
[†] O. W. Richardson, Phil. Trans. A. vol. cci. p. 497 (1903).
[‡] H. A. Wilson, Phil. Trans. A. vol. ccii. p. 243 (1903).

those of θ^{-1} we ought to obtain a straight line.   In diagram II. the ordinates are the values of $\log_{10} c - \frac{1}{2} \log_{10} \theta$, the abscissæ being the corresponding values of θ^{-1}.

It will be seen that seven points representing the largest values of θ^{-1} (*i. e.* the lowest temperatures) lie very nearly on a straight line.   The fact that the points above $1300°$ C. do not lie on this line is probably due to disturbing causes which affect the resistance to be measured.   At these high temperatures, the lime would probably have some chemical action on the platinum electrodes, causing, possibly, a decrease in the contact resistances.   It was, for instance, always found that the lime was fixed to the electrodes after it had been strongly heated, but not when lower temperatures were used.   This explanation would account for the fact that the value found for the conductivity at high temperatures was greater than it should be on the present theory.   Diagram II. may therefore be taken as showing that the increase of conductivity of lime with temperature follows the same law as the emission of corpuscles from its surface.

When the above series of experiments had been completed, tests were made to see if any polarization could be detected which would show that some part of the conduction through lime was electrolytic.   The lime was connected to a battery for a few seconds and a current allowed to pass through it. The battery was then cut out and a delicate d'Arsonval galvanometer substituted.   This indicated a small polarization current in the opposite direction to the original current and dying away logarithmically with the time.   The polarization current when the lime was at $800°$ C. was very small, but it increased as the temperature was raised, the increase being no doubt due to the decrease in the resistance of the lime.   A battery of 100 volts was next connected up in series with the slab of lime and galvanometer, the latter being shunted for measuring larger currents.   The current through the galvanometer gradually decreased, and, if the E.M.F. was left on for a few seconds and the key connected to the lime reversed, a larger deflexion in the opposite direction was obtained which also decreased with time.   But if the current was allowed to pass for several minutes in one direction through the lime, the galvanometer deflexion decreased to about a tenth of its original value, and, on reversing the key, the deflexion obtained in the opposite direction was only slightly greater than the small value just before reversing.   This increased with time slowly at first, but soon very rapidly, and then more slowly again as a maximum deflexion was reached. the current then decreased with time, at a gradually

decreasing rate until about the same magnitude as the direct current just before reversing. This behaviour may be explained by supposing that a layer of oxygen is formed by electrolysis against the anode, and that this increases the resistance of the circuit in addition to causing the polarization E.M.F. When the circuit is completed for only a short time before the key connected to the lime is reversed, very little gas can have collected at the anode, and this gives a back-E.M.F., but does not greatly increase the resistance. On reversing, therefore, a large deflexion in the opposite direction is obtained. On the other hand, when time is allowed for much gas to collect at the anode, then, on reversing, the deflexion in the opposite direction is slightly larger than the small value just before reversing, owing to the back-E.M.F. This increases as the oxygen recombines with the calcium which is now formed at that electrode, and the oxygen which is being formed at what is now the anode combines with the calcium already liberated there by the direct current. Thus the resistance gradually decreases until all the collected gas is used up; a maximum current then passes and the deflexion afterwards decreases as oxygen collects against the other electrode.

These experiments were carried out at about 1000° C. At higher temperatures the current did not die away to so small a value; and on reversing the key connected to the lime a large deflexion in the opposite direction was obtained which gradually died away, and did not increase even when the direct current had been flowing for a long time. It is probable that this is due to the oxygen escaping more easily from the electrode at the higher temperature and not forming a layer over the anode, and thus increasing the resistance of the circuit as it did in the experiments at 1000° C.

Several specimens of lime were experimented on and gave results practically identical with those described above, both as regards the variation of conductivity with temperature and the electrolytic effects of the current. It thus appears that the conduction of electricity through heated lime is carried on partly by electrolytic ions as in a liquid electrolyte; but the fact that the increase of conductivity with temperature corresponds, so closely, with the increase in the rate of emission of electrons, seems to indicate that most of the conduction is metallic.

Magnesia.

A piece of magnesia ·84 cm. × ·83 cm. and ·090 cm. thick was used. The resistance of this at the temperature of the laboratory was too large to be measured by the method

used, being over 100 megohms. The values of the resistance at higher temperatures and also of the conductivity calculated from the resistance and dimensions of the slab are given in the following table.

Temperature Centigrade.	Resistance in ohms.	Conductivity in mhos.
471	35,000,000	3.69×10^{-9}
564	3,000,000	4.34×10^{-8}
630	800,000	1.61×10^{-7}
743	215,000	6.00×10^{-7}
828	88,000	1.46×10^{-6}
933	31,000	4.16×10^{-6}
970	22,300	5.79×10^{-6}
1055	12,450	1.04×10^{-5}
1191	4,900	2.63×10^{-5}
1204	6,800	1.90×10^{-5}
1280	12,900	1.00×10^{-5}
1341	290,000	4.45×10^{-7}

Diagram III.—Showing increase of conductivity of MgO with temperature.
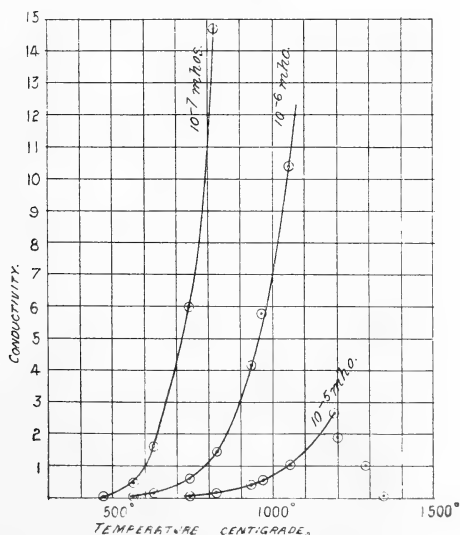
In diagram III. the values of the conductivity given in the above table are plotted against the corresponding

temperatures. The curves obtained are similar to those given by lime.

It will be observed, both from the table and diagram, that after 1200° C. the conductivity suddenly altered its behaviour, and began to decrease with increasing temperature. At these temperatures the resistance was not constant, but increased as time went on as well as with increasing temperature. If is, however, well known that the crystalline structure of magnesia alters when heated to high temperatures, the alteration being accompanied by a change in density. Several specimens of magnesia were experimented on and they all behaved in the same manner, giving a minimum resistance at from 1200° C. to 1250° C. It therefore appears probable that this sudden change in the conductivity is due to an alteration of the condition of the oxide into a less conducting state, the resistance continually increasing while the change is going on.

In order to see whether the increase of conductivity with temperature follows the same law as the emission of corpuscles

Diagram IV.
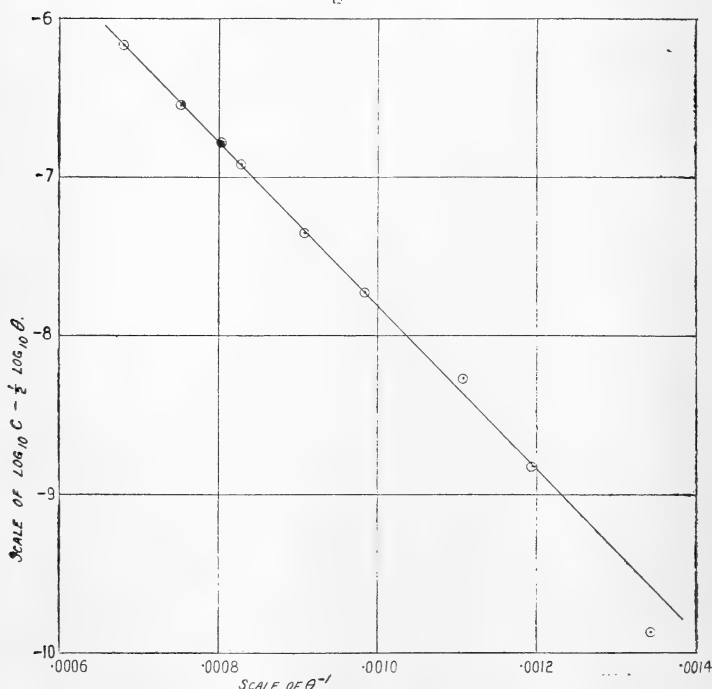
from the surface, the values of $\log_{10} c - \frac{1}{2} \log_{10} \theta$ were plotted against those of θ^{-1}, as in the case of lime. This is done in

diagram IV., which shows that the points lie very nearly indeed on a straight line.

Experiments showed that the conduction of electricity through magnesia was accompanied by a little electrolysis; but all the effects of this were much smaller than in the case of lime.

The results obtained with the other oxides experimented on are very similar to those for calcium and magnesium, the temperature-conductivity curves possessing in all cases precisely the same characteristics as those connecting the emission of corpuscles with the temperature of the oxide when used as the cathode in a discharge-tube. The results obtained with these oxides will therefore be given more briefly.

Baryta.

This was difficult to work with on account of the strong chemical action it has on platinum when heated, and because of the ease with which it absorbs moisture and carbonic-acid gas from the air. Also, if heated above a dull red heat, it takes up more oxygen from the air and forms the dioxide. The numbers given in the following table are therefore not so reliable as those given for lime or magnesia. The dimensions of the piece of barium oxide used were ·853 cm. by ·843 cm. and ·467 cm. thick.

Temperature Centigrade.	Resistance in ohms.	Conductivity in mhos.
307	646,000	$1\cdot00\times10^{-5}$
311	359,000	$1\cdot81\times10^{-6}$
312	274,000	$2\cdot37\times10^{-6}$
320	184,000	$3\cdot53\times10^{-6}$
319	218,000	$2\cdot98\times10^{-6}$
345	78,400	$8\cdot29\times10^{-6}$
355	10,760	$6\cdot03\times10^{-5}$
369	6,892	$9\cdot43\times10^{-5}$
386	602	$1\cdot08\times10^{-3}$
428	35·7	$1\cdot82\times10^{-2}$
450	21·4	$3\cdot04\times10^{-2}$
469	17·6	$3\cdot70\times10^{-2}$
497	14·1	$4\cdot61\times10^{-2}$

When the conductivities in the above table are plotted against the corresponding temperatures, curves similar in type to those given by lime and magnesia are obtained : but

the various readings do not fall so well on the curves as in the former case. This is no doubt due to the causes already mentioned, and especially to the action of the baryta on the platinum electrodes. The effect of this could be seen in the blackening of the oxide through some considerable distance from each surface of contact. Much importance cannot, therefore, be attached to these values of the conductivity of baryta.

Experiments showed that, as with lime and magnesia, some electrolysis was caused by the passing of the current. Some experiments were afterwards made to see whether any evolution of gas at the anode could be detected. These will be described later.

Plumbic Oxide.

Some lead monoxide was melted in an atmosphere of oxygen (to prevent reduction) and cast into a slab. This was filed up to ·758 cm. × ·733 cm. and ·158 cm. in thickness. The following table contains the values of the conductivity at various temperatures after the resistance had become small enough to be measured.

Temperature Centigrade.	Resistance in ohms.	Conductivity in mhos.
384	7,000,000	$3 \cdot 86 \times 10^{-8}$
420	2,350,000	$1 \cdot 15 \times 10^{-7}$
462	650,000	$4 \cdot 15 \times 10^{-7}$
504	250,000	$1 \cdot 08 \times 10^{-6}$
539	128,000	$2 \cdot 11 \times 10^{-6}$
572	72,000	$3 \cdot 75 \times 10^{-6}$
600	45,000	$6 \cdot 00 \times 10^{-6}$
646	13,500	$2 \cdot 00 \times 10^{-5}$
687	4,050	$6 \cdot 67 \times 10^{-5}$
700	2,870	$9 \cdot 39 \times 10^{-5}$
757	730	$3 \cdot 695 \times 10^{-4}$
787	330	$8 \cdot 175 \times 10^{-4}$

The relation between the conductivity and the temperature is shown graphically in diagram V.

When a current of ·001 ampere was sent through this oxide no sign of electrolysis could be detected, either as a liberation of gas or as a polarization E.M.F. This current is much greater than those used in the determinations of the conductivity of the oxide. On sending a current of ·015

ampere through the oxide for a short time, and then connecting it up to a delicate galvanometer, a feeble polarization current was indicated ; but this was much smaller than those

Diagram V.—Showing increase of conductivity of *Plumbic Oxide* with temperature.
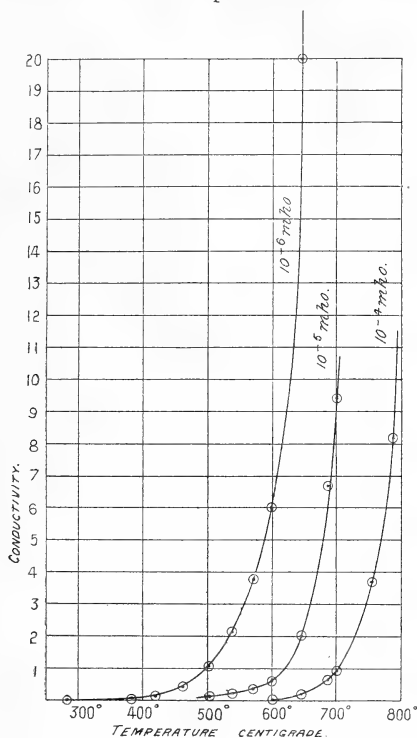
observed in the experiments with the oxides of the alkaline earths. The temperature of the plumbic oxide during these experiments was 780° C.; the polarization was still smaller at lower temperatures.

Bismuth Trioxide.

Slabs of bismuth trioxide were prepared in a manner similar to that employed with plumbic oxide. Some pure trioxide was melted in a stream of oxygen and cast into a slab which was filed up to the required size. The sample from which the numbers given in the following table were obtained was ·77 cm. × ·77 cm. and ·202 cm. thick.

Temperature Centigrade.	Resistance in ohms.	Conductivity in mhos.
225	80,000,000	$4 \cdot 27 \times 10^{-9}$
261	20,000,000	$1 \cdot 71 \times 10^{-8}$
315	2,550,000	$1 \cdot 34 \times 10^{-7}$
342	800,000	$4 \cdot 27 \times 10^{-7}$
376	259,000	$1 \cdot 32 \times 10^{-6}$
424	49,000	$6 \cdot 97 \times 10^{-6}$
458	22,000	$1 \cdot 55 \times 10^{-5}$
487	15,300	$2 \cdot 23 \times 10^{-5}$
515	9,500	$3 \cdot 59 \times 10^{-5}$
535	7,150	$4 \cdot 77 \times 10^{-5}$
566	4,700	$7 \cdot 27 \times 10^{-5}$
598	3,400	$1 \cdot 040 \times 10^{-4}$
645	2,050	$1 \cdot 664 \times 10^{-4}$

The above values of the conductivity are plotted against the corresponding temperatures in diagram VI.

Diagram VI.—Showing increase of conductivity of *Bismuth Trioxide* with temperature.
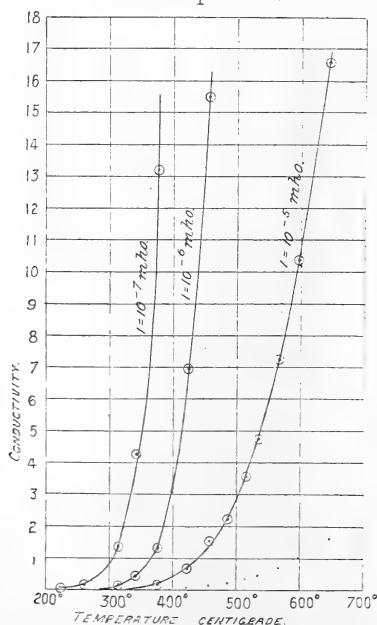
Electrolysis was more easily detected with this oxide than with plumbic oxide, a small polarization current being obtained even at 200° C.

Cupric Oxide.

Some pure oxide of copper was melted up by means of the oxy-hydrogen flame, and the melted mass ground into shape on a carborundum wheel. Its dimensions were :—1·006 cm. × ·994 cm. and 2·179 mms. thick. Two series of experiments were made with this specimen, the first up to 400° C. and the second to over 1000° C. The results obtained in both series are given in the following table :—

Temperature Centigrade.	Resistance in ohms.	Conductivity in mhos.
12·2	462,400	$4·71 \times 10^{-7}$
59	91,560	$2·38 \times 10^{-6}$
66	74,560	$2·92 \times 10^{-6}$
134	12,360	$1·76 \times 10^{-5}$
157	5,930	$3·67 \times 10^{-5}$
268	556·5	$3·91 \times 10^{-4}$
303	300·8	$7·24 \times 10^{-4}$
385	96·2	$2·26 \times 10^{-3}$
463	36·3	$6·00 \times 10^{-3}$
12	645,100	$3·38 \times 10^{-7}$
487	27·8	·00784
564	11·13	·02424
618	5·87	·03711
691	2·34	·0931
733	1·455	·1497
786	·845	·258
873	·347	·628
944	·196	1·111
974	·121	1·80
1038	·021	10·37

The resistance of copper oxide at high temperatures is so small, that it was necessary to subtract from the measured resistance the resistance of the platinum leads. The resistance of these at various temperatures was found by a preliminary experiment in which the two electrodes were placed in contact

and heated up just as in the experiment proper. The values of the conductivity are plotted against the temperatures in diagram VII.

Diagram VII.—Showing increase of conductivity of *Cupric Oxide* with temperature.
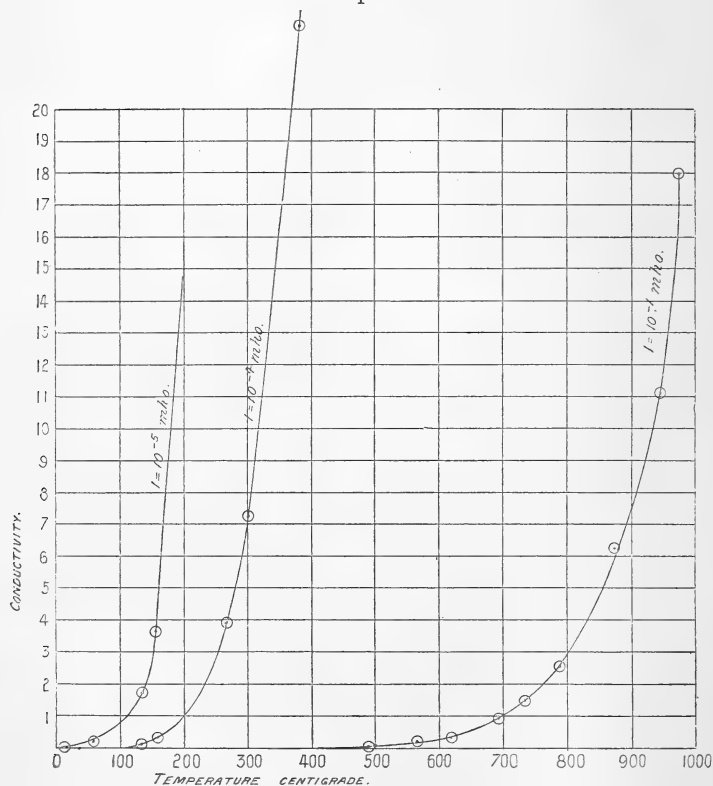
No sign of electrolysis could be found when a current was sent through this oxide at the ordinary laboratory temperature. A current of $1·625 \times 10^{-8}$ ampere was sent through the oxide. This slowly decreased to $1·528 \times 10^{-8}$ ampere in 10 minutes, but on cutting out the battery no trace of polarization current could be observed. The decrease of current was not due to decomposition such as had been observed in the case of lime ; for on reversing the direction of the current through the copper oxide a steady and equal deflexion of the galvanometer in the opposite direction was obtained. Thus we may conclude that the passage of a current of electricity through copper oxide at the ordinary laboratory temperature is not accompanied by electrolysis. At higher temperatures it was

impossible to test for a polarization current because of a thermoelectric effect due to one face of the oxide being slightly warmer than the other. A current of about ·25 ampere was sent through the oxide at 700° C. The galvanometer deflexion remained steady except on one occasion when it decreased slightly. In no case did the current fall off as it had done in the experiments with lime, so that we may conclude that at high temperatures, also, the conduction of electricity through copper oxide is not accompanied by electrolysis.

Sodium Peroxide.

The results obtained with this oxide were not very satisfactory on account of the readiness with which it deliquesces when exposed to air. Only a small range of temperature could be used, as the oxide melts at about 300° C. The numbers in the following table refer to a piece of oxide 1 sq. cm. in area and 2 mms. thick.

Temperature Centigrade.	Resistance in ohms.	Conductivity in mhos.
20	5000	4×10^{-5}
59	2500	8×10^{-5}
109	1200	1.67×10^{-4}
198	600	3.3×10^{-4}
244	400	5.0×10^{-4}
284	200	1.0×10^{-3}

Diagram VIII.
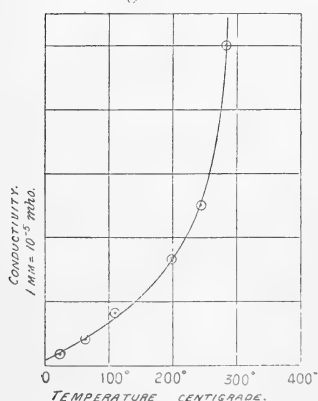
The above values of the conductivity are plotted against the corresponding temperatures in diagram VIII.

2 M 2

Quartz.

A rectangular slab of fused quartz was used in these experiments. Its two faces were ground parallel and roughly polished, and the edges were ground square. The area of the slab was ·946 × ·939 sq. cm. and it was ·1514 cm. in thickness. The two faces through which the current was to be passed were platinized by means of a colloidal solution of platinum, and thus good contact over the surfaces was assured. The platinized slab was held between the two platinum electrodes in the heating furnace and its resistance at various temperatures measured. The following values were obtained :

Temperature Centigrade.	Resistance in ohms.	Conductivity in mhos.
486	95,000,000	$1·79 \times 10^{-9}$
580	12,000,000	$1·42 \times 10^{-8}$
652	3,500,000	$4·86 \times 10^{-8}$
716	1,570,000	$1·08 \times 10^{-7}$
703	1,980,000	$8·59 \times 10^{-8}$
795	750,000	$2·27 \times 10^{-7}$
869	409,000	$4·16 \times 10^{-7}$
942	250,000	$6·80 \times 10^{-7}$
1013	170,000	$1·00 \times 10^{-7}$
1093	121,000	$1·40 \times 10^{-6}$
1288	65,200	$2·61 \times 10^{-6}$

Diagram IX.—Showing increase of conductivity of *Quartz* with temperature.
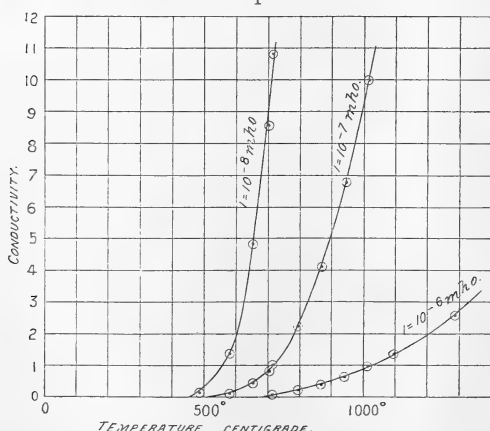
The rapid increase of conductivity with rise of temperature is readily seen from diagram IX., in which the above values

of the conductivity are plotted against the corresponding temperatures.

When the experiments already described had been performed, it was thought to be desirable to investigate further the electrolysis which was found to accompany the conduction in the case of some of the oxides, and to apply a more delicate test for it in those cases in which no trace of a polarization current could be detected. The oxides chosen for these experiments were lime, baryta, and cupric oxide ; lime and baryta because they were the oxides with which electrolysis had been most easily detected, and the cupric oxide because it behaved as if its conductivity were entirely metallic.

Wehnelt has shown * that if glowing lime (or certain other oxides) be used as a cathode in a discharge-tube, a strong current can be made to pass. If the cathode consists of a strip of platinum covered with lime, the whole of the current must pass through the oxide. If electrolysis takes place, oxygen will be liberated at the free surface of the oxide and calcium next to the platinum strip. The gas pressure in the discharge-tube will thus increase with the passage of the current, and the amount of gas liberated in a given time will be proportional to the current passing. Wehnelt found in his experiments that there was "no notable increase of pressure," which seems to indicate that the passage of the current through the oxide was unaccompanied by electrolysis. In the following experiments much larger cathodes and stronger currents were employed, so that if the conduction of electricity is electrolytic, the amount of oxygen evolved would be quite easily detected.

A strip of platinum foil 2 cms. long and ·3 cm. wide, having a thermocouple of fine platinum and platinum-rhodium wires fixed to its middle point, was covered with lime and used as the cathode in a discharge-tube, the anode being a disk of platinum 2 cms. in diameter placed parallel to the strip and about 2 cms. away from it. The discharge-tube was connected to a mercury-pump and McLeod gauge, and the whole apparatus was pumped down to a low pressure. The platinum strip and lime were heated, by means of the alternating current from a small transformer, to a white heat for several hours. The heating was accompanied by a considerable evolution of gas from the oxide. This was pumped out of the apparatus at intervals, and the heating was continued until the pressure remained fairly constant. The transformer

* *Ann. der Physik*, xiv. pp. 425–468 (1904) ; Phil. Mag. vi. pp. 80–90 (1905).

was then removed and the strip was heated to a lower temperature by means of the direct current from a battery of 10 E.P.S. motor cells. The cathode was also connected to the negative pole of a battery of small accumulator-cells, the positive pole of which was connected to the anode through a current-measuring instrument—a Thomson galvanometer for small currents and a milliammeter for larger ones.

In making the experiments, the required temperature of the cathode (indicated by the thermocouple) was first obtained by adjusting the resistance in the heating circuit. The pressure of gas in the apparatus was then measured by the McLeod gauge, and the apparatus left for 15 minutes, after which the pressure was again read and the discharge started through the tube. The current was allowed to pass for 15 minutes. It was measured at intervals during this time and the mean value was calculated. The pressure was then read, and the apparatus was left to itself for another 15 minutes. In this way any liberation of gas due merely to the heating of the cathode could be allowed for, the temperature of the cathode being maintained the same throughout the experiments. It was found that the passage of the discharge slightly increased the temperature of the cathode, and at very high temperatures (1400° C.) the arrangement was unstable, for at the higher temperature the lime gives a larger supply of corpuscles for carrying the current, which is increased, and thus still further increases the temperature of the cathode. In order that the current passing through the tube should not become sufficient to raise the temperature of the cathode above the melting-point of platinum, a small fuse cut from a sheet of thin tin-foil was placed in the circuit.

The results obtained varied with the length of time during which the oxide had been heated before the observations were made. It was found in one case, when there was still a fair amount of gas being given off on merely heating the cathode, that the extra liberation of gas when a current of $2 \cdot 66 \times 10^{-4}$ ampere was sent through the tube was within 8 per cent. of that required by Faraday's law on the supposition that the conduction through the lime was entirely electrolytic[*]. On heating the cathode to about 1400° C. for several hours longer, there was much more gas given off from it, and on pumping this out and repeating the experiment it was found that the evolution of gas had practically ceased, and that on now sending a discharge through the tube the pressure was altered

[*] It is interesting to note in this connexion that Skinner (Phys. Rev. July 1905) has found that the evolution of hydrogen from a fresh cathode obeys Faraday's Law.

only very slightly, the alteration being in most cases an increase, but in a few cases a decrease. The following numbers serve to illustrate the results obtained :—

Temperature of cathode $= 1100°$ C.
Potential-difference between the electrodes $= 40$ volts.

Time.	Pressure in mm. of Hg.	Current in amperes.
3·54 P.M.	·0018	none
4·3 „	·00185	·03
4·21 „	·0021	none
4·31 „	·00224	

mm.
Increase of pressure per minute while discharge was passing $= ·000014$
Mean increase per minute when no discharge was passing $\,..\;= ·000010$

If the conductivity of the lime was entirely electrolytic, the passage of ·03 ampere would have liberated oxygen at a rate sufficient to increase the pressure of gas in the apparatus by ·0803 mm. per minute. So the actual increase of pressure while the discharge is passing is only $\frac{1}{20,000}$ of what would be expected if the conduction were entirely electrolytic. The experiment may therefore be taken as showing that in the conduction of electricity through heated lime only a very small fraction of the total current is carried by means of electrolytic ions.

A result similar to this was obtained by using barium oxide on the cathode in place of lime. Some experiments were also made with a cupric-oxide cathode, and no increase of gas-pressure in the apparatus could be observed when a discharge was allowed to pass.

Conclusion.

The point of most interest in connexion with the conduction of electricity through the metallic oxides is the method by which the current is carried. Nernst [*] has given reasons for supposing the conductivity to be electrolytic, the chief of these being (*a*) that definite traces of electrolysis were found in some cases ; the chemical composition at the cathode after the current had been allowed to flow through the oxide for a long time being found to be different from that at the anode; (*b*) the conductivity of a mixture of oxides was found to be much greater than that of each oxide separately. Nernst states that the products of electrolysis are continually recombining, and so the supply of electrolytic ions is kept up.

[*] Nernst, *Zeitschr. f. Elektroch.* vi. pp. 41–43 (1899).

The conclusion from the present experiments is just the opposite of this view, and is that the conductivity of the oxides investigated is mainly metallic. It may be that the signs of electrolysis obtained, in some cases, by Nernst correspond to the signs of electrolysis found with some of the oxides in the present experiments. With mixtures of oxides it is probable that signs of electrolysis would be more readily given—for instance, the wandering of colour towards the cathode in the case of mixtures containing coloured oxides (*e. g.* oxide of iron) may be due, as Nernst suggests, to the coloured oxide being in solution in the rest and giving rise to the ordinary phenomena of electrolysis ; but I do not think it is justifiable to conclude that because such traces of electrolysis can be detected therefore the whole conductivity is electrolytic.

In the present paper it is shown that some oxides (*e. g.* CuO) show no trace of electrolysis ; and I think that if the conduction of electricity through these oxides is carried on by means of electrolytic ions, some evidence of the liberation of the products of electrolysis would have been obtained in the experiment in which a current was passed through oxide of copper whilst it was heated in a vacuum.

The distinction between electrolytic and metallic conductivity is generally based on the sign of the temperature coefficient of the conductivity of the conductor in question. If the conductivity decreases with rise of temperature, the conduction is taken to be metallic; if it increases with rise of temperature, the conduction is considered to be electrolytic. This rule is, however, one with so many exceptions (some of which have already been stated at the commencement of this paper) that it is useless for deciding the question.

The effect of increased temperature on the conductivity of electrolytes has been studied by many observers. For aqueous solutions the experiments over the largest range of temperature are those by Noyes and Coolidge [*]. These observers have studied the conductivity of solutions of sodium and potassium chloride up to 306° C. They find that for very dilute solutions ($\cdot 0005$ normal) the conductivity increases at a rate closely proportional to the increase of temperature. With more concentrated solutions the conductivity curves rise more slowly and attain a maximum, after which increase of temperature decreases the conductivity. The authors show that this is due to a decrease in the number of dissociated

* Noyes and Coolidge, Proc. Amer. Acad. of Arts & Sci. vol. xxxix. p. 160 (1903-94).

molecules with rise of temperature, which is greater the more concentrated the solution.

Experiments on the conductivity of solutions in other solvents show similar results. Walden and Centnerschwer [*] have investigated solutions in liquid SO_2 from $-76°$ C. to the critical point, $157°$ C. In all cases it was found that the conductivity first increases with rise of temperature and then falls off, following roughly a parabolic formula.

An increase of conductivity with rise of temperature is also found in the case of fused salts. The conduction through these is electrolytic, and the amount of the ions liberated by the current has been shown to follow Faraday's law of electrolysis [†]; but the increase of conductivity of fused salts with temperature is not nearly so great as that obtained in the case of heated metallic oxides. The increase of conductivity with temperature of all the oxides examined in the present paper is much more rapid than in any known case of electrolytic conductivity. On heating a slab of cupric oxide from $12°$ C. to $385°$ C., its conductivity increased 5000 times; and on heating lime from $763°$ C. to $1466°$ C. the conductivity increased to nearly 10^6 times its value at the lower temperature.

In cases of electrolytic conductivity the increase of conductivity with rise of temperature is due to the increased mobility of the ions at the higher temperature. As the oxides used in these experiments remained solid throughout the observations, it does not seem reasonable to suppose that the mobility of the current-carriers could have increased by the enormous amounts given above. The only alternative explanation is that the *number* of the carriers is enormously increased, a supposition which is not compatible with the idea that the conductivity is electrolytic; for in cases of electrolytic conductivity increase of temperature decreases the degree of ionization, the decrease being more rapid the more concentrated the solution.

If, on the other hand, we consider the conductivity to be metallic and the current to be carried by the negatively charged corpuscles which are wandering about in the interior of the oxide, formed by the dissociation of the metallic atoms: then the explanation is at once to hand, for Wehnelt's work has shown that the number of these corpuscles emitted by the oxide (and, therefore, probably the number contained by it)

* Walden and Centnerschwer, *Zeitschr. phys. Chem.* xxxix. p. 513 (1902).

† Lorentz, *Zeitschr. f. Elektroch.* vii. p. 277 (1900); p. 753 (1901).

increases rapidly with the temperature. Moreover, in the present paper it has been shown that the conductivity of the oxide increases in the same manner as the rate of emission of corpuscles from its surface.

The fact that mixtures of certain oxides conduct better than either of their constituents, is capable of explanation on the corpuscular theory of metallic conduction. For there is ample proof that the ease with which a corpuscle can escape from an atom depends on its surroundings. In an atom placed by itself and left to its own resources, a corpuscle has nothing to help it to escape except its own velocity; but if it is surrounded by other atoms, the other corpuscles exert forces on it and so help it to escape. A corpuscle, for instance, might escape from an atom in the solid or liquid state when it could not escape in the gaseous state. Liquid mercury is a good conductor of electricity, and if there are the same number of atoms in mercury vapour as in the liquid we should expect a much higher conductivity in the vapour on account of the larger free path of the corpuscles; but neither Maxwell nor Hittorf could find any trace of conductivity through hot mercury vapour. Professor Thomson * found that at high temperatures mercury vapour was a better insulator than air under similar conditions; while Strutt † has shown that up to a full red heat the conductivity of saturated mercury vapour is only about one ten-millionth part of that of the liquid. It seems not improbable, therefore, that there might, in certain cases, be many more free corpuscles in a mixture of two oxides than in either oxide separately. Even neglecting this influence of the surroundings on the freedom of the corpuscles, we should expect to obtain a mixture of better conductivity than either of its constituents by mixing two metallically conducting bodies, one containing a large number of free corpuscles, but in which the mobility of the corpuscles was small, and the other in which the mobility of the corpuscles was very great, but containing only a few. On mixing a little of the first substance with the second a good conducting mixture would be obtained.

I do not, therefore, think that Nernst's reasons for considering the conductivity of heated metallic oxides to be electrolytic are conclusive. The experiments described in this paper seem to show that by far the greater part of the current is carried by negatively charged corpuscles, as in metals. In the case of those oxides or mixtures of oxides in which a

* J. J. Thomson, Phil. Mag. [5] xxix. p. 364.
† Hon. R. J. Strutt, Phil. Mag. [6] iv. p. 596 (1902).

feeble polarization current or other signs of electrolysis have been observed, we have both metallic and electrolytic conductivity associated in the same substance.

These experiments were carried out at the Cavendish Laboratory, and I gladly take this opportunity of thanking Professor Thomson for his advice and interest during the investigation.

XLIV. *A Note on Talbot's Lines.*
By JAMES WALKER, *M.A., Oxford* *.

IT is sometimes felt that Stokes's explanation of the " polarity " of Talbot's lines is mathematical rather than physical, so that those who are unable to follow the analysis still require an adequate reason for the fact that the retarding plate must be inserted on the one side of the aperture rather than on the other.

Prof. Schuster† has indeed supplied this want by considering the source of light to be due to a succession of impulsive velocities ; but there is perhaps still room for an elementary explanation of the phenomenon on the old familiar lines of regarding the light as resolved into a congeries of monochromatic constituents.

1. When a stream of monochromatic light, coming from a distant slit, falls upon the object-glass of a telescope that is limited by a rectangular aperture with its sides parallel to the luminous line, it is easily shown that the diffraction-pattern in the focal plane of the telescope is characterised by a series of dark lines, arranged at equal distances on either side of the geometrical image of the slit. When half the aperture is covered by a retarding plate, the minima of an even order retain their former positions, but those of an odd order are displaced towards the side on which the plate is placed by an amount depending upon the retardation introduced by the plate.

Let us now suppose that the primitive light is white and that its monochromatic constituents have been separated before reaching the aperture by a prism or a grating, so that the different colours occupy different angular positions in the field of view.

The minima of an even order will now disappear ; for on account of the dispersion there will be a gradual shift of the

centre of the system and consequently of these minima on passing from one wave-length to the next[*].

The case of the minima of an odd order is, however, different; for as the wave-length alters, there is not only a shift due to the dispersion of the light, but also a dispersion due to the varying displacement caused by the change in the retardation introduced by the plate, and if these balance one another, the bands corresponding to different wave-lengths will be superposed and will therefore be rendered visible.

It is clear then that the plate must be so placed that the dispersion of the bands that it produces must oppose the dispersion of the light introduced by the prism or the grating. Thus the retardation of phase caused by the plate increasing with decreasing wave-length, the plate must be placed over the right or the left half of the aperture, according as the dispersion in the focal plane of the telescope from red to blue is from right to left or from left to right.

2. The following analysis of what takes place may perhaps tend to make the foregoing explanation somewhat clearer.

Let us suppose that the light, falling normally upon the aperture, is nearly monochromatic, as is the case when a small portion of the spectrum is considered, and that the slit of a spectroscope is placed in the focal plane of the telescope in a direction perpendicular to the bands of the diffraction pattern.

Then the light will be drawn out into a short spectrum, and the aperture being uncovered, this will be traversed by bands running along it, that are nearly straight and parallel to the sides of the spectrum (fig. 1).
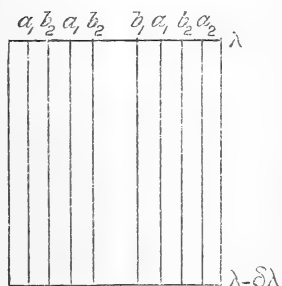
Fig. 1.   Fig. 2.
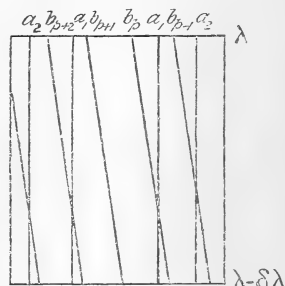
When half of the aperture is covered by the retarding plate, the bands (a) of an even order will retain their positions,

[*] The slight change in the distance between the minima as the wave-length alters, may be left out of account when a small region of the spectrum is considered.

but those (*b*) of an odd order will be shifted, and as the
amount of the displacement varies with the wave-length, will
be made to slant across the spectrum. Thus the right half
being that which is covered, the appearance will be that
represented diagrammatically in figure (2).

Now the effect of a dispersion of the light will be to move
each horizontal line of the spectrum in its own direction by
an amount depending upon its distance from the end—in
fact to give the figure a kind of shear. If this motion be to
the right, the result will be that both sets of bands will slant
across the spectrum, but if it be to the left, the bands (*a*)
will be made to slant, while the others (*b*) will be placed
more nearly along the spectrum and, the shift being properly
adjusted, will become exactly parallel to the sides.

Hence in the former case no bands will be seen when the
focal plane is viewed directly with an eyepiece, but in the
latter case the bands (*b*) will become visible.

3. The best thickness of the retarding plate for a given
part of the spectrum may now be determined.

It is obvious that the resultant disturbances in any direction
from the two halves of the aperture have the same amplitude,
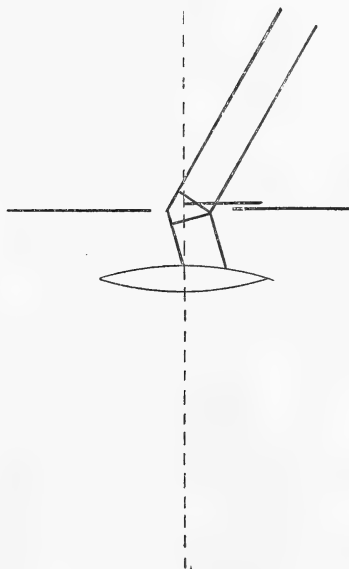
and that their phases are those of the secondary waves
emanating from the central elements. Since the source of
light is a line parallel to the sides of the aperture, directions
perpendicular to these sides alone need be considered, and if

ϕ be the angle of incidence, the disturbance in a direction θ may be represented by

$$A \cos\left(\frac{2\pi}{\lambda} vt - \Delta\right) + A \cos \frac{2\pi}{\lambda} \{vt - h(\sin\theta + \sin\phi)\}$$

$$= 2A \cos\left\{\frac{\pi}{\lambda}h \left(\sin\theta + \sin\phi\right) - \frac{\Delta}{2}\right\}$$

$$\times \cos\left[\frac{2\pi}{\lambda} \left\{vt - \frac{h(\sin\theta + \sin\phi)}{2}\right\} - \frac{\Delta}{2}\right],$$

$2h$ being the width of the aperture, Δ the retardation of phase introduced by the plate, and θ, ϕ being regarded as positive when measured on the side of the normal on which the plate is inserted.

Hence the intensity is

$$4A^2 \cos^2\left\{\frac{\pi}{\lambda}h(\sin\theta + \sin\phi) - \frac{\Delta}{2}\right\},$$

and there will be a primary series of minima given by $A=0$, being those due to a rectangular aperture of width h, and a secondary series of minima in directions given by

$$h(\sin\theta + \sin\phi) = \left\{\frac{\Delta}{\pi} \pm (2n-1)\right\}\frac{\lambda}{2}, \quad n = 1, 2, \ldots$$

Let the telescope be directed so that light of wave-length λ_0 is incident normally upon the aperture, and let p be the order of the secondary minimum next to the focal point. Then θ being very small,

$$h\theta = \left\{\frac{\Delta}{\pi} - (2p-1)\right\}\frac{\lambda_0}{2},$$

and for light of wave-length $\lambda_0 + \delta\lambda$

$$h(\theta + \delta\theta + \delta\phi) = \left\{\frac{\Delta}{\pi} - (2p-1)\right\}\frac{\lambda_0 + \delta\lambda}{2} + \frac{\lambda_0}{2\pi}\frac{d\Delta}{d\lambda_0}\delta\lambda$$

whence

$$= \left\{\frac{\Delta}{\pi} - (2p-1)\right\}\frac{\lambda_0}{2} + \frac{\lambda_0}{2\pi}\frac{d\Delta}{d\lambda_0}\delta\lambda,$$

$$h(\delta\theta + \delta\phi) = \frac{\lambda_0}{2\pi}\frac{d\Delta}{d\lambda_0}\delta\lambda.$$

Hence for coincidence of the bands due to the monochromatic constituents λ_0 and $\lambda_0 + \delta\lambda$, we must have

$$\frac{d\Delta}{d\lambda_0} = \frac{2\pi h}{\lambda_0}\frac{d\phi}{d\lambda_0},$$

which gives the best thickness of the plate.

Also $\frac{d\Delta}{d\lambda}$ and $\frac{d\phi}{d\lambda}$ must have the same sign: whence $\frac{d\Delta}{d\lambda}$ being negative, the angle of incidence must increase on the positive side of the normal with decreasing wave-length.

XLV. *Genesis of Ions by Collision and Sparking-Potentials in Carbon dioxide and Nitrogen. By* H. E. HURST, *B.A., B.Sc., Hertford College, Oxford* *.

A N explanation of the theory of ionization by collisions of positive and negative ions with molecules of a gas in a uniform field of force was given in a paper on the "Genesis of Ions in a Gas" (J. S. Townsend, Phil. Mag. Nov. 1903) and the experiments were continued in a further paper (Dec. 1904). The apparatus previously used for these experiments with air and hydrogen has recently been used to verify the theory in the case of nitrogen and carbon dioxide, and the results of these experiments are recorded in the present paper. A detailed description of the apparatus and of the method of conducting experiments will be found in the first-mentioned paper. Briefly, the apparatus consisted of two parallel plate electrodes, one of which was zinc. In the first experiments on nitrogen and those on carbon dioxide, the other electrode was a quartz plate coated on one side with silver. Fine lines were ruled on this to allow ultra-violet light, produced by a spark between aluminium terminals, to fall on the zinc electrode. In the second set of experiments on nitrogen, a zinc plate with fine slits was substituted for the silvered quartz. These plates can be arranged at distances from ·1 to 1·2 cm. apart. By means of an electrometer the currents between these plates were measured for different values of the electric intensity between them, their distance apart, and the pressure of the gas in which they were immersed. In the previous experiments the electrometer was kept at zero potential while the current was passing, by means of a condenser and potentiometer arranged as an induction balance. In the experiments here recorded the same effect was obtained by finding the rise of potential during half the time of an experiment, and then starting with the potential of the electrometer and electrode connected to it below zero by this amount. Hence during half the time of an experiment the potential-difference between the electrodes is slightly too large, and during the other half too small by nearly equal amounts. The same precautions with regard to insulation were taken as were described in the previous paper.

It has been shown that if n_0 negative ions are liberated from the zinc plate by the ultra-violet light, and travel through the gas under the action of the electric force between the plates, the number n which arrive at the positive electrode

* Communicated by Prof. J. S. Townsend, F.R.S.

is given by

$$n = n_0 \frac{(\alpha - \beta)\epsilon^{(\alpha-\beta)d}}{\alpha - \beta\epsilon^{(\alpha-\beta)d}},$$

where α is the number of new ions of one kind, either positive or negative, produced by a negative ion in travelling through 1 centimetre of the gas, β is the number produced by a positive ion in travelling through 1 centimetre, and d is the distance between the parallel plate electrodes. The values of α and β depend upon the electric intensity X between the plates, and also on the pressure p of the gas. They may be determined for any values of X and p, by measuring the currents for different distances between the electrodes, X and p being kept constant.

It was also shown that at a distance d, given by $\alpha - \beta\epsilon^{(\alpha-\beta)d} = 0$, sparking takes place when the difference of potential between the plates is Xd, when ultra-violet light falls on the negative electrode. If a is the value of d satisfying this equation

$$a = \frac{\log \alpha - \log \beta}{\alpha - \beta}.$$

The carbon dioxide used in these experiments was prepared by the action of pure hydrochloric acid on marble. Before entering the drying apparatus it passed through glass wool and over calcium chloride. It was dried by passing through sulphuric acid and standing over phosphorus pentoxide. In the following tables q is proportional to the currents determined experimentally, for fixed values of X and p, and different distances between the plates. The distance d between the plates is given in centimetres, and X the electric intensity in volts per centimetre. α and β are calculated so that the formula

$$n = n_0 \frac{(\alpha - \beta)\epsilon^{(\alpha-\beta)d}}{\alpha - \beta\epsilon^{(\alpha-\beta)d}}$$

shall be in agreement with the currents at three different distances. n_0 is taken as unity. Below the tables are given the values of a determined from the equation

$$a = \frac{\log \alpha - \log \beta}{\alpha - \beta},$$

Xa, and the sparking-potential V determined experimentally. It will be seen that the theoretical values of n agree with the values of the current q determined experimentally. In some of the tables β is too small to be determined with the

apparatus used. In order to determine β in these cases it would be necessary to have an apparatus which admitted of larger distances between the plates. When β is small the formula

$$n = n_0 \frac{(\alpha - \beta)\epsilon^{(\alpha-\beta)d}}{\alpha - \beta\epsilon^{(\alpha-\beta)d}}$$

reduces to $n = n_0\epsilon^{ad}$. When the sparking-distance a was more than $1\cdot2$ cm., it was not possible to determine the sparking-potential, since the plates could not be separated by more than that amount.

Carbon dioxide.

Pressure 2 mms.

	d	$\cdot1$	$\cdot2$	$\cdot3$
$X = 1400$	q	8·32	70·7	747
$\alpha = 21\cdot21$ $\beta = \cdot0085$	n	8·32	71·7	755

$a = \cdot369.$ $Xa = 516.$ $V = 517.$

Pressure 1 mm.

	d	$\cdot2$	$\cdot3$	$\cdot4$	$\cdot5$	$\cdot6$
$X = 700$	q	8·54	24·7	73·1	226	787
$\alpha = 10\cdot68$ $\beta = \cdot0041$	n	8·54	24·8	73·8	228	790

$a = \cdot736.$ $Xa = 515.$ $V = 509.$

Pressure 1 mm.

	d	$\cdot2$	$\cdot3$	$\cdot4$	$\cdot5$
$X = 875$	q	10·7	36·1	129	643
$\alpha = 11\cdot83$ $\beta = \cdot0138$	n	10·7	36	130	645

$a = \cdot571.$ $Xa = 500.$ $V = 495.$

Pressure 1 mm.

	d	·2	·3	·4
X=1050	q	13	51	289
$a = 12\cdot68$ $\beta = \cdot0353$	n	13	50·4	278

$a = \cdot465.$ $Xa = 488.$ $V = 491.$

Pressure ·5 mm.

	d	·2	·4	·6	·8
X=525	q	3·64	13·3	53·1	303
$a = 6\cdot41$ $\beta = \cdot0174$	n	3·64	13·3	52·9	301

$a = \cdot929.$ $Xa = 488.$ $V = 485.$

Pressure ·5 mm.

	d	·2	·3	·4	·5	·6
X=700	q	3·9	7·8	16·6	37·4	109·2
$a = 6\cdot74$ $\beta = \cdot06$	n	3·9	7·85	16·5	37·3	107

$a = \cdot706.$ $Xa = 494.$ $V = 497.$

Pressure ·5 mm.

	d	·2	·3	·4	·5
X=875	q	4·58	9·63	23·2	70·8
$a = 7\cdot5$ $\beta = \cdot079$	n	4·58	10·14	24·2	71·2

$a = \cdot613.$ $Xa = 537.$ $V = 530.$

Pressure ·25 mm.

	d	·2	·4	·6	·8
X=525	q	2·05	4·39	9·82	25·8
$\alpha = 3·56$ $\beta = ·089$	n	2·05	4·35	9·77	26·1

$a=1·062.$ $Xa=558.$ $V=564.$

Pressure 4 mms.

	d	·2	·3	·4	·5	·6
X=700	q	6·41	16·2	41	103	263
$\alpha = 9·28$ β small	$n=\epsilon^{\alpha d}$	6·41	16·2	41·2	104	262

Pressure 4 mms.

	d	·2	·4	·6	·8	1·0
X=350	q	1·64	2·72	4·41	7·28	11·7
$\alpha = 2·47$ β small	$n=\epsilon^{\alpha d}$	1·64	2·69	4·41	7·25	11·9

Pressure 2 mms.

	d	·2	·4	·6
X=700	q	9·6	95	883
$\alpha = 11·3$ β small	$n=\epsilon^{\alpha d}$	9·6	92·5	881

The values of α here obtained for the smaller values of $\frac{X}{p}$ agree very closely with those obtained by Prof. Townsend (Phil. Mag. April 1903). Some doubt has been expressed as to the validity of the experiments performed on carbon dioxide, as the gas is electrolysed by spark-discharge into

2 N 2

carbon monoxide and oxygen, some of the oxygen being set free as ozone, which would oxidize the zinc plate. In the experiments recorded above, all the currents q were measured before determining any spark-potentials, and if sparking took place during these experiments it was by accident, and was only momentary.

The largest currents used were of the order 10^{-10} ampere, so that if electrolysis takes place before a spark passes, an extremely small amount of gas is electrolysed. Hence the oxidation effect on the electrodes is small, and in the case of the experiments on air the only noticeable effect of oxidation was, that the currents produced by a given intensity of the light were proportionately less. The ratios of the currents were unaltered. That is to say, oxidation causes less ions to be given off by the zinc plate under the action of ultra-violet light. As to the change of constitution of the gas, the results of the experiments show that the spark-potential can be predicted from measurements of the currents before a spark passes, so obviously it is of no importance whether the gas remains as carbon dioxide or after passage of the current contains traces of ozone. If the gas were electrolysed in quantities the last experiments of a series should not be consistent with the first, but no measurable inconsistency was ever noticed.

The nitrogen used in the first set of experiments was obtained from air by passing air and ammonia over red-hot copper, the ammonia being in excess of that required to reduce the copper oxide formed. The nitrogen so produced was again passed over red-hot copper, and then over caustic potash into the drying apparatus, where it passed through sulphuric acid, and was finally dried by remaining in contact with phosphorus pentoxide.

The following tables give the results of experiments made on gas prepared by this method.

Nitrogen prepared by First Method.
Pressure 4 mms.

	d	$\cdot 2$	$\cdot 3$	$\cdot 4$	$\cdot 5$
X $= 700$	q	4·15	8·58	18·5	43·6
$a = 7\cdot08$ $\beta = \cdot049$	n	4·15	8·58	18·6	44·3

$a = \cdot708.$ $Xa = 496.$ $V = 494.$

Pressure 4 mms.

	d	·2	·4 ...	·6..	·8
X=525	q	2·16	4·77	10·8	25·7
$\alpha = 3\cdot83$ $\beta = \cdot033$	n	2·16	4·73 .	... 10·6	25·4

$$a = 1\cdot236. \qquad Xa = 649.$$

Pressure 2 mms.

	d	·2	·3	·4	·5
X=525	q	4·17	8·73	19·7	50·4
$\alpha = 7\cdot19$ $\beta = \cdot06$	n	4·17	9·07	20	49·7

$$a = \cdot67. \qquad Xa = 352. \qquad V = 349.$$

Pressure 2 mms.

	d	·2	·4	·6	·8	1·0
X=350	q	2·01	4·07	8·5	18·7	45·4
$\alpha = 3\cdot54$ $\beta = \cdot025$	n	2·01	4·16	8·73	18·8	44·7

$$a = 1\cdot41. \qquad Xa = 493.$$

Pressure 1 mm.

	d	·2	·4	·6	·7	·8
X=350	q	2·44	6·46	18·9	37·2	112
$\alpha = 4\cdot42$ $\beta = \cdot099$	n	2·44	6·30	18·8	37·4	107

$$a = \cdot88. \qquad Xa = 308. \qquad V = 303.$$

Pressure 1 mm.

	d	·2	·3	·4	·5
X=525	q	3·72	7·66	16·9	48·5
$a = 6·41$ $\beta = ·15$	n	3·72	7·55	16·95	48·1

$a = ·601.$ $Xa = 316.$ $V = 311.$

Pressure ·5 mm.

	d	·2	·4	·6	·8	1·0
X=262·5	q	1·92	3·82	7·61	17·1	49·5
$a = 3·23$ $\beta = ·074$	n	1·92	3·77	7·68	17·2	49·8

$a = 1·195.$ $Xa = 314.$

Pressure 4 mms.

	d	·2	·6	1·0
X=350	q	1·31	2·25	3·85
$a = 1·35$ β small	$n = \epsilon^{ad}$	1·31	2·25	3·86

Pressure 8 mms.

	d	·2	·6	1·0
X=350	q	1·03	1·12	1·18
$a = ·17$ β small	$n = \epsilon^{ad}$	1·03	1·10	1·18

Pressure 16 mms.

d	·1	·3	·5
X=700 q	1·03	1·075	1·17
$a = ·32$ β small $n = \epsilon^{ad}$	1·03	1·1	1·17

In order to test whether hydrogen was present in the gas in appreciable quantities, some more nitrogen was prepared by the same method, and its specific gravity carefully determined.

The mean of two experiments gave ·971 as its sp. gr., which was probably correct to ·2 per cent., as an error of 3 milligrams in weighing would only have produced 1 per cent. error in the result. The sp. gr. of pure nitrogen is ·972.

In the following experiments the nitrogen used was prepared by a similar method to that used by Lord Rayleigh for obtaining nitrogen from air. Air was passed through strong ammonia, over hot copper, through dilute sulphuric acid to remove excess of ammonia, then over hot copper oxide to a gas-holder, where it stood over water containing sulphuric acid. This gas was afterwards passed through a solution of caustic potash, and again over heated freshly reduced copper in a long tube, and copper oxide in another long tube. The copper oxide was kept red-hot, and the first portions of nitrogen were used to wash the receiver, which was exhausted each time. After an hour some nitrogen was collected and the receiver sealed. From the receiver the gas passed into the drying apparatus as before. Lord Rayleigh tested the efficacy of this method of removing hydrogen, by allowing a constant stream of hydrogen to mix with his gas before it passed over copper oxide, and found no hydrogen in the gas at the finish.

In order to avoid difficulties arising from the burning away of the silver on the positive electrode, a zinc plate perforated with fine slits was used. The following results were obtained.

Pressure 4 mms.

d	·2	·3	·4	·5	·6
X=700 q	4·2	8·86	19·2	44·2	123
$a = 7·12$ $\beta = ·044$ n	4·2	8·85	18·9	43·4	123

$a = ·72.$ $Xa = 502.$ $V = 507.$

Pressure 2 mms.

	d	·2	·3	·4	·5
X=525	q	3·94	7·95	17	42·7
$\alpha = 6·76$ $\beta = ·08$	n	3·94	8·05	17·2	41·9

$a = ·665.$ $Xa = 349.$ $V = 344.$

Pressure 1 mm.

	d	·2	·3	·4	·5
X=525	q	3·58	7·15	15·9	54·3
$\alpha = 6·18$ $\beta = ·194$	n	3·58	7·20	16·3	52·1

$a = ·58.$ $Xa = 304.$ $V = 300.$

A satisfactory agreement may be found between these tables and those for the nitrogen which was first used.

Besides these experiments a number of determinations of the spark-potential were made, for different values of the pressure and distance between the plates.

The effect of mixing small quantities of oxygen with the gas was also determined. The sparking-potentials were determined by connecting in series with the parallel plates a battery of small cells, a voltmeter, and a resistance-box.

In the experiments on carbon dioxide and the first sample of nitrogen the total external resistance was about 50,000 ohms, but in the later experiments on nitrogen 68,000 or 98,000 ohms. This increase of resistance raised the sparking-potential about 2 volts.

In all the determinations the lowest potential which gave a spark when ultra-violet light of small intensity fell on the negative electrode is given, and this is the potential V in the preceding tables. But in some of the experiments the potential required to give a spark without the action of ultra-violet light was also determined. After one determination had been made and a spark had passed, it was found necessary to wait a

few seconds, and in no case was sparking allowed to be more than momentary. It was found that unless these conditions were complied with, other phenomena occurred which made the determination of the sparking-potential uncertain. These phenomena were especially noticeable in the neighbourhood of the minimum sparking-potential, and at values of pd less than the critical value, p being the pressure and d the distance between the plates.

The results of all the experiments are collected in the following tables, in which are given the potentials required to produce sparks with ultra-violet light acting, the potentials required without light, and in some cases the potentials calculated for the distance

$$a = \frac{\log \alpha - \log \beta}{\alpha - \beta}.$$

Carbon dioxide.

Pressure. p	Distance between plates d.	pd.	Calculated potential. Xd.	Spark potential with light. V.	Spark potential without light.
mm.	cm.		volts.	volts.	volts.
2	·369	·738	516	517	
1	·736	·736	516	509	525
1	·571	·571	500	495	509
1	·465	·465	488	491	
·5	·929	·464	488	485	
·5	·706	·353	494	497	
·5	·613	·306	537	530	542
·25	1·062	·265	558	564	

Nitrogen prepared by first method.

p.	d.	pd.	Xd.	V (with light).	Spark potential without light.
mm.	cm.		volts.	volts.	volts.
4	·708	2·832	496	494	
4	1·236	4·944	649		
2	1·41	2·82	493		
2	·67	1·34	352	349	
1	·88	·88	308	303	305
1	·601	·601	316	311	311
·5	1·195	·597	314		

Nitrogen prepared by second method.

p.	d.	pd.	Xd.	V (with light).	Spark potential without light.
4	·72	2·88	502	507	512
4	·708	2·832		504	504
4	·6	2·4		460	464
2	·58	1·16		332	334
2	·65	1·30		339	343
2	·665	1·33	349	344	346
2	·85	1·70		382	384
2	·67	1·34		329	345
1	·4	·4		330	336
1	·5	·5		310	314
1	·58	·58	304	300	304
1	·601	·601		298	303
1	·868	·868		300	306
1	1·0	1·0		310	314

Mixture of Nitrogen and 1·1 per cent. of Oxygen.

p.	d.	pd.	Xd.	V (with light).	Spark potential without light.
4	·72	2·88		516	518
4	·708	2·832		514	520
4	·6	2·4		472	476
2	·58	1·16		335	343
2	·65	1·30		342	350
2	·85	1·70		392	398
2	·665	1·330		341	349
1	·601	·601		...	302

Mixture of Nitrogen and 5 per cent. Oxygen.

p.	d.	pd.	Xd.	V (with light).	Spark potential without light.
1	·4	·4		369	371
	·5	·5		334	338
	·58	·58		320	324
	·601	·601		...	320
	·65	·65		316	320
	·70	·70		312	316
	·8	·80		316	320
	·868	·868		320	324
	1·0	1·0		324	332

The sparking-potentials for nitrogen do not agree with those obtained by Strutt (Phil. Trans. 193, p. 377, 1900). He found that the sparking-potential of nitrogen varied considerably with different samples of gas, although prepared by the same method. This he attributed to the presence of

traces of oxygen, since if the gas was passed repeatedly through a liquid alloy of sodium and potassium, a constant minimum potential of 251 volts was obtained. The volume of the apparatus used was small, and before determining a sparking-potential the gas was sparked through vigorously. If the apparatus had previously been used for the experiments on hydrogen, it is possible that the sparking would drive out some hydrogen occluded in them, which in a small volume might form an appreciable percentage of the gas present. This would tend to lower the spark-potential and make its determination uncertain. In the determinations of spark-potentials described in this paper it was found that in the neighbourhood of the minimum spark-potential, the spark-potential was sometimes affected to the extent of 6 or 8 volts by the previous passage of a current through the gas, and before the potential could be again determined some time must elapse if the results were to be concordant, so that all potentials were determined without sparking having taken place immediately before.

It will be seen by comparison of the tables given for nitrogen prepared by the methods mentioned, that there is not a great difference between the results obtained in both cases. The tables of sparking-potentials show also that a small percentage of oxygen only produces small differences in the sparking-potentials, and that 5 per cent. produces more effect than 1 per cent. These facts seem to show that small quantities of impurities do not produce disproportionate effects. If this be true, the variations in the spark-potential of nitrogen are not produced by minute traces of impurity but are due to other causes. Further experiments in this direction are to be attempted shortly.

The accompanying curves, figs. 1 and 2, show the relation between the sparking-potential and the product of the pressure of the gas p and the distance between the plates d, for carbon dioxide and nitrogen. As the spark-potential determined with ultra-violet light falling on the negative electrode was in most cases nearly the same as that determined without the light, one curve only has been drawn in each case, through the points representing the potentials obtained with the light.

The theoretical values of the spark-potential calculated from the formula before given are also indicated.

Fig. 1.—Carbon dioxide.
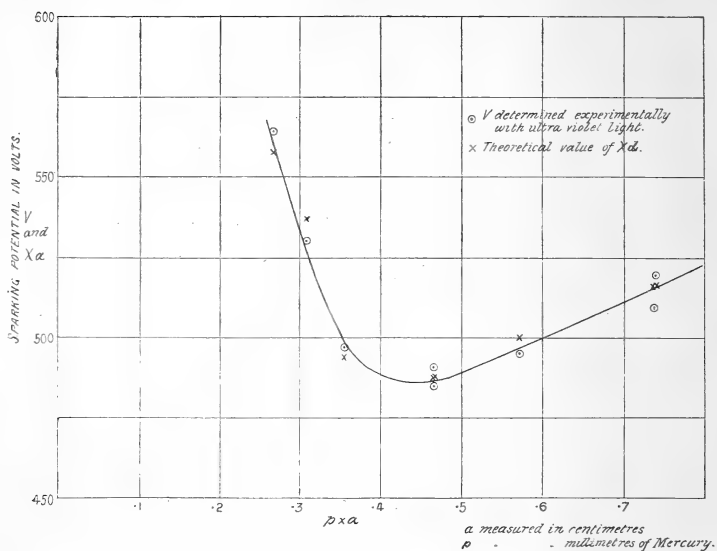
Fig. 2.—Nitrogen.
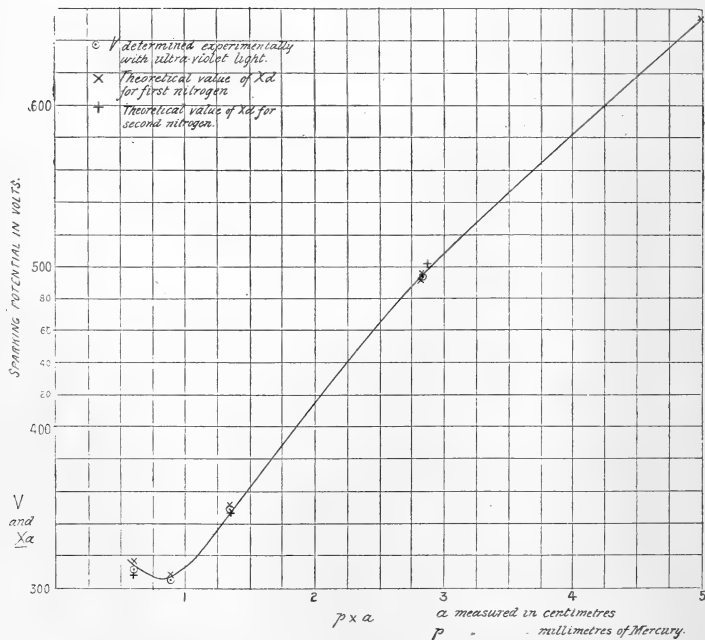
Fig. 3 shows the curves obtained with the different samples of nitrogen and also the effect of small quantities of oxygen.

Fig. 3.

pressure × distance pd.
p measured in mms. of mercury.
d measured in cms.
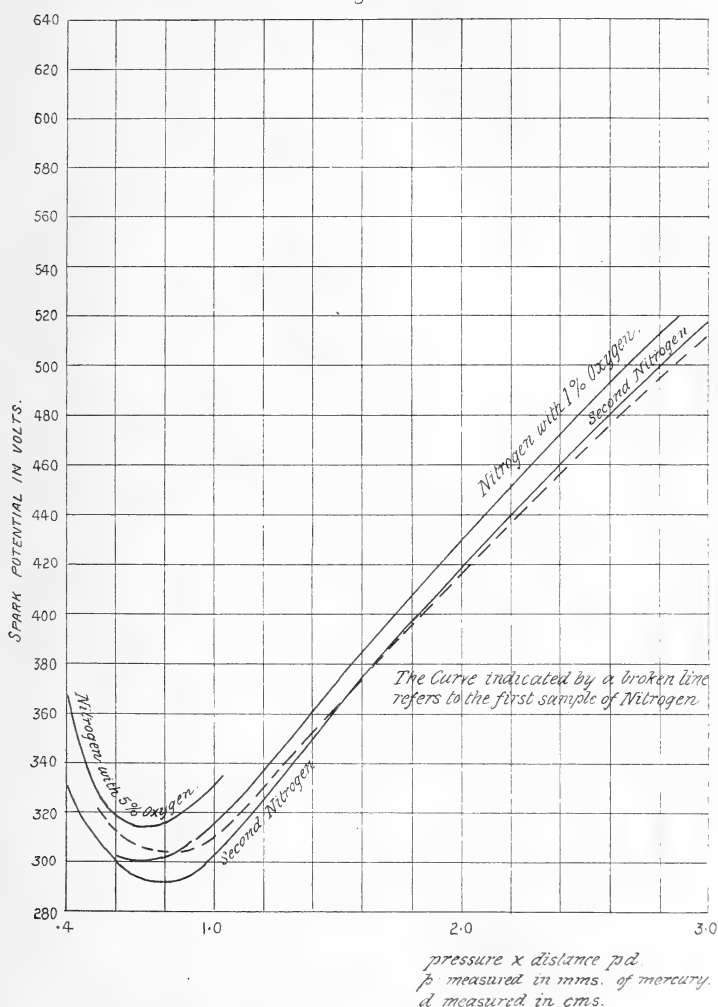
The theoretical investigations are intended to apply to cases where the pressure is not less than the critical pressure, and it will be noticed that the effect of small impurities is negligible at the higher pressures.

The minimum spark-potential for the first nitrogen is about 302 volts, for the second about 295 volts. The gas containing

5 per cent. of oxygen gives a minimum spark-potential of about 312 volts.

In the tables before given it will be noticed that the theoretical values of n, calculated from the formula

$$n = n_0 \frac{(\alpha - \beta)\epsilon^{(\alpha-\beta)d}}{\alpha - \beta\epsilon^{(\alpha-\beta)d}},$$

agree with the values of the current q determined experimentally, so that the theory furnishes an explanation of the currents which take place in a gas before sparking occurs. In preceding papers on this subject it was shown that $\frac{\alpha}{p}$ and $\frac{\beta}{p}$ are functions of $\frac{X}{p}$, and the accompanying curves, figs. 4 and 5, show the relations between these variables. In

Fig. 4.
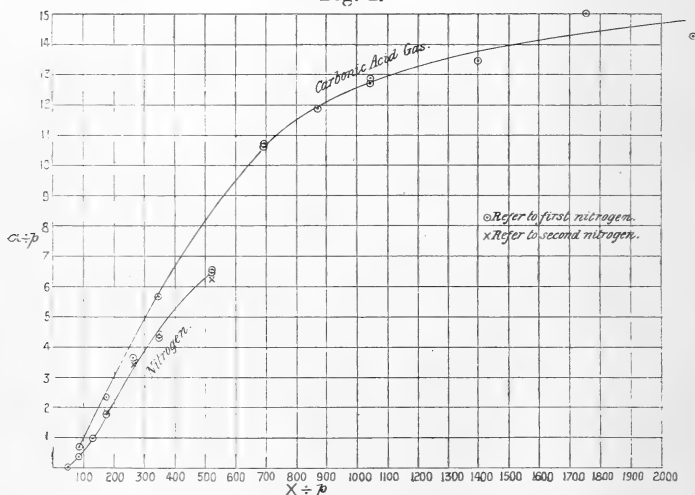
those for nitrogen, the curves relate to the first set of experiments, and the values for the second experiments were put in after the curves were drawn. It will be seen that the discrepancy between the two sets is comparable with the errors of experiment, so that we shall give the following conclusions drawn from the results of the first experiments, since it seems that the presence of small quantities of impurities only produces small errors. The larger values of α can be represented by the formula $\frac{\alpha}{p} = N\epsilon^{-\frac{NVp}{X}}$

(J. S. Townsend, Phil. Mag. April 1903), obtained on the assumption that new ions are formed by collision when the colliding ion has a velocity due to a fall of potential $\not< V$.

N is the average number of collisions of a negative ion in moving through a centimetre of gas at 1 mm. pressure. In

Fig. 5.
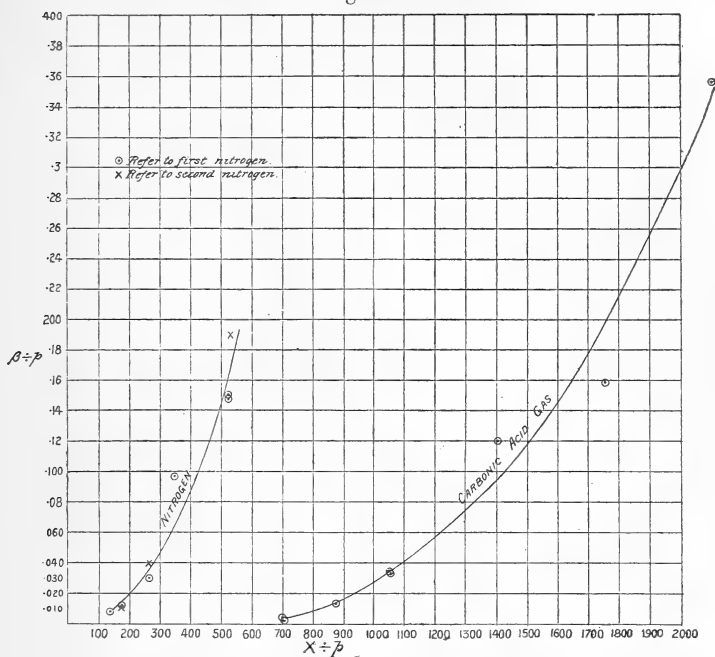
the case of carbon dioxide when $N = 17 \cdot 4$ and $V = 18 \cdot 8$ volts, the values of α calculated from the above formula agree with those obtained from the curve for values of $\dfrac{X}{p}$ between 2000 and 350. For nitrogen the numbers $N = 13 \cdot 6$ and $V = 28 \cdot 5$ give values of α agreeing with experiment for values of $\dfrac{X}{p}$ between 175 and 525. For the smaller values of $\dfrac{\alpha}{p}$ it is necessary to suppose that some new ions are formed when the velocity on collision is less than that due to the critical fall of potential V.

The values of N and V for nitrogen do not differ much from those for air given in the paper previously mentioned, and the curve there given falls between those for nitrogen and carbon dioxide. It is not possible to represent $\dfrac{\beta}{p}$ within the range of experiments performed by a formula of the type $N \epsilon^{-\frac{NVp}{X}}$. Perhaps, as in the case of the smaller

values of $\dfrac{\alpha}{p}$, one could be obtained on the hypothesis that some collisions after velocities less than a critical velocity are productive of new ions. This view obtains some support from the fact that the curves for $\dfrac{\beta}{p}$ are very similar to those portions of the $\dfrac{\alpha}{p}$ curves corresponding to the smaller values of $\dfrac{X}{p}$.

It will be noticed that the values of α for CO_2 and nitrogen are of the same order of magnitude, but that β is much smaller in CO_2 than in nitrogen. It has been shown that all ions have the same charge. Now, if we assume that a positive ion is approximately of the same size as the molecules of the gas in which it is formed, and a negative ion very small compared with these, a positive ion in travelling a given distance will make about four times as many collisions as a negative ion in travelling the same distance. Therefore a negative ion in a gas at 1 mm. pressure will make approximately the same number of collisions per centimetre as a positive ion in the same gas at ·25 mm. pressure. In CO_2 at 1 mm. pressure the number of collisions made by a negative ion in going 1 cm. is 17·4, and when the electric intensity is 175, α is 2·4. Now at $\frac{1}{4}$ mm. pressure β is ·001 for the same force. So that for an equal number of collisions under the same conditions, a negative ion makes about 2400 times as many new ions in CO_2 as a positive ion. In nitrogen a negative ion would make about 17·4 collisions per centimetre at a pressure of 1·28 mms. For the same electric intensity 175, α is 1·28. At ·32 mm. pressure the number of collisions of a positive ion in nitrogen would be approximately the same as in the cases already considered, and for intensity 175, β is about ·058. Under these conditions then, a negative ion in nitrogen produces 22 times as many new ions as a positive ion in going a given distance. Hence the positive ion in nitrogen differs less from the negative ion than does the positive ion in CO_2. Since all negative ions are the same, this will be the case if negative ions are small compared with positive ions, and positive ions in nitrogen smaller than those in carbon dioxide. There is a considerable difference between the values of β for nitrogen and those for air, β being larger in nitrogen than in air. From this it would seem that β for oxygen is small.

In conclusion, I wish to express my thanks to Professor Townsend for his advice and assistance during the course of the experiments, and also to Dr. H. B. Baker for his criticism of some points connected with the preparation of the gases.

XLVI. *The Retardation of the Velocity of the α Particles in passing through Matter.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN the Philosophical Magazine of July 1905, I gave an account of some preliminary experiments on the retardation of the velocity of the α particles from radium C in passing through matter.

Using an active wire coated with radium C as a homogeneous source of radiation, the velocity of the α particles was found to decrease in passing through aluminium, and the lowest value of the velocity observed was $\cdot 64V_0$, where V_0 is the initial velocity of projection of the α particles from the bare wire.

I have recently repeated these observations under much better experimental conditions, using more active wires, and with the photographic plate closer to the source of rays. I have, in consequence, been able to measure the velocity of the α particles after their passage through a greater thickness of aluminium. The lowest value of the velocity which could be measured was $\cdot 43V_0$. The photographic effect observed on the plate was very feeble, and certainly less than 4 per cent. of that found for the unscreened wire. The thickness of aluminium in this case was equal in absorbing power to almost 7 cms. of air—the maximum range of the ionization of the α particle from radium C.

As I pointed out in my previous paper, such a result suggests that the ionization produced by the α particle falls off rapidly when its velocity falls below a certain critical value. The deduction from experiments of this kind are, however, complicated by a scattering of the α rays in their passage through matter. This is initially small, but increases with the diminution of the velocity of the α particle.

I have determined the value of e/m for the α particle from radium C after passing through a screen equal in absorbing power to 5·5 cms. of air, and found it the same as for the α particle from the bare wire. This experiment shows that the α particle retains its charge and mass unaltered over a greater part of its range in air.

Further experiments are in progress to see if it is possible to detect by special methods the presence of the α particles after passing through an absorbing screen equal to 7 cms. of air. It is hoped that, by such methods, it may be possible to

settle definitely whether this critical velocity of the α particle is only apparent or has a real existence.

As these experiments will take some time to complete, I have taken this opportunity of stating briefly the results so far obtained.

<div style="text-align:center">I am, Gentlemen,</div>

McGill University, Montreal,   Yours faithfully,
    February 27, 1906.   E. RUTHERFORD.

XLVII. *On the Inversion-Points for a Fluid passing through a Porous Plug and their use in Testing Proposed Equations of State. By* ALFRED W. PORTER, *B.Sc., Fellow of, and Assistant-Professor of Physics in, University College, London* [*].

IN a paper in the Phil. Mag. vol. xlv. p. 227 (1898), Rose-Innes showed that the results obtained by Joule and Kelvin for the change in temperature of a gas in passing irreversibly through a porous plug could be represented as well by the formula $A - \dfrac{B}{T}$ as by the formula adopted by Joule and Kelvin, viz. : $\dfrac{A}{T^2}$.

Rose-Innes' formula, if a valid empirical formula, would indicate that a point of inversion must exist ; that is to say, that there must exist a temperature below which cooling will occur, while heating takes place if the gas is above this temperature.

The fact proved later that by sufficient cooling hydrogen, which heats at ordinary temperatures, cools at lower temperatures, demonstrates that for it an inversion-point exists; and still later (*Krakauer Protocolle,* 1901 ; Drude's *Annalen,* 1902) Olszewski has experimentally shown that hydrogen suffers neither heating nor cooling when the temperature is $-80°\cdot5$ C. and the pressures on the two sides of the plug are about 113 atmospheres and that of the atmosphere.

The object of the present paper is to examine the subject more minutely ; and the outcome of it is that it is shown that any of the usually adopted equations of state of a gas require that there shall be not one inversion-point only, but a whole series of such points corresponding to all possible pressures.

Instead of dealing with finite changes of pressure we shall

consider the difference of pressure on the two sides of the plug as being infinitesimal; considerable simplification results from doing this. Of course, it must be recognized that this condition does not hold good for the circumstances of the Olszewski experiment, the discussion of which is undertaken in section (*d*).

The equation characteristic of the cooling-effect is

$$T\left(\frac{\partial v}{\partial t}\right)_p - v = C^p\left(\frac{\partial T}{\partial p}\right)_{(U+pv)},$$

where v=specific volume, p=pressure, T=temperature, C_p the specific heat at constant pressure, and U the intrinsic energy of the fluid.

If no change of temperature takes place for an infinitesimal change of pressure

$$T\left(\frac{\partial v}{\partial T}\right)_p - v = 0 ;$$

a relation which is independent of the calorimetric properties of the fluid. For any given equation of state this equation completely determines the inversion-point corresponding to each temperature. We proceed to connect it with van der Waals's equation and with an equation of Dieterici.

(a) *Van der Waals's Equation.*

We will write the equation in its *reduced* form, *i. e.*, pressures, volumes, and temperatures will be expressed as fractions α, β, and γ of their critical values. The results obtained will then be the same for every fluid obeying this equation owing to the applicability of the law of corresponding states.

The reduced equation is

$$\left(\alpha + \frac{3}{\beta^2}\right)(3\beta - 1) = 8\gamma,$$

and the equation characteristic of the inversion-points becomes

$$\gamma\left(\frac{\partial \beta}{\partial \gamma}\right) - \beta = 0,$$

or

$$\frac{-8\gamma}{3\beta - 1} + \frac{6}{\beta^2}(3\beta - 1) = 0 ;$$

whence the inversion temperature is given by the equation

$$\gamma = \frac{3(3\beta - 1)^2}{4\beta^2}.$$

2 O 2

From this and the previous equation γ can be eliminated, with the result

$$\alpha = \frac{9}{\beta^2}(2\beta - 1).$$

This formula connects the reduced pressure and volume which correspond to an inversion-point. The simplest mode of calculation is to obtain γ for a series of assumed values of β, and then to calculate α by means of the last equation.

The curve obtained by plotting γ against α is shown in the following diagram (fig. 1).

Fig. 1.

Reduced Inversion-points according to van der Waals's Equation.
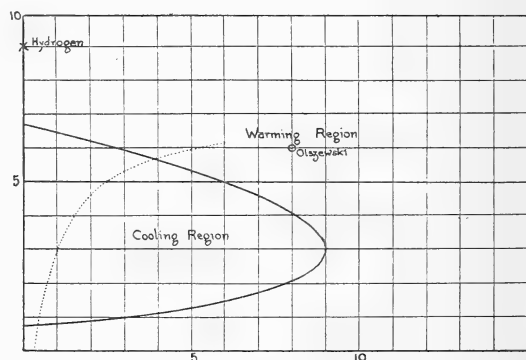
| Continuous curve | { Ordinates—Reduced Temperature.
Abscissæ—Reduced Pressure. |
| Dotted curve | { Ordinates—Reduced Temperature.
Abscissæ—Reduced Volume. |

The form of this curve involves the following consequences:

(1) For all pressures from zero to nine times the critical pressure there are two inversion temperatures which range from a little below the critical temperature to about 6·7 times the critical value.

(2) At pressures higher than nine times the critical value there is no inversion-point.

The curve separates the regions for which cooling and warming occur from one another. The cooling region is of

very limited extent. It corresponds to positive values of

$$T \frac{\partial v}{\partial t} - v.$$

On the same figure is plotted the reduced temperature against the reduced volume corresponding to the series of inversion-points; whence it can be seen that the inversion-point which corresponds to the maximum pressure occurs when the volume is equal to the critical volume, the temperature being three times and the pressure nine times their critical values.

Olszewski's so-called inversion-point is inserted in the same figure (reduced pressure = 8, reduced temperature 5·95). It is obvious from the diagram that in his experiment the gas during its expansion must first of all have risen in temperature and afterwards cooled down to the same value as at first. The point obtained is therefore only a fictitious inversion-point. We shall return to a consideration of this question later on in this paper.

(b) *Dieterici's Equation.*

In Drude's *Annalen* for 1901 (v. p. 51), Dieterici published an equation of state, viz.:

$$p = \frac{RT}{v-b} \epsilon^{\frac{-A}{RTv}},$$

which fits remarkably well throughout a very wide range both above and below the critical point. Indeed, it is the most satisfactory of all which contain three constants only.

The law of corresponding states is true for this equation, and when reduced it becomes

$$\alpha = \frac{\gamma \epsilon^{2\left(1-\frac{1}{\gamma\beta}\right)}}{2\beta-1}.$$

The reduced temperature of the inversion-point is given in terms of the reduced volume by the equation

$$\gamma = \frac{4(2\beta-1)}{\beta};$$

and it is given in terms of the reduced pressure by the equation

$$\alpha = (8-\gamma)\epsilon^{\frac{5\gamma-8}{2\gamma}}.$$

The values of α and β corresponding to various inversion-points are tabulated below :—

Reduced temperature of inversion-point.	Reduced volume of inversion-point.	Reduced pressure of inversion-point.
0	·5	0
1	4/7	$7\epsilon^{-\frac{3}{2}} = 1\cdot561$
2	2/3	$6\epsilon^{\frac{1}{2}} = 9\cdot894$
3	4/5	$5\epsilon^{\frac{7}{6}} = 16\cdot11$
4	1	$4\epsilon^{\frac{3}{2}} = 17\cdot93$
5	4/3	$3\epsilon^{\frac{17}{10}} = 16\cdot422$
6	2	$2\epsilon^{\frac{11}{6}} = 12\cdot47$
7	4	$\epsilon^{\frac{27}{14}} = 6\cdot89$
8	∞	0

These numbers are plotted on fig. 2.

Fig. 2.

Reduced Inversion-points according to Dieterici's Characteristic Equation.

Continuous curve { Ordinates —Reduced Temperature.
                  { Abscissæ—Reduced Pressure.
Dotted curve { Ordinates—Reduced Temperature.
             { Abscissæ—Reduced Volume.
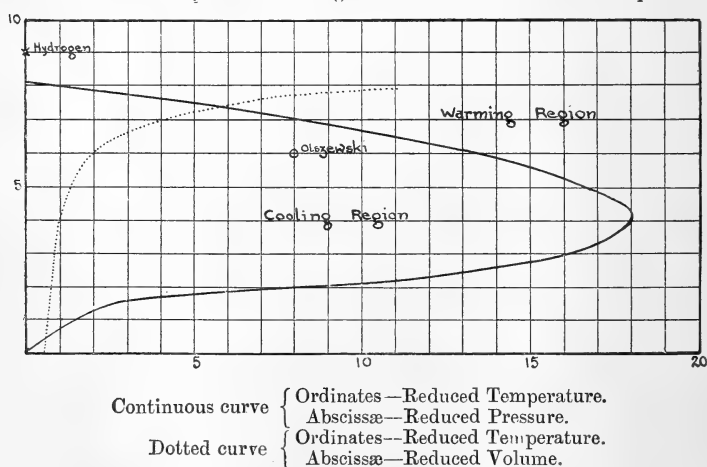
(c) *General Discussion.*

The curves which we have obtained corresponding to different equations of state both exhibit the same general

consequences with regard to the series of inversion-points. At the same time they exhibit the extreme sensitiveness of phenomena depending upon the Joule-Kelvin effect in discriminating between different equations of state. On each of them are shown the points representing hydrogen (H) under usual conditions (*i. e.* about 20° C. and a few atmospheres pressure). This point for hydrogen is calculated from the values of the critical temperature recently determined by Olszewski, viz., 32°·2 absolute (Drude's *Annalen,* 1905, x. p. 986). It will be seen that both van der Waals's and Dieterici's equations given above indicate a heating of hydrogen in expanding at ordinary temperatures. The inversion-point corresponding to a few atmospheres' pressure is about 6·7 times the critical temperature according to van der Waals's equation, *i. e.* at about 216° abs. ; while according to Dieterici's it is at 8 times the critical temperature, *i. e.* at about 258° abs. Thus for moderate pressures, if the latter equation holds good, hydrogen should undergo cooling at temperatures much higher than has hitherto been thought. An accurate determination of this temperature for various pressures will discriminate between these two equations, even at points in the neighbourhood of atmospheric temperature and pressure, in a way which no direct measurements of p, v, and t can ever be expected to do.

Dieterici had previously published another equation similar to van der Waals's, but in which the molecular pressure term is given by $a/v^{\frac{5}{2}}$ instead of a/v^2. This equation holds good near the critical point only, at which point it is very satisfactory.

The law of corresponding states applies and the reduced values for the inversion-points are easily shown to be

$$\gamma = \frac{4(4\beta - 1)^2}{9\beta^{\frac{5}{3}}},$$

$$\alpha = \frac{16(5\beta - 2)}{3\beta^{\frac{5}{3}}}.$$

The same general conclusions can be derived from these values. The equation does not profess, however, to have an equally general validity as the others, and consequently will not be further discussed.

It is a notable fact with regard to each of these equations that the maximum pressure which gives any inversion-point is that for which the volume is the critical volume. A further connexion is shown between any inversion-curve and the corresponding equation of state by the following considerations.

Consider any two points A and B in the same vertical on inversion-curves, such as figs. 1 or 2 ; for these the pressure is the same. They will therefore correspond to two points A′ and B′ on a constant-pressure line upon a v-T diagram.

Fig. 3.
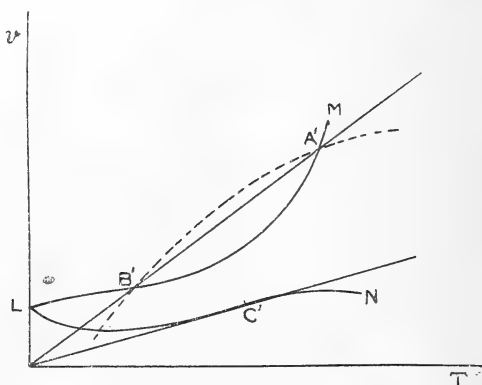
But for both A and B, $T^2 \dfrac{\partial}{\partial T}\left(\dfrac{v}{T}\right) = 0$. Therefore a curve $v = Tf(p)$, obtained by integrating this equation, passes through A′ and B′ on the v-T diagram ; the form of this curve is immaterial to the present argument. That is to say, the two inversion volumes at a given pressure are proportional to the corresponding inversion temperatures, since $f(p)$ is the same for both ; or A′ and B′ lie on a straight line passing through the origin as shown in fig. 3.

For the same reason, if there were n inversion temperatures for the same pressure, the constant-pressure line on a v-T diagram would be cut n times by a line through the origin.

Again, in general for any fluid we may write

$$T^2 \frac{\partial}{\partial T}\left(\frac{v}{T}\right) = F(p,\, T),$$

and $F(p,\, T) = 0$ for inversion-points. But the maximum pressure for which inversion occurs corresponds to

$$\frac{dp}{dT} = 0,$$

or

$$-\frac{\partial F}{\partial T}\Big/\frac{\partial F}{\partial p} = 0\,;$$

which can only be when $\dfrac{\partial F}{\partial T} = 0$, excluding infinite values of $\dfrac{\partial F}{\partial p}$.

By differentiation we get

$$\frac{\partial F}{\partial T} = \frac{\partial}{\partial T}\left(T^2 \frac{\partial}{\partial T}\left(\frac{v}{T}\right)\right) = T\frac{\partial^2 v}{\partial T^2}.$$

Hence the maximum inversion-pressure corresponds to $\dfrac{\partial^2 v}{\partial T^2} = 0$ for the corresponding constant-pressure line on a v-T diagram, that is to say, it corresponds to a point of inflexion on such a diagram.

(d) *Discussion of Olszewski's Observation.*

In Olszewski's determination of his inversion-point for hydrogen a large fall of pressure was used; and consequently his point does not correspond in definition with those referred to in this paper. The criterion of all points such as his is that the temperature does not fall *on the whole* in the actual flow, even though the difference of pressure is large, and this corresponds to the condition

$$U + pv = \text{constant},$$

where U is the intrinsic energy of the fluid. If then it is possible to obtain the values of $U + pv$ for various temperatures and plot constant-pressure values against these temperatures, the resulting curves will necessarily intersect one another, the point of intersection of any two constant-pressures lines representing the inversion-point in Olszewski's sense. The value of pv can be determined for some fluids from Amagat's determinations; but the value of U involves calorimetric data as well, and these are very imperfectly known.

For a substance obeying van der Waals's equation

$$dU = dQ - pdv = C_v dT + \left(T\frac{\partial p}{\partial T} - p\right)dv$$

$$= C_v dT + \frac{a}{v^2}dv,$$

and

$$\frac{\partial C_v}{\partial v} = T\frac{\partial^2 p}{\partial T^2} = 0.$$

Consequently $C_v =$ a function of temperature alone.

and
$$U - U_0 = f(T) - \frac{a}{v} + \frac{a}{v_0}$$

where v_0 is conveniently taken as the volume for which T is zero, viz. b.

Hence for a van der Waals's fluid $U + pv$ can be calculated, with the exception of a part of it which is a function of temperature alone, without knowing the specific heat at constant volume.

Writing the above in terms of critical volume and temperature, and adding pv, *i. e.* $\frac{\alpha\beta}{9}\frac{a}{b}$, we have

$$U - U_0 + pv - F(\gamma T_c) = \frac{a}{b}\left(\frac{1}{3\beta} - 1 + \frac{\alpha\beta}{9}\right).$$
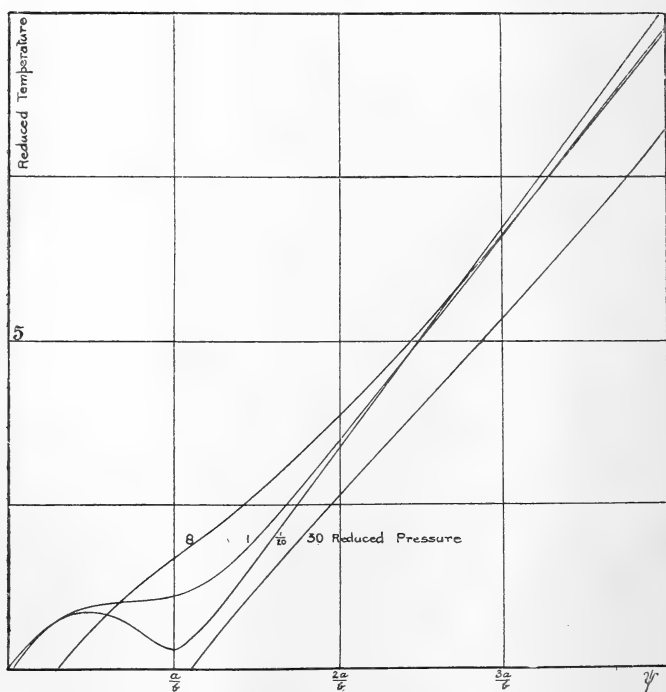
Fig. 4.

The quantity on the right-hand side (ψ) is plotted horizontally against reduced temperature vertically in fig. 4, and the fact is brought out in this diagram that successive constant-

pressure lines do not intersect at one point; all within a certain range must therefore have a caustic curve as envelope, and it should be noted that each one of these meets the caustic twice. The actual calculation of the caustic is cumbrous, and as no particular importance attaches to it the calculation has not been made. In Olszewski's experiment with hydrogen the initial pressure was eight times the critical pressure, and the final was atmospheric, *i. e.* roughly $\frac{1}{20}$ the critical value. These constant-pressure lines are shown, amongst others, on the diagram; their intersection can be calculated more accurately by successive approximations from the equation. When this calculation is made the point is found to be at about 5·8 times the critical temperature.

Now Olszewski's experimental value is nearly six times the critical temperature. There is thus very good agreement between these two results. Inspection of the diagram (fig. 1) shows that while the gas decreased in pressure as it flowed from one side to the other of the throttle, it must (if van der Waals's equation were valid) at first have heated until the pressure had fallen to about three times the critical ; in the subsequent part of the expansion it must have cooled by an equal amount, thus bringing about zero change on the whole.

On the other hand, inspection of the curve of inversion‧points corresponding to Dieterici's equation shows no possibility of this compensation occurring; for Olszewski's observed point lies *within* the cooling region. Thus the evidence afforded by this single experimental value is much more in favour of the validity of van der Waals's equation *in the region involved* than of that of either of the two other equations.

(e) *Ramsay and Young Fluids.*

For any fluid obeying Ramsay and Young's linear law we have

$$p = BT - A,$$

where B and A are functions of the volume alone. All the three equations which we have considered satisfy this law ; it is therefore of interest to find the general equation for the inversion-points of such a fluid. We at once obtain

$$T\frac{\partial v}{\partial T} - v = \frac{-T\left(B + v\frac{dB}{dv}\right) + v\frac{dA}{dv}}{T\frac{dB}{dv} - \frac{dA}{dv}}.$$

This is zero whenever

$$T = \frac{v\dfrac{d\mathrm{A}}{dv}}{\mathrm{B} + v\dfrac{d\mathrm{B}}{dv}} = \frac{v\dfrac{d\mathrm{A}}{dv}}{\dfrac{d}{dv}(\mathrm{B}v)},$$

unless T is the critical point at which

$$T\frac{d\mathrm{B}}{dv} - \frac{d\mathrm{A}}{dv} = 0.$$

Another example of a characteristic equation following Ramsay and Young's law is the five-constant equation of Rose-Innes. In this equation

$$\mathrm{B} = \frac{\mathrm{R}}{v}\left(1 - \frac{e}{v + \kappa - g/v^2}\right), \quad \mathrm{A} = \frac{l}{v(v + \kappa)},$$

where e, κ, g, l are constants, and R is the ordinary gas constant.

From these we obtain

$$v\frac{d\mathrm{A}}{dv} = -\frac{vl(2v + \kappa)}{(v + \kappa)^2 v^2}$$

and

$$\frac{d(\mathrm{B}v)}{dv} = -\frac{\mathrm{R}e(1 + 2g/v_3)}{(v + \kappa - g/v^2)^2}.$$

The inversion-points are therefore given by

$$T_i = \frac{2l}{\mathrm{R}e} \cdot \frac{1 + \kappa/2v}{1 + 2g/v^3}\left\{1 - \frac{g}{v^2(v + \kappa)}\right\}^2.$$

Hence when v is very large the inversion-point approaches the value $2l/\mathrm{R}e$; while as v diminishes T_i *at first increases* since $\kappa/2v$ is the dominant small term.

For isopentane the values of the constants as given by Rose-Innes (Phil. Mag. vol. xliv. p. 81) are

$$l = 5420800, \quad \kappa = 3{\cdot}636,$$
$$e = 7{\cdot}473, \quad g = 6{\cdot}2318,$$
$$\mathrm{R} = 1/{\cdot}001158.$$

The critical constants as calculated from these data and as observed by Young are

Critical temperature	. .	191°·7 C., 187·8 C.
„ pressure	. . .	26250 mm.
„ volume	. . .	4·5, 4·266 cm³. per gr.

The value of T_i for infinite volume is therefore 1666 absolute, that is about 3·6 times the calculated critical

temperature. For temperatures higher than this value it is easy to show that there would be a heating effect. Hence we have in this case a curve of inversion-points whose upper portion rises with increasing pressure instead of diminishing as it would do according to the other equations, and is at the same time much lower if we take the evidence afforded by observations on isopentane.

The fact that the slope of the inversion-point curve corresponding to Rose-Innes' equation is for large volumes of opposite sign to that of the other equations, suggested the idea that possibly the equation rigorously valid for an actual gas might lead to a curve parallel to the pressure axis; this would imply a *single inversion-point for all pressures*. It is easy to obtain an equation for which such would be the case. For, provided that the fluid satisfies Ramsay and Young's linear law, the condition that there shall be one inversion-point only is that

$$v \frac{dA}{dv} \bigg/ \frac{d}{dv}(Bv) = \text{constant} = T_i;$$

or

$$T \frac{d(Bv)}{dv} = v \frac{dA}{dv}.$$

For the sake of illustration we will follow Rose-Innes in taking $A = \dfrac{l}{v(v+\kappa)}$, and adapt his value of B to satisfy the above condition; then

$$T_i \frac{d(Bv)}{dv} = -\frac{(2v+\kappa)l}{v(v+\kappa)^2}.$$

Integrating this equation we obtain

$$T_i Bv = -\frac{l}{\kappa} \left\{ \log v - \log(v+\kappa) - \frac{\kappa}{v+\kappa} - \text{const.} \right\};$$

and the equation of state becomes

$$p = \frac{l}{\kappa T_i} \frac{T}{v} \left[\text{const.} + \log \frac{v+\kappa}{v} + \frac{\kappa}{v+\kappa} \right] - \frac{l}{v(v+\kappa)}.$$

The value of $l \times \text{const.}/\kappa T_i$ is R, the ordinary gas constant.

How nearly such an equation, having one inversion-point only, is capable of representing the behaviour of isopentane is evident from fig. 5, where pv for the critical isotherm is plotted against the cube root of the reciprocal of the volume. The circles represent experimental values, while the dotted line represents the above equation with values of the constants chosen so as nearly to satisfy four of the experimental values.

The equation to this isotherm, with the numerical values of the coefficients inserted, is

$$pv = 397930 + 3892300 \log_{10} \frac{v + 1 \cdot 870}{v} - \frac{5554710}{v + 1 \cdot 870}.$$

Fig. 5.—Critical Isotherm of Isopentane.
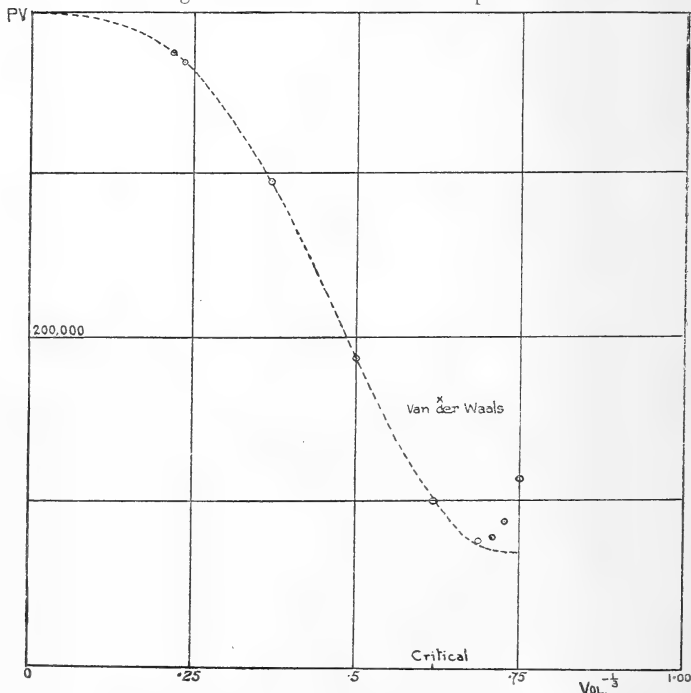
Fluid satisfying dotted curve would have one inversion-point only.
Observed values ⊙.

The comparison between experimental and calculated values may be more easily made by means of the following table.

v.	$v^{\frac{1}{3}}$.	pv observed.	p observed.	p calculated. Porter.	p calculated. Dieterici.
2·4	·7503	117790	49080	29627	42730
3	·6931	79380	26460	25433	26780
4	·6299	100080	25020	25020	25320
8	·5	189680	23710	23778	23400
20	·3684	296800	14840	14752	14560
80	·2321	369120	4614	4614	4604
100	·2154	375000	3750	3748	3740

Thus, from the last two columns, it is seen that this equation, which only indicates one inversion-point, actually fits the experimental values better than Dieterici's second equation throughout a range from large values of v to values below the critical volume which is $4\cdot266$. Below this, however, it is hopelessly out of competition. At the critical volume, however, it is far superior to van der Waals's equation, as is shown by the cross on the diagram, which gives the value of pv for the critical point as calculated from van der Waals's. The constant inversion-point comes out as $1\frac{1}{3} \times$ critical temperature for isopentane; but it should be noted that the equation does not satisfy the law of corresponding states.

(f) *General Condition for the uniqueness of the Inversion-Point.*

We have worked out the special case of a Ramsay and Young fluid. In the general case where no restriction is introduced it must be possible to write

$$\frac{\partial}{\partial T}\left(\frac{v}{T}\right) = (T-\kappa)\,f(p, T),$$

where $f(p, T)$ is a function of p and T which never vanishes unless when $T=\kappa$, and which is not infinite when $T=\kappa$. The constant inversion-point is of course $T=\kappa$.

Integrating this equation, we obtain

$$\frac{v}{T} = (T-\kappa)\frac{\partial y}{\partial T} - y + \psi(p)$$

$$= (T-\kappa)\frac{\partial}{\partial T}\left(\frac{y}{T-\kappa}\right) + \psi(p)$$

as the most general equation of state which gives a unique inversion-point. In this expression y is restricted to be a function of p and T, whose first two differential coefficients have no zero unless when $T=\kappa$, and no infinity when $T=\kappa$. Further, $\psi(p)$ is any function of p alone.

Conclusion.

The chief points brought out in this paper are the following :—

(1) The value of the inversion temperature of the Joule-Kelvin effect is probably a function of the pressure; and even for the same pressure two inversion-points, in general, may exist.

(2) Different equations of state, all of which are fair approximations to the behaviour of a real gas, indicate very different values for these inversion temperatures.

(3) The sensitiveness of the positions of these points to change in the characteristic equation of the fluid makes a knowledge of their actual position, as determined experimentally, a very valuable means of discriminating between the relative validity of any proposed equations of state.

In conclusion attention may be drawn to the fact that all the results given in this paper are *exact* consequences of the equations of state to which they relate.

XLVIII. *Some Ellipsoidal Potentials, Æolotropic and Isotropic.* By R. HARGREAVES, *M.A.*[*]

IN a former paper on Æolotropic Potential, the potential functions corresponding to normal distribution on a conducting ellipsoid and to uniform volume-distribution were treated. The functions are $\int_\lambda^\infty \dfrac{d\lambda}{\sqrt{J}}$ and $\int_\lambda^\infty \dfrac{(1-u_a)\,d\lambda}{\sqrt{J}}$, where J is the cubic in λ which replaces the isotropic form $(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$, and $u_a=1$ represents the æolian quadric which replaces a confocal. If the additional factor $\dfrac{\partial u_a}{\partial x}$ occurs under the integral sign a vector type is given, which was also considered in the second, but not in the first case.

It is proposed now to deal with the more general types

$$\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}\,[u_a^s, \quad \text{or} \quad u_a^s(1-u_a), \quad \text{or} \quad (1-u_a)^s],$$

and with types in which the factor $\dfrac{\partial u_a}{\partial x}$ is attached to the several forms. The second and third forms belong to volume distributions with densities depending on powers of u_a; the first has also a normal surface-density.

The energies attaching to these potentials take the simple form of numerical multiples of that of a conductor. They are obtained by the use of what (in the paper cited) were called wave-forms of the potential. Two theorems are required for the purpose: one to connect the wave-forms with

those written above, and the other to deal with moments and products of inertia of any even order for an ellipsoid. The evaluation of energy turns on a similarity in the forms of the two theorems, and they prove to be closely related.

A potential is then considered which is a linear function of members of the second type, namely,

$$\int_\lambda^\infty (1-u_a)(k_1 + k_2 u_\alpha + k_3 u_\alpha^2 + \ldots k_n u_\alpha^{n-1}) \frac{d\lambda}{\sqrt{J}}.$$

The energy is a quadric in the k's, persymmetric in form ; while the whole charge is a linear function of the k's. For any value of n this potential may be determined so as to give a minimum of energy subject to constancy of total charge. The minima have simple values which decrease as n increases, ranging from that for uniform volume-distribution ($n=1$) to that for normal distribution on a conductor ($n=\infty$). Thus the potentials form an interesting series of links connecting the two potentials commonly considered with reference to an ellipsoid ; and each potential, with a distribution depending on n constants, in some measure simulates that of a structure containing n parts brought into relation by a minimum condition.

To give greater generality the work is written in æolotropic form ; but most of the energies evaluated vary as $\int_0^\infty \frac{d\lambda}{\sqrt{J}}$, and the only difference between isotropic and æolotropic forms consists in the relation of the constants in the cubic J to the axes of the ellipsoid.

§ 1. Of the three forms *

$$\psi_s = \frac{\rho}{4} \int_\lambda^\infty \frac{u_\alpha^s d\lambda}{\sqrt{J}}, \quad U_s = \frac{\rho}{4} \int_\lambda^\infty \frac{u_\alpha^s (1-u_\alpha) d\lambda}{\sqrt{J}},$$

$$V_s = \frac{\rho}{4} \int_\lambda^\infty \frac{(1-u_\alpha)^s d\lambda}{\sqrt{J}}, \quad . \quad . \quad (1)$$

the first will be taken as fundamental, results for U_s and V_s being directly deducible from those for ψ_s. Since u_a may be written $=1$ when it is not under the sign of integration, we have

$$\frac{d\psi_s}{dx} = -\frac{\rho}{4\sqrt{J}} \frac{d\lambda}{dx} + \frac{s\rho}{4} \int_\lambda^\infty u_\alpha^{s-1} \frac{\partial u_a}{\partial x} \frac{d\lambda}{\sqrt{J}}.$$

---

* The notation is that of the previous paper on ' Æolotropic Potential ' in Phil. Mag. April 1905, quoted as Æ.

Thus the surface-discontinuity or density is the same for all the functions ψ and is that of ψ_0; while potentials U_s and V_s containing differences of powers of u_a have only a volume-density. A second differentiation gives

$$\frac{d^2\psi^s}{dx\,dy} = -\frac{\rho}{4\sqrt{J}}\left(\frac{d^2\lambda}{dx\,dy} - \frac{\dot{J}}{2J}\frac{d\lambda}{dx}\frac{d\lambda}{dy}\right) - \frac{s\rho}{4\sqrt{J}}\frac{\partial u_a}{\partial x}\frac{d\lambda}{dy}$$

$$+ \frac{s\rho}{4}\int_\lambda^\infty \left\{2\gamma' u_a^{s-1} + \overline{s-1}\,u_a^{s-2}\frac{\partial u_a}{\partial x}\frac{\partial u_a}{\partial y}\right\}\frac{d\lambda}{\sqrt{J}}.$$

When $\nabla_e^2\psi_s$ is formed, the sum of terms such as those in the first bracket vanishes, and the total outside the sign of integration is $-s\rho/\sqrt{J}$. Under the sign of integration the bracket gives

$$2u_a^{s-1}\Sigma(p\alpha + 2p'\alpha') - 4(s-1)u_a^{s-2}\dot{u}_a,$$

or

$$2u_a^{s-1}\dot{J}/J - 4(s-1)u_a^{s-2}\dot{u}_a \text{ quoting Æ. (76) and (86)}.$$

Hence the integral

$$= -s\rho\int_\lambda^\infty \frac{\partial}{\partial\lambda}\left(\frac{u_a^{s-1}}{\sqrt{J}}\right)d\lambda = s\rho\,u_a^{s-1}/\sqrt{J}.$$

Externally the lower limit is variable, and the two terms are cancelled in virtue of $u_a = 1$; thus $\nabla_e^2\psi_s = 0$. Internally the lower limit is constant and $= 0$, and there are no terms outside the integral sign; since $\lambda = 0$ makes $J = 1$, $u_a = u_a$, the integral is $s\rho u_a^{s-1}$, that is $\nabla_e^2\psi_s = s\rho\,u_a^{s-1}$ and the volume-density is $-s\rho\,u_a^{s-1}$. The surface-density σ

$$= -\left[l\left(p\frac{d\psi_0}{dx} + r'\frac{d\psi_0}{dy} + q'\frac{d\psi_0}{dz}\right) + m\left(r'\frac{d\psi_0}{dx} + q\frac{d\psi_0}{dy} + p'\frac{d\psi_0}{dz}\right)\right.$$

$$\left. + n\left(q'\frac{d\psi_0}{dx} + p'\frac{d\psi_0}{dy} + r\frac{d\psi_0}{dz}\right)\right]$$

$$= \frac{\rho}{4}\left[l\left(p\frac{d\lambda}{dx} + r'\frac{d\lambda}{dy} + q'\frac{d\lambda}{dz}\right) + \dots\right]$$

$$= \frac{\rho\varpi}{8}\left[\frac{\partial u_a}{\partial x}\left(p\frac{d\lambda}{dx} + r'\frac{d\lambda}{dy} + q'\frac{d\lambda}{dz}\right) + \dots\right] = \frac{\rho\varpi}{2},$$

ϖ being a perpendicular from the centre on a tangent plane and $l\,m\,n$ direction-cosines of a normal.

Since $\dfrac{d\psi_s}{dx} = \dfrac{d\psi_0}{dx} + \dfrac{\partial\psi_s}{\partial x}$, and $\dfrac{d\psi_s}{dx}$ and $\dfrac{d\psi_0}{dx}$ are external

potentials, so also is $\dfrac{\partial \psi_s}{\partial x}$. It is more convenient to use an independent notation for the vector type, and we take

$$\chi_s = \int_\lambda^\infty \frac{u_a^s (\alpha x + \gamma' y + \beta' z) d\lambda}{\sqrt{J}} . \quad . \quad . \quad (2)$$

Since

$$\frac{d\chi_s}{dx} = -\frac{\rho}{4\sqrt{J}}(\alpha x + \gamma' y + \beta' z)\frac{d\lambda}{dx}$$

$$+ \frac{\rho}{4}\int_\lambda^\infty \left[\alpha\, u_a^s + s u_a^{s-1}\,(\alpha x + \gamma' y + \beta' z)\frac{\partial u_a}{\partial x} \right]\frac{d\lambda}{\sqrt{J}},$$

the surface-discontinuity is the same as for χ_0, and the surface-density is $\dfrac{\rho \varpi}{2}(ax + c'y + b'z)$. The volume-density is $-s\rho(ax + c'y + b'z)u_a^{s-1}$ as appears from the connexion with ψ_{s+1}; or by work following closely the lines of Æ. p. 442 we show that

$$\nabla_e^2 \chi_s = -s\rho(\alpha x + \gamma' y + \beta' z)/\sqrt{J}$$

$$-s\rho\int_\lambda^\infty \frac{\partial}{\partial \lambda}\left\{ \frac{u_a^{s-1}(\alpha x + \gamma' y + \beta' z)}{\sqrt{J}} \right\} d\lambda$$

$$= -s\rho(\alpha x + \gamma' y + \beta' z)/\sqrt{J} + s\rho\, u_a^{s-1}(\alpha x + \gamma' y + \beta' z)/\sqrt{J},$$

from which

$\nabla_e^2 \chi_s = 0$ externally and $= s\rho\,(ax + c'y + b'z)\,u_a^{s-1}$ internally.

§ 2. We proceed to the formulæ by which the above are transferred to wave-forms. In the latter it is more convenient to work with $\mu = \lambda\Delta_a$ and $\Delta(\mu) = \Delta_a^2 J(\lambda)$, identity in final results following from $d\mu/\sqrt{\Delta(\mu)} = d\lambda/\sqrt{J}$. The fundamental formula for transference is

$$\frac{\Delta_a^s}{2\pi}\int \frac{(lx + my + nz)^{2s}\, d\omega}{u_p(u_\Lambda + \mu u_p)^{s+1/2}} = \int_\mu^\infty \frac{u_a^s d\mu}{\sqrt{\Delta(\mu)}} = \int_\lambda^\infty \frac{u_a^s d\lambda}{\sqrt{J}} = \psi_s, \quad (3)$$

and a particular case is

$$\frac{\Delta_a^s}{2\pi}\int \frac{(lx + my + nz)^{2s}\, d\omega}{u_p\, u_\Lambda^{s+1/2}} = \int_0^\infty \frac{u_a^s d\lambda}{\sqrt{J}}; \quad . \quad . \quad (3\,b)$$

the general case applies to external potentials, the particular to internal potentials, the latter specially required for energy. Now (3) is an immediate integral, with regard to μ, of

$$\frac{1}{4\pi}\int \frac{(lx + my + nz)^{2s}\, d\omega}{(u_\Lambda + \mu u_p)^{s+3/2}} = \frac{1}{2s+1}\left(\frac{u}{\Delta_a}\right)^s \frac{1}{\sqrt{\Delta(\mu)}}$$

$$= \frac{u_{\Lambda(\mu)}^s}{(2s+1)\{\Delta(\mu)\}^{s+1/2}}, \quad . \quad (4)$$

2 P 2

the particular case with $\mu = 0$ being

$$\frac{1}{4\pi} \int \frac{(lx + my + nz)^{2s}\, d\omega}{u_{\mathrm{A}}^{s+3/2}} = \frac{u_a^s}{(2s+1)\Delta_a^{s+1}} : \quad . \quad (4\,b)$$

u_a and u_α on the right hand have $(x\,y\,z)$ for variables, u_{A} and u_p on the left have $(l\,m\,n)$. We shall deduce (4) from

$$\frac{1}{4\pi} \int \frac{d\omega}{(u_{\mathrm{A}} + \mu u_p)^{3/2}} = \frac{1}{\sqrt{\Delta(\mu)}}, \quad . \quad . \quad . \quad (5)$$

an integral obtained indirectly in the previous paper, and first give an independent proof of (5). The volume integral

$$\iiint dx\, dy\, dz \Big/ \sqrt{\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2}} = 2\pi\alpha\beta\gamma,$$

where the range of integration is the volume contained by the ellipsoid $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$, is well known. Put $(x,\, y,\, z) = r\,(l,\, m,\, n)$, and use a polar element of volume $r^2 dr d\omega$, where $d\omega$ is an element of area on a unit sphere ; then integrate with regard to r, the boundary value being $1 = \mathrm{R}^2\Big(\frac{l^2}{\alpha^2} + \frac{m^2}{\beta^2} + \frac{n^2}{\gamma^2}\Big)$. The result is

$$\frac{1}{4\pi} \int d\omega \Big/ \Big(\frac{l^2}{\alpha^2} + \frac{m^2}{\beta^2} + \frac{n^2}{\gamma^2}\Big)^{3/2} = \alpha\beta\gamma,$$

or

$$\frac{1}{4\pi} \int d\omega / (al^2 + bm^2 + cn^2)^{3/2} = \frac{1}{\sqrt{abc}}.$$

If the ellipsoid is referred to general axes, then $u_a\,(l,\, m,\, n)$ takes the place of $al^2 + bm^2 + cn^2$, and abc must be replaced by Δ_a. If again for a we write $\mathrm{A} + \mu p$, for a', $\mathrm{A}' + \mu p'$, then Δ_a is replaced by $\Delta(\mu)$, a determinant with the elements $\mathrm{A} + \mu p, \ldots$ and this is (5).

The step to (4) is made by using an operator

$$\Sigma\Big(x^2\frac{d}{d\mathrm{A}} + yz\frac{d}{d\mathrm{A}'}\Big) \text{ say } \mathrm{O}.$$

Since

$$\frac{d}{d\mathrm{A}}\,u_{\mathrm{A}}(l,\, m,\, n) = l^2, \quad \frac{d}{d\mathrm{A}'}\,u_{\mathrm{A}} = 2mn, \ldots$$

we have

$$\mathrm{O}\,.\,u_{\mathrm{A}}\,(l,\, m,\, n) = (lx + my + nz)^2.$$

And since

$$\frac{d}{dA}\Delta(\mu)=A(\mu)=\frac{\alpha\Delta(\mu)}{\Delta_a}, \text{ and } \frac{d}{dA'}\Delta(\mu)=2A'(\mu)=\frac{2\alpha'\Delta(\mu)}{\Delta_a},\dots$$

we have

$$O\,.\,\Delta(\mu)=u_{A(\mu)}(x,y,z)=\frac{u_a\Delta(\mu)}{\Delta_a}.$$

Also $u_{A(\mu)}$ written at length is

$$\Sigma x^2\{(BC-A'^2)+\mu(qC+rB-2p'A')+\mu^2P\}$$
$$+2\Sigma yz\{B'C'-AA'+\mu(q'C'+r'B'-pA'-p'A)+\mu^2P'\},$$

and therefore

$$\left.\begin{aligned}
\frac{d}{dA}u_{A(\mu)}&=(C+\mu r)y^2+(B+\mu q)z^2-2(A'+\mu p')yz,\\
\frac{1}{2}\frac{d}{dA'}u_{A(\mu)}&=(C'+\mu r')zx+(B'+\mu q')yx-(A+\mu p)yz-(A'+\mu p')x^2.
\end{aligned}\right\}\ (6)$$

When the whole operator is used all terms are cancelled, and we have the three results

$$\left.\begin{aligned}
O\,.\,u_A(l,m,n)&=(lx+my+nz)^2,\quad O\,.\,\Delta(\mu)=u_{A(\mu)}(x,y,z)=\frac{u_a\Delta(\mu)}{\Delta_a},\\
\text{and}\qquad O\,.\,u_{A(\mu)}(x,y,z)&=0.
\end{aligned}\right\}\ (7)$$

Hence the operation applied to the first and third members of (4) yields a repetition of (4) with $s+1$ for s; and (4) follows by repeated use of the operator starting from (5).

The formula of transference when $\alpha x+\gamma'y+\beta'z$ appears under the integral sign is got from (4) by use of the operator

$$O_x'\equiv x\frac{d}{dA}+\frac{y}{2}\frac{d}{dC'}+\frac{z}{2}\frac{d}{dB'},$$

for which

$$\left.\begin{aligned}
O_x'\,.\,u_A&=l(lx+my+nz),\quad O_x'\,.\,\Delta(\mu)=\frac{\Delta(\mu)}{\Delta_a}(\alpha x+\gamma'y+\beta'z),\\
O_x'\,.\,u_{A(\mu)}&=0.
\end{aligned}\right\}\ (8)$$

This gives

$$\frac{1}{4\pi}\int\frac{l(lx+my+nz)^{2s-1}\,d\omega}{(u_A+\mu u_p)^{s+3/2}}=\frac{u_a^{s-1}(\alpha x+\gamma'y+\beta'z)}{(2s+1)\Delta_a^s\sqrt{\Delta(\mu)}}.\quad (9)$$

Integrate with regard to μ from μ to ∞ and raise s by 1, then

$$\frac{\Delta_a^{s+1}}{2\pi}\int\frac{l(lx+my+nz)^{2s+1}\,d\omega}{u_p(u_A+\mu u_p)^{s+3/2}}=\int_\mu^\infty\frac{u_a^s(\alpha x+\gamma'y+\beta'z)\,d\mu}{\sqrt{\Delta(\mu)}},\ (10)$$

and in particular

$$\frac{\Delta_a^{s+1}}{2\pi} \int \frac{l(lx+my+nz)^{2s+1}\,d\omega}{u_p\,u_A^{s+3/2}} = \int_0^\infty \frac{u_a^s\,(\alpha x+\gamma'y+\beta'z)d\mu}{\sqrt{\Delta(\mu)}} \quad (10b)$$

§ 3. The fundamental integral for moments of inertia in an ellipsoid is

$$\frac{1}{\tau_0} \int (lx+my+nz)^{2s}\,d\tau = \frac{3}{(2s+1)(2s+3)} \left\{ \frac{u_A(l,m,n)}{\Delta_a} \right\}^s \quad (11)$$

The coefficients of like power arrangements in $l\,m\,n$ are equal on the two sides, so that all moments and products of inertia of degree $2s$ are comprised in the formula. A way of expressing the results for individual terms, which also applies to (4), will be given later ; but for our main purpose the form (11) and some collateral results are required. In immediate connexion with (11) are

$$\frac{1}{\tau_0} \int (lx+my+nz)^{2s}\,u_a^{s'}\,d\tau = \frac{3}{(2s+1)(2s+2s'+3)} \left(\frac{u_A}{\Delta_a}\right)^s \quad (11\,b)$$

and the surface integral

$$\frac{1}{\tau_0} \int (lx+my+nz)^{2s}\,\varpi dS = \frac{3}{2s+1} \left(\frac{u_A}{\Delta_a}\right)^s, \quad . \quad (11\,c)$$

where ϖ is a perpendicular from the centre on a tangent plane. Write $x, y, z = r(\lambda, \mu, \nu)$, use a polar element of volume $r^2 dr\,d\omega$ and integrate to the surface value

$$1 = R^2\,u_a(\lambda,\ \mu,\ \nu).$$

For (11)

$$\int (lx+my+nz)^{2s}\,d\tau = \iint r^{2s+2}(l\lambda+m\mu+n\nu)^{2s}\,dr\,d\omega$$

$$= \frac{1}{2s+3} \int R^{2s+3}\,(l\lambda+m\mu+n\nu)^{2s}\,d\omega;$$

for (11 b) $\quad \displaystyle\int (lx+my+nz)^{2s}\,u_a^{s'}\,d\tau$

$$= \frac{1}{2s+2s'+3} \int R^{2s+2s'+3}(l\lambda+m\mu+n\nu)^{2s}\,u_a^{s'}(\lambda,\mu,\nu)d\omega$$

$$= \frac{1}{2s+2s'+3} \int R^{2s+3}\,(l\lambda+m\mu+n\nu)^{2s}\,d\omega;$$

and for (11 c)

$$\int (lx+my+nz)^{2s}\,\varpi dS = \int R^{2s+3}\,(l\lambda+m\mu+n\nu)^{2s}\,d\omega,$$

since $\varpi\,dS = R^3\,d\omega$.

Thus the three forms all depend on one angular integral which, in view of the value of R, and $\tau_0 = 4\pi/3\sqrt{\Delta_a}$, is

$$\frac{1}{4\pi} \int \frac{(l\lambda + m\mu + n\nu)^{2s}\, d\omega}{\{u_a(\lambda, \mu, \nu)\}^{s+3/2}} = \frac{\{u_A(l, m, n)\}^s}{(2s+1)\Delta_a^{s+1/2}}.$$

This integral is a variant of $(4\,b)$ got by writing a for A.

A modification of (11) by a factor $ax + c'y + b'z$ is required for the χ functions. Operate with $a\dfrac{d}{dl} + c'\dfrac{d}{dm} + b'\dfrac{d}{dn}$ on (11), and raise s by 1, thus

$$\frac{1}{\tau_0} \int (ax + c'y + b'z)(lx + my + nz)^{2s+1}\, d\tau$$

$$= \frac{3l}{(2s+3)(2s+5)} \left(\frac{u_A}{\Delta_a}\right)^s; \quad . \quad . \quad (12)$$

then

$$\frac{1}{\tau_0} \int (ax + c'y + b'z)(lx + my + nz)^{2s+1} u_a^{s'}\, d\tau$$

$$= \frac{3l}{(2s+3)(2s+2\,s'+5)} \left(\frac{u_A}{\Delta_a}\right)^s, \quad (12\,b)$$

and

$$\frac{1}{\tau_0} \int (ax + c'y + b'z)(lx + my + nz)^{2s+1}\, \varpi\, dS$$

$$= \frac{3l}{2s+3} \left(\frac{u_A}{\Delta_a}\right), \quad . \quad . \quad . \quad . \quad (12\,c)$$

are connected with (12) in the manner shown above for $(11\,b)$ and $(11\,c)$.

§ 4. The evaluation of energy now follows by combining the typical integrals, thus :—

$$\frac{1}{\tau_0} \int u_a^{s'}\, d\tau \int_0^\infty \frac{u_a^s\, d\lambda}{\sqrt{J}} = \frac{1}{\tau_0} \int u_a^{s'}\, d\tau \int \frac{\Delta_a^s(lx + my + nz)^{2s}\, d\omega}{2\pi u_p u_A^{s+1/2}} \quad \text{by } (3\,b)$$

$$= \frac{3}{(2s+1)(2s+2s'+3)} \int \frac{d\omega}{2\pi u_p u_A^{1/2}} \quad \text{by } (11\,b)$$

$$= \frac{3\phi_0}{(2s+1)(2s+2s'+3)}. \quad . \quad . \quad . \quad (13)$$

using ϕ_0 as in Æ. (135) for

$$\int_0^\infty \frac{d\lambda}{\sqrt{J}} = \int_0^\infty \frac{d\mu}{\sqrt{\Delta(\mu)}} = \int \frac{d\omega}{2\pi u_p u_A^{1/2}}.$$

We may also write (13) as

$$\int u_a^{s'} \psi_s\, d\tau = \frac{3\rho\tau_0 \phi_0}{4(2s+1)(2s+2s'+3)}, \quad . \quad . \quad (13\,b)$$

and the surface integral is

$$\int \psi_s \varpi \, dS = \frac{3\rho\tau_0\phi_0}{4(2s+1)}. \quad . \quad . \quad . \quad (13\,c)$$

Now the potential $\psi_{s'}$ has a surface-density $\rho\varpi/2$, and a volume-density $-s'\rho u_a^{s'-1}$. The term in energy due to the interaction of ψ_s and $\psi_{s'}$ is therefore

$$E(s, s') = \tfrac{1}{2}\int \rho\varpi\psi\,dS - \int s'\rho u_a^{s'-1}\psi_s\,d\tau$$

$$= \frac{3\rho^2\tau_0}{8}\left(\frac{1}{2s+1} - \frac{2s'}{2s+1}\frac{}{2s+2s'+1}\right) = \frac{3\rho^2\tau_0\phi_0}{8(2s+2s'+1)} \quad (14)$$

The energy due to ψ_s alone is $3\rho^2\tau_0\phi_0/16\,(4s+1)$, *i. e.* we write $s'=s$ and halve the result. The energy belonging to $k_s\psi_s + k_{s'}\psi_{s'}$ is

$$\frac{3\rho^2\tau_0\phi_0}{16}\left[\frac{k_s^2}{4s+1} + \frac{2k_sk_{s'}}{2s+2s'+1} + \frac{k_{s'}^2}{4s'+1}\right],$$

but it is clearly sufficient to give as in (14) a single composite term. The total charge attaching to ψ_s is

$$\tfrac{1}{2}\int \rho\varpi dS - \int s\rho u_a^{s-1}\,d\tau = \frac{3\rho\tau_0}{2} - \frac{3s\rho\tau_0}{2s+1} = \frac{3\rho\tau_0}{2(2s+1)}.$$

For normal distribution on a conductor $s=0$, $e=3\rho\tau_0/2$, and the energy is $e^2\phi_0/12\tau_0$.

From (14) may be derived the composite term in the energy of $k_s U_s + k_{s'} U_{s'}$, or of $k_s(\psi_s - \psi_{s+1}) + k_{s'}(\psi_{s'} - \psi_{s'+1})$. For this term we are concerned with the products

$$\rho_{s'}\psi_s - \rho_{s'+1}\psi_s - \rho_{s'}\psi_{s+1} + \rho_{s'+1}\psi_{s+1},$$

and the application of (14) yields

$$\frac{3\rho^2\tau_0\phi_0}{8}\left[\frac{1}{2s+2s'+1} - \frac{2}{2s+2s'+3} + \frac{1}{2s+2s'+5}\right]$$

$$= \frac{3\rho^2\tau_0\phi_0}{2s+2s'+1\ \ 2s+2s'+3\ \ 2s+2s'+5}, \quad (15)$$

which multiplied by $k_s k_{s'}$ is the composite term required.

To deal with the function V_s or $\dfrac{\rho}{4}\displaystyle\int \dfrac{d\lambda}{\sqrt{J}}(1-u_a)^s$, the two summations for the indices of ρ and ψ may be taken

separately. The first is

$$\frac{3\rho^2\tau_0\phi_0}{8}\left[1 - \frac{s'C_1}{3} + \frac{s'C_2}{5} - \cdots\frac{-1|^{s'}}{2s'+1}\right]$$

$$= \frac{3\rho^2\tau_0\phi_0}{8} \cdot \frac{2 \cdot 4 \ldots 2s'}{3 \cdot 5 \ldots 2s'+3};$$

and the second is

$$\frac{3\rho^2\tau_0\phi_0}{8} \times 2 \cdot 4 \ldots 2s'\left[\frac{1}{1 \cdot 3 \ldots 2s'+1} - \frac{sC_1}{3 \cdot 5 \ldots 2s'+3}\right.$$

$$\left. + \cdots\frac{-1|^s}{2s+1 \cdot 2s+3 \ldots 2s+2s'+1}\right]$$

$$= \frac{3\rho^2\tau_0\phi_0}{8} \cdot \frac{2 \cdot 4 \ldots \overline{2s+2s'}}{3 \cdot 5 \ldots \overline{2s+2s'+1}}; \quad . \quad . \quad (16)$$

and this quantity multiplied by $k_s\,k$ is the composite term in the energy of $k_s\,V_s + k_{s'}\,V_{s'}$. The summations for (16) are made by a series of steps in which the combination formula $C_{s\,r} = C_{s-1\,r} + C_{s-1\,r-1}$ is used, and the difference of adjoining terms taken. The densities attaching to U_s and V_s are respectively $\rho(\overline{s+1}\,u_a^s - s u_a^{s-1})$ and $s\rho(1-u_a)^{s-1}$, and the corresponding total charges $\dfrac{3\rho\tau_0}{2s+1\ 2s+3}$ and $\dfrac{3\rho\tau_0}{2} \cdot \dfrac{2 \cdot 4 \ldots 2s}{3 \cdot 5 \ldots 2s+1}$; they follow from those of ψ_s, or may be got by independent work. The energy formula for U_s may be got directly, viz. for the composite term

$$\mathrm{E}(s, s') = \int \rho_{s'}\mathrm{U}_s d\tau$$

$$= \int \rho\left(\overline{s'+1}u_a^{s'} - s'u_a^{s'-1}\right)d\tau \cdot \frac{\rho}{4}\int_0^\infty \frac{u_a^s(1-u_a)d\lambda}{\sqrt{\mathrm{J}}}, \text{which by (13)}$$

$$= \frac{3\rho^2\tau_0\phi_0}{4}\left[(s'+1)\left\{\frac{1}{2s+1\ 2s+2s'+3} - \frac{1}{2s+3\ 2s+2s'+5}\right\}\right.$$

$$\left. - s'\left\{\frac{1}{2s+1\ 2s+2s'+1} - \frac{1}{2s+3\ 2s+2s'+3}\right\}\right]$$

$$= 3\rho^2\tau_0\phi_0/(2s+2s'+1)(2s+2s'+3)(2s+2s'+5),$$

in agreement with (15). Moreover, the energy of ψ_s may be calculated from that of U_s by treating ψ_s as $U_s + U_{s+1} + \ldots$ *ad inf.*, and making a double summation of the formula just obtained. Either U_s or V_s for $s=0$ gives the case of uniform volume distribution $e=\rho\tau_0$, and energy $= e^2\phi_0/10\tau_0$.

§ 5. We now give corresponding results for χ_s. Using the integrals, ($10\,b$) for the first step, and ($12\,b$) for the second,

$$\frac{1}{\tau_0}\int u_a^{s'}(ax+c'y+b'z)d\tau\int_0^\infty \frac{u_a^s(ax+\gamma'y+\beta'z)d\lambda}{\sqrt{J}}$$

$$=\frac{1}{\tau_0}\int u_a^{s'}(ax+c'y+b'z)d\tau\int\frac{\Delta_a^{s+1}l(lx+my+nz)^{2s+1}\,d\omega}{2\pi u_p u_A{}^{s+3/2}}$$

$$=\frac{3}{(2s+3)(2s+2s'+5)}\int\frac{\Delta_a l^2 d\omega}{2\pi u_p u_A{}^{3/2}}$$

$$=\frac{3\mathrm{L}_0}{(2s+3)(2s+2s'+5)}\quad \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad (17)$$

where L_0 as in Æ. (135) stands for

$$\int_0^\infty\frac{\alpha d\lambda}{\sqrt{J}}=\int\frac{\Delta_a l^2 d\omega}{2\pi u_p u_A{}^{3/2}}.$$

This and the corresponding surface integral may be written

$$\int u_a^{s'}(ax+c'y+b'z)\chi_s d\tau=\frac{3\rho\tau_0\mathrm{L}_0}{4(2s+3)(2s+2s'+5)}\quad\cdot\quad(17\,b)$$

and

$$\int(ax+c'y+b'z)\chi_s\varpi dS=\frac{3\rho\tau_0\mathrm{L}_0}{4(2s+3)}.\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad(17\,c)$$

The surface-density due to $\chi_{s'}$ is $\dfrac{\rho\varpi}{2}\;(ax+c'y+b'z)$, and the volume-density $-s'\rho u_a^{s'-1}(ax+c'y+b'z)$; thus applying ($17\,b$ and c) the composite term in energy attaching to indices s and s' is

$$\mathrm{E}(s,s')=\frac{3\rho^2\tau_0\mathrm{L}_0}{8(2s+3)}\left[1-\frac{2s'}{2s+2s'+3}\right]=\frac{3\rho^2\tau_0\mathrm{L}_0}{8(2s+2s'+3)}.\quad(18)$$

If functions with $\gamma'x+\beta y+\alpha'z$ and $\beta'x+\alpha'y+\gamma z$ are also under consideration, the three functions may be distinguished as $\chi_s(x),\ \chi_s(y),\ \chi_s(z)$. In the composite term for $\chi_s(y)$ and $\chi_{s'}(y)$, M_0 appears for L_0 above, but if a composite term for $\chi_s(x)$ and $\chi_{s'}(y)$ is in question $\mathrm{N}_0{}'$ appears, viz.,

$$\mathrm{N}_0{}'=\int\frac{\Delta_a lm d\omega}{2\pi u_p u_A{}^{3/2}}=\int_0^\infty\frac{\gamma' d\lambda}{\sqrt{J}}.$$

For the density of $\chi_{s'}(y)$ has the factor $c'x+by+a'z$; and if this is taken with $\chi_s(x)$ a factor m is introduced in place of l in the second step towards (17), where ($12\,b$) is quoted ; while the contrary arrangement, density of $\chi_s(x)$ with potential

$\chi_{s'}(y)$, introduces the factor m in the first step where $(10\,b)$ is quoted, with an identical result.

For the volume potentials of magnetic type

$$\frac{\rho}{4}\int_\lambda^\infty u_a^s(1-u_a)(\alpha x+\gamma'y+\beta'z)\frac{d\lambda}{\sqrt{\mathrm{J}}}$$

and

$$\frac{\rho}{4}\int_\lambda^\infty (1-u_a)^s(\alpha x+\gamma'y+\beta'z)\frac{d\lambda}{\sqrt{\mathrm{J}}},$$

which we may call $\mathrm{U}_{s'}(x)$, $\mathrm{V}_{s'}(x)$ the composite terms in energy are respectively

$$3\rho^2\tau_0\mathrm{L}_0/(2s+2s'+3)(2s+2s'+5)(2s+2s'+7) \quad . \ (19)$$

and

$$\frac{3\rho^2\tau_0\mathrm{L}_0}{8}\cdot\frac{2.4\ldots\overline{2s+2s'}}{3.5\ldots\overline{2s+2s'+3}}\cdot \quad . \quad . \quad . \quad . \quad . \quad . \ (20)$$

In lieu of total charge in connexion with ψ, we may here take a magnetic moment* defined for $\chi_s(x)$ by

$$m_x=\int x\rho_s d\tau=\frac{3\rho\tau_0}{2(2s+3)};$$

for the function $\mathrm{U}_{s'}(x)$ it is $\dfrac{3\rho\tau_0}{2s+3\,.\,2s+5}$. Thus an exact

correspondence is given between the two types of potential.

§ 6. We now consider with reference to a potential

$$\mathrm{P}=k_1\mathrm{U}_0+k_2\mathrm{U}_1+\ldots k_n\mathrm{U}_{n-1}$$

$$=\frac{\rho}{4}\int_\lambda^\infty (1-u_a)(k_1+k_2u_a+k_3u_a^2+\ldots k_n u_a^{n-1})\frac{d\lambda}{\sqrt{\mathrm{J}}},$$

the problem of determining the k's so as to make the energy E a minimum subject to the constancy of total charge e, E and e being given by

$$\mathrm{E}=\frac{3\rho^2\tau_0\phi_0}{2}\left[\frac{k_1^2}{1\,.\,3\,.\,5}+\frac{2k_1k_2}{3\,.\,5\,.\,7}+\frac{k_2^2+2k_1k_3}{5\,.\,7\,.\,9}+\ \cdots\right]\Big\} \quad .. \ (21)$$

$$e=3\rho\tau_0\left[\frac{k_1}{1\,.\,3}+\frac{k_2}{3\,.\,5}+\ \cdots\right]$$

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* In the case of Æ. (117) where these functions are used, it is, however, a question of total charge, and $e=3\rho\tau_0/2$ for the conductor ($s=0$), $e=\rho\tau_0$ for the uniform volume distribution ($s=1$). The energies for χ_0 and for χ_1 are then respectively $e^2\mathrm{L}_0/36\tau_0$ and $e^2\mathrm{L}_0/70\tau_0$; thus if we seek the energy of convection currents due to surface charge instead of body charge, we must write in Æ. (117) $e^2/36\mathrm{V}^2$ outside the square bracket.

If a similar problem is stated for

$$Q = \frac{\rho}{4} \int_\lambda^\infty (1 - u_a)(k_1 + k_2 u_a + \ldots k_n u_a^{n-1})(ax + \gamma'y + \beta'z) \frac{d\lambda}{\sqrt{J}},$$

ϕ_0 is replaced by L_0 and e by m_x, and the coefficients open with $\dfrac{1}{3.5.7}$ and $\dfrac{1}{3.5}$ instead of $\dfrac{1}{1.3.5}$ and $\dfrac{1}{1.3}$. By writing $k = \dfrac{e\xi}{3\rho\tau_0}$ for each subscript the minimum problem is presented with pure numbers, and we may generalize it by making the series of coefficients start at a different place. Thus using

$$a_p = 1/(2p-1)(2p+1)(2p+3) \quad \text{and} \quad b_p = 1/(2p-1)(2p+1),$$

it is required to find ω the minimum value of

$$\left.\begin{aligned}
\Omega &= \xi_1\eta_1 + \xi_2\eta_2 + \ldots \quad \ldots \; \xi_n\eta_n \\
\text{where} \qquad \eta_1 &= a_p\xi_1 + a_{p+1}\xi_2 + \ldots \quad a_{p+n-1}\xi_n \\
\eta_2 &= a_{p+1}\xi_1 + a_{p+2}\xi_2 + \ldots a_{p+n}\xi_n \\
&\;\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\eta_n &= a_{p+n-1}\xi_1 + \ldots \quad \quad a_{p+2n-2}\xi_n
\end{aligned}\right\} \quad . \; (22)$$

define the η's, and $1 = b_p\xi_1 + b_{p+1}\xi_2 + \ldots \; b_{p+n-1}\xi_n$

is an equation of condition. From the point of view of potential theory, interest attaches to the value of the minimum, which proves to be

$$\omega = \frac{2p-1}{2} \cdot \frac{\overline{n+1}\;\overline{2n+2p-1}}{n\;\overline{2n+2p+1}} \quad . \; . \; . \quad (23)$$

in the general case; and to the nature of the distribution which gives the minimum. For $p = 1$ the case of (21),

$$\omega = (n+1)(2n+1)/2n(2n+3) \quad \text{and} \quad E = e^2\phi_0\omega/6\tau_0;$$

while the density for the potential P is

$$\frac{e}{3\tau_0}\left[\xi_1 + \xi_2(-1 + 2u_a) + \xi_3(-2u_a + 3u_a^2)\ldots + \xi_n\left(-\overline{n-1}\,u_a^{n-2} + nu_a^{n-1}\right)\right]$$

or

$$\left.\begin{aligned}
&\frac{e}{3\tau_0}\left[\xi_1 - \xi_2 + 2(\xi_2 - \xi_3)u_a + 3(\xi_3 - \xi_4)u_a^2 + \ldots + n\xi_n u_a^{n-1}\right] \\
&\text{or say} \\
&\frac{e}{3\tau_0}\left[x_1 + x_2 u_a + x_3 u_a^2 + \ldots + x_n u_a^{n-1}\right].
\end{aligned}\right\} \quad (24)$$

The conditions of no variation are

$$\eta_1 = \omega b_1, \quad \eta_2 = \omega b_2, \ldots \text{with } 1 = b_p \xi_1 + \ldots b_{p+n-1} \xi_n ;$$

i. e.

$$\left.\begin{array}{c} a_p \xi_1 + \ldots \qquad a_{p+n-1}\ \xi_n = \omega b_p \\[4pt] \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\[4pt] a_{p+n-1}\xi_1 + \ldots a_{p+2n-2}\xi_n = \omega b_{p+n-1} \\[4pt] b_p \xi_1 + \ldots \qquad b_{p+n-1}\ \xi_n = 1 \end{array}\right\} \quad \cdot \quad \cdot \quad \cdot \quad (25)$$

Write the first as

$$\omega b_p = (\xi_1 - \xi_2)a_p + (\xi_2 - \xi_3)(a_p + a_{p+1})$$
$$+ (\xi_3 - \xi_4)(a_p + a_{p+1} + a_{p+2}) + \ldots + \xi_n(a_p + \ldots a_{p+n-1}),$$

and note that

$$a_p + a_{p+1} + \ldots a_{p+n-1} = \tfrac{1}{4}(b_p - b_{p+n})$$

$$= \frac{1}{4}\left\{ \frac{1}{2p-1}\frac{1}{2p+1} - \frac{1}{2p+2n-1}\frac{1}{2p+2n+1} \right\}$$

$$= \frac{1}{8}\left\{ \frac{1}{2p-1} - \frac{1}{2p+1} - \frac{1}{2p+2n-1} + \frac{1}{2p+2n+1} \right\}$$

$$= \frac{n}{4}\left\{ \frac{1}{2p-1}\frac{1}{2p+2n-1} - \frac{1}{2p+1}\frac{1}{2p+2n+1} \right\}.$$

The first equation of (25) is then

$$\frac{1}{2p-1}\left[\frac{x_1}{2p+1} + \frac{x_2}{2p+3} + \cdots \frac{x_n}{2p+2n-1} \right] - \frac{1}{2p+1}\left[\frac{x_1}{2p+3} \right.$$
$$\left. + \cdots \frac{x_n}{2p+2n+1} \right] = 4\omega b_p = 2\omega\left(\frac{1}{2p-1} - \frac{1}{2p+1} \right),$$

and others of this type are got by increasing p by 1, 2, 3...

Since $b_p + b_{p+1} + \ldots b_{p+n-1} = \dfrac{n}{2p-1 \cdot 2p+2n-1}$, the equation of condition treated in this way becomes

$$\frac{x_1}{2p+1} + \frac{x_2}{2p+3} + \cdots \frac{x_n}{2p+2n-1} = 2p-1. \quad \cdot \quad (26)$$

The group of equations is then equivalent to

$$\frac{x_1}{2p+1} + \cdots \frac{x_n}{2p+2n-1} = 2\omega + \mathrm{C}(2p-1),$$

$$\frac{x_1}{2p+3} + \cdots \frac{x_n}{2p+2n+1} = 2\omega + \mathrm{C}(2p+1)\ldots,$$

together with (26). The first with (26) gives $2\omega + C(2p-1)$
$=2p-1$, and the right hand of the $q+1\overline{|}^{\text{th}}$ equation

$$= 2\omega + C(2p+2q-1) = 2p+2q-1 - \frac{4q\omega}{2p-1},$$

which if the stated minimum (23) is correct

$$= (2p-1)\left[1 - \frac{2q}{n(2p+2n+1)}\right].$$

The group of equations is then

$$\frac{x_1}{2p+2q+1} + \frac{x_2}{2p+2q+3} + \cdots \frac{x_n}{2p+2q+2n-1}$$

$$= \frac{2p-1}{n(2p+2n+1)}\left[n(2p+2n+1)-2q\right] \quad . \quad (27)$$

where q has in succession the values 0, 1, 2 ..n; if these are
satisfied the original conditions are satisfied and ω has the
value in (23).

If $P_n^r(\mu)$ is used for the rth differential coefficient of the
zonal harmonic $P_n(\mu)$, the solution of (27) is comprised in the
statement of the following as an identity, viz.:

$$x_1 + x_2\mu^2 \cdot \frac{2p+1}{1} + x_3\mu^4\frac{\overline{2p+1}\ \overline{2p+3}}{1\cdot3} + x_4\mu^6\frac{\overline{2p+1}\ \overline{2p+3}\ \overline{2p+5}}{1\cdot3\cdot5} + \cdots$$

$$\equiv \frac{(2p-1)P_{2n+p}^{2+p}}{n(2p+2n+1)1\cdot3\ldots\overline{2p-1}}. \quad (28)$$

The numerical coefficients on the left hand are cleared by an
integration $\int_0^\mu f d\mu$, followed by $p-1$ integrations

$$I \cdot f = \int_0^\mu \mu f d\mu,$$

and the result is

$$x_1\mu^{2p-1} + x_2\mu^{2p+1} + \ldots x_n\mu^{2p+2n-1} \equiv \frac{(2p-1)}{n(2p+2n+1)} I^{p-1}\cdot P_{2n+p}^{1+p}; \quad (29)$$

for $\quad p=1, \ x_1\mu + x_2\mu^3 + \ldots = P_{2n+1}''/n(2n+3), \quad . \quad . \quad . \quad (29b)$

and for $\quad p=2, \ x_1\mu^3 + x_2\mu^5 + \ldots = 3(\mu P_{2n+2}'' - P_{2n+2}')/n(2n+5). \ (29c)$

The left-hand member of (27) is got from the left-hand
member of (29) by multiplying by μ^{2q+1} and integrating from

$\mu = 0$ to 1, *i.e.* we require

$$\int_0^1 P''_{2n+1}\mu^{2q+1}\,d\mu = n(2n+3) - 2q, \quad \text{for } p = 1,$$

$$\int_0^1 (\mu P''_{2n+2} - P'_{2n+2})\mu^{2q+1}\,d\mu = n(2n+5) - 2q, \text{ for } p = 2,$$

and

$$\int_0^1 \mu^{2q+1}(I^{p-1}\cdot P^{1+p}_{2n+p})d\mu = n(2p+2n+1) - 2q \qquad (30)$$

for the general case.

Now

$$\int_0^1 P''_{2n+1}\,\mu^{2q+1}d\mu = P'_{2n+1}(1) - (2q+1)\int_0^1 \mu^{2q}P'_{2n+1}\,d\mu,$$

and

$$\int_0^1 \mu^{2q}P'_{2n+1}\,d\mu = P_{2n+1}(1) - 2q\int_0^1 \mu^{2q-1}\,P_{2n+1}\,d\mu\ ;$$

the last term vanishes for $q = 0, 1, 2, \dots n$, but no further, and therefore

$$\int_0^1 P''_{2n+1}\mu^{2q+1}\,d\mu = (n+1)(2n+1) - (2q+1) = n(2n+3) - 2q.$$

Again

$$\int_0^1 (\mu P''_{2n+2} - P'_{2n+2})\mu^{2q+1}d\mu = P'_{2n+2}(1) - (2q+3)\int_0^1 \mu^{2q+1}\,P'_{2n+2}\,d\mu$$

while

$$\int_0^1 \mu^{2q+1}\,P'_{2n+2}\,d\mu = P_{2n+2}(1) - (2q+1)\int_0^1 \mu^{2q}\,P_{2n+2}\,d\mu.$$

The last integral vanishes for $q = 0, 1, 2, \dots n$, but no further, and therefore

$$\int_0^1 (\mu P''_{2n+2} - P'_{2n+2})\mu^{2q+1}\,d\mu = (n+1)(2n+3) - (2q+3) = n(2n+5) - 2q.$$

The solution is therefore fully verified for the cases $p = 1$ and 2, which belong to the two problems stated, the cases of potentials P and Q. For the general case a mode of proof will be briefly indicated. Carry out the operation I by using $P'_{s+1} - \mu P'_s = (s+1)P_s$ differentiated any number of times, then in the result $I^{p-1}\cdot P^{p+1}_{2n+p}$ yields

$$P''_{2n+2p-1} - (p-1)(2n+2p+1)P'_{2n+2v-2} + \text{a residue},$$

The residue can be expressed as a linear function of the P's of odd degree from P_{2n+3} to $P_{2n+2p+1}$, and the multiplication of these by μ^{2q+1} and integration from 0 to 1 yields a zero result from $q=0$ to $q=n$. The integral on the left hand of (30) is then

$$\int_0^1 \mu^{2q+1}\left[P''_{2n+2p-1}-(p-1)(2n+2p+1)P'_{2n+2p-2}\right]d\mu = (n+p)(2n+2p-1)$$

$$-(2q+1)-(p-1)(2n+2p+1)=n(2n+2p+1)-2q \text{ as required.}$$

A comparison of (29 b) with (24) shows that the solution gives directly the density. Now a zonal harmonic has a point of inflexion between each maximum and minimum, therefore $\mu^{-1}P''_{2n+1}$ vanishes for $n-1$ values of μ^2; hence the density vanishes for these values of u_a. The distribution is one in which there are n concentric ellipsoidal shells of alternate positive and negative charges. When n is indefinitely increased the charges on all but those for which u_a is nearly $=1$ are indefinitely small, and the case becomes that of a conductor. A simple independent proof for the case of n infinite shows the character of the limit. Since

$$a_p+a_{p+1}+\ldots\text{ad inf.} = \frac{b_p}{4}$$

the equation of condition may be written

$$\tfrac{1}{4}=\xi_1(a_p+a_{p+1}+\ldots)+\xi_2(a_{p+1}+a_{p+2}+\ldots)+\ldots=\eta_1+\eta_2+\ldots$$

i. e. it can be expressed in terms of the η's. But

$$\delta\Omega=\xi_1\delta\eta_1+\ldots \quad \text{subject to} \quad 0=\delta\eta_1+\delta\eta_2+\ldots$$

makes the ξ's all equal; and it is then easy to show that each $=2(2p-1)$, and that $\omega=(2p-1)/2$. The potential P (case $p=1$) is then

$$\frac{e}{6\tau_0}\int_\lambda^\infty (1-u_a)(1+u_a+u_a^2+\ldots)\frac{d\lambda}{\sqrt{J}} = \frac{e}{6\tau_0}\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}},$$

and the potential Q (case $p=2$) is

$$\frac{m_x}{2\tau_0}\int_\lambda^\infty (\alpha x+\gamma'y+\beta'z)(1-u_a)(1+u_a+u_a^2+\ldots)\frac{d\lambda}{\sqrt{J}}$$

$$= \frac{m_x}{2\tau_0}\int_\lambda^\infty \frac{(\alpha x+\gamma'y+\beta'z)}{\sqrt{J}}.$$

The higher derived functions which would correspond to higher values of p in the minimum problem are not here constructed.

§ 7. In conclusion we return to the general theorem of moments, give an independent proof, and then show how to

deal with individual terms. We set out from the sphere, for which

$$\frac{1}{\tau_0}\int x^{2\nu_1}y^{2\nu_2}z^{2\nu_3}d\tau$$

$$= \frac{1}{\tau_0}\int_0^R\int_0^\pi\int_0^{2\pi} r^2\sin\theta\,dr\,d\theta\,d\phi\,(r\sin\theta\cos\phi)^{2\nu}\,(r\sin\theta\sin\phi)^{2\nu_2}\,(r\cos\theta)^{2\nu_3}$$

$$= 3R^{2\nu_1+2\nu_2+2\nu_3}\times\frac{1\,.\,3\,\ldots\,\overline{2\nu_1-1}\,.\,1\,.\,3\,\ldots\,\overline{2\nu_2-1}\,.\,1\,.\,3\,\ldots\,\overline{2\nu_3-1}}{1\,.\,3\,\ldots\,2\nu_1+2\nu_2+2\nu_3+3}\,;$$

hence for an ellipsoid referred to principal axes

$$ax^2+by^2+cz^2=1,\text{ with } A=bc,\,\ldots$$

$$\frac{1}{\tau_0}\int x^{2\nu_1}y^{2\nu_2}z^{2\nu_3}\,d\tau$$

$$=3\left(\frac{A}{\Delta_a}\right)^{\nu_1}\left(\frac{B}{\Delta_a}\right)^{\nu_2}\left(\frac{C}{\Delta_a}\right)^{\nu_3}\times\frac{1\,.\,3\ldots\overline{2\nu_1-1}\,.\,1\,.\,3\ldots\overline{2\nu_2-1}\,.\,1\,.\,3\,\ldots\,\overline{2\nu_3-1}}{1\,.\,3\,\ldots\,2\nu_1+2\nu_2+2\nu_3+3}.$$

Therefore, if $2s=2\nu_1+2\nu_2+2\nu_3$,

$$\frac{1}{\tau_0}\int(lx+my+nz)^{2s}d\tau = \Sigma\frac{\lfloor 2s\,l^{2\nu_1}\,m^{2\nu_2}n^{2\nu_3}}{\lfloor 2\nu_1\lfloor 2\nu_2\rfloor\lfloor 2\nu_3\rfloor\tau_0}\int x^{2\nu_1}y^{2\nu_2}z^{2\nu_3}d\tau$$

$$= \frac{3}{(2s+1)(2s+3)}\Sigma\frac{\lfloor s}{\lfloor\nu_1\lfloor\nu_2\rfloor\lfloor\nu_3}\left(\frac{Al^2}{\Delta_a}\right)^{\nu_1}\left(\frac{Bm^2}{\Delta_a}\right)^{\nu_2}\left(\frac{Cn^2}{\Delta_a}\right)^{\nu_3}$$

$$= \frac{3}{(2s+1)(2s+3)}\left(\frac{Al^2+Bm^2+Cn^2}{\Delta_a}\right)^s.$$

Now suppose $x\,y\,z$ transformed by a linear substitution to other axes $x_1\,y_1\,z_1$ which are not principal axes of the ellipsoid, and $l\,m\,n$ transformed by the inverse substitution to $l_1m_1n_1$ (*cf.* Salmon, Higher Algebra, p. 102). The transformation makes $lx+my+nz=l_1x_1+m_1y_1+n_1z_1$, leaves Δ_a unaltered, and gives $A_1l_1^2+\ldots+2A_1'm_1n_1\ldots$ for $Al^2+Bm^2+Cn^2$.

Hence dropping the subscripts we have the general form

$$\frac{1}{\tau_0}\int(lx+my+nz)^{2s}d\tau= \frac{3}{(2s+1)(2s+3)}\left(\frac{u_A}{\Delta_a}\right)^s.\quad\cdot\quad(11)$$

In order to deal with specified products of inertia, a mode of writing the coefficients in the expansion of a ternary quadric is required, say

$$\left.\begin{array}{c}\{u_a(x,y,z)\}^s=\Sigma x^{\nu_1}y^{\nu_2}z^{\nu_3}\,K(\nu_1,\,\nu_2,\,\nu_3;\,a)\\[4pt]\text{with}\qquad 2s=\nu_1+\nu_2+\nu_3\end{array}\right\}\quad\cdot\quad(31)$$

Then comparing corresponding terms in the expansion of (11),

$$\frac{1}{\tau_0}\int x^{\nu_1} y^{\nu_2} z^{\nu_3} d\tau = \frac{3}{(2s+3)}\frac{|\nu_1|}{|2s+1|}\frac{|\nu_2|}{\cdot}\frac{|\nu_3|}{\Delta_a^s} \cdot \frac{K(\nu_1, \nu_2, \nu_3 ; A)}{\Delta_a^s} \quad . \quad . \quad (32)$$

The particular cases $s=1$ and 2 were given in Æ. p. 443 without proof. Making a similar application to the angular integral (3), the type of an individual term is

$$\frac{1}{2\pi}\int \frac{l^{\nu_1} m^{\nu_2} n^{\nu_3} d\omega}{u_p(u_A + \mu u_p)^{s+\frac{1}{2}}} = \frac{|\nu_1|}{|2s \cdot \Delta_a^s|}\frac{|\nu_2|}{}\frac{|\nu_3|}{} \int_\mu^\infty \frac{K(\nu_1, \nu_2, \nu_3 ; a) \, d\mu}{\sqrt{\Delta(\mu)}} . (33)$$

The special cases $s=1$ and 2 were also given in Æ. p. 445 and p. 459.

The calculation of the coefficients K is facilitated by the use of sequence equations, obtained by differentiation of (31). Thus differentiation with regard to x gives the first of

$$\nu_1 K(\nu_1, \nu_2, \nu_3 ; a) = (\nu_1 + \nu_2 + \nu_3)[aK(\nu_1 - 2, \nu_2, \nu_3) + c'K(\nu_1 - 1, \nu_2 - 1, \nu_3) \\ + b'K(\nu_1 - 1, \nu_2, \nu_3 - 1)]$$

$$\nu_2 K(\nu_1, \nu_2, \nu_3 ; a) = (\nu_1 + \nu_2 + \nu_3)[c'K(\nu_1 - 1, \nu_2 - 1, \nu_3) + bK(\nu_1, \nu_2 - 2, \nu_3) \\ + a'K(\nu_1, \nu_2 - 1, \nu_3 - 1)]$$

$$\nu_3 K(\nu_1, \nu_2, \nu_3 ; a) = (\nu_1 + \nu_2 + \nu_3)[b'K(\nu_1 - 1, \nu_2, \nu_3 - 1) + a'K(\nu_1, \nu_2 - 1, \nu_3 - 1) \\ + cK(\nu_1, \nu_2, \nu_3 - 2)].$$

If in $K(\nu_1, \nu_2, \nu_3)$ one of the indices is negative $K=0$. There is a second group of relations of which the type is

$$A(\nu_1 + 1)K(\nu_1 + 1, \nu_2, \nu_3 ; a) + C'(\nu_2 + 1)K(\nu_1, \nu_2 + 1, \nu_3 ; a) \\ + B'(\nu_3 + 1)K(\nu_1, \nu_2, \nu_3 + 1 ; a) \\ = (\nu_1 + \nu_2 + \nu_3 + 1)K(\nu_1 - 1, \nu_2, \nu_3 ; a),$$

got either by inverting the first group, or independently by the use of

$$\left(A\frac{d}{dx} + C'\frac{d}{dy} + B'\frac{d}{dz}\right)u_a^s = 2s\Delta_a . xu_a^{s-1}.$$

XLIX. *The Absorption of the γ Rays of Radioactive Substances. By* A. S. EVE, *M.A., Lecturer in Mathematics, McGill University, Montreal* *.

XLIX. *The Absorption of the γ Rays of Radioactive Substances. By* A. S. EVE, *M.A., Lecturer in Mathematics, McGill University, Montreal* *.

THE following investigations were made in order to ascertain whether the γ rays could be taken as an accurate measure of the total amount of radioactive matter present in a given substance. It was, therefore, necessary to ascertain whether the γ rays of various substances were absorbed to an equal degree under the same conditions.

* Communicated by Prof. E. Rutherford, F.R.S.

When the activity of a substance is measured by the α or β rays, the effect observed depends on the density of the active substance. In the case of β rays it is necessary to cut off the α rays by screens of aluminium or some light material, and these screens absorb the β rays of the different active substances to an unequal extent, so that the β rays are not satisfactory as a test of activity.

There seemed good reason to anticipate that the γ rays would afford a most satisfactory measure of the total active matter in a given body, because these rays under the ordinary experimental conditions are but slightly absorbed by the matter from which they are emitted. It is possible to measure the rays of the substance in bulk, also, and this is an advantage when considerable quantities of ore are under investigation. Moreover, previous experiments indicated that the γ rays from thorium, uranium, and radium were absorbed to an equal extent in passing through the same thicknesses of lead. It was found, however, in the course of these experiments, that radium and thorium do emit γ rays which have identical coefficients of absorption, whilst the γ rays from uranium are weak and easily absorbed. Thus the γ ray method can be employed to great advantage in comparing radium and thorium, but uranium and actinium cannot be compared with them, or with one another.

Apparatus.—An electroscope, 30 centimetres high and 20 centimetres in diameter, made of zinc ·45 mm. thick, contained the usual gold-leaf system on a long insulated central rod. The fall of potential was measured by the movement of the gold-leaf read by a microscope with a graduated scale in the eyepiece. The radioactive substance under examination was placed on a platform about 7 cm. below the electroscope. Layers of lead were introduced between the active body and the electroscope. Corrections were made in the usual manner for the natural leak of the instrument. In every case lead was used as the absorbing material.

Substances Investigated.

(1) Radium Bromide.
(2) Uraninite from Joachimsthal, Bohemia. (1 kilo.)
(3) Uranium Nitrate, prepared by Eimer and Amend. (1 kilo.)
(4) Thorium Nitrate, prepared by Eimer and Amend. (2 kilos.)
(5) Radio-thorium, lent to me by the discoverer, Dr. Hahn, who was working in the McGill Physics Building.
(6) Actinium, Giesel's preparation, activity about 300.

(7) Actinium, Debierne's preparation, activity about 700.
    This substance, belonging to Sir William Ramsay,
    was kindly lent to me by Dr. Hahn.

Results.—If I_0 be the intensity of radiation on one side
of a lead plate, and I be the intensity on the other side after
absorption, we have $I = I_0 e^{-\lambda x}$, where x is the thickness of the
plate, and λ is the coefficient of absorption. This formula,
deduced from $\dfrac{dI}{dx} = -\lambda I$, assumes that the absorption by a
thin layer per unit thickness is proportioned to the intensity.
It has been found by McClelland, Wigger, and others, that
in the case of radium λ is not constant over wide ranges,
but the less penetrating γ rays are first absorbed, and the
subsequent values of λ are therefore smaller. In the present
experiments, the range of the thicknesses between which the
values of λ have been obtained will be stated, without
including the minute correction for the air traversed, or for
the ·45 mm. of zinc which formed the base of the electro-
scope, or for the glass vessels, such as test-tubes, in which
some substances were placed.

Radium.—The results obtained for radium are in fair
agreement with those found by McClelland, using the
electrometer method (Phil. Mag. July 1904).

<div align="center">γ rays.</div>

Thickness of lead in cm.	λ (cms.)$^{-1}$.
·64 to 1·21	·57
1·21 to 1·79	·56
1·79 to 2·36	·46
2·4 to 3·0	·46

McClelland found that λ varied from ·64 to ·44 through
the same range. The result is shown in fig. 1, where the
abscissæ denote the thickness of lead traversed, and the
ordinates give the logarithms of the intensities as measured
by the gold-leaf electroscope. It will be seen that the radium
line thus plotted is not straight, and that λ has a gradual
slight decrease in value.

Uraninite.—The specimen from Joachimsthal gave values
of λ almost identical with those obtained from radium
bromide, although in one case the radium was concentrated,
and in the other was distributed through a kilogram of
pitchblende. It will be found later that the γ rays from

uranium are almost absorbed by ·64 cm. of lead, so that the
γ rays from uraninite are mainly due to radium, and are a
measure of the radium present, not more than 20 per cent.
being lost by self-absorption.

Fig. 1.
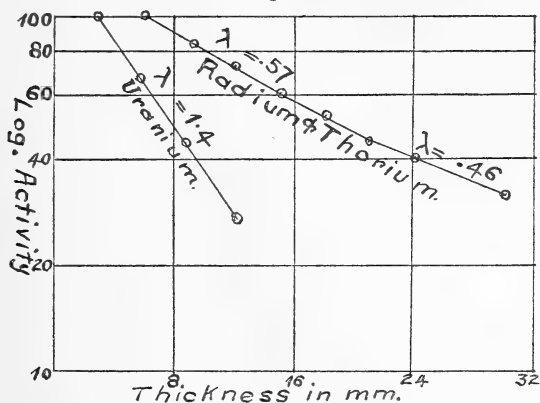
Thorium.—One kilogram of thorium nitrate was sealed in
a flat glass cylinder, 16 cm. in diameter. The values of
λ were found to be practically the same as those for radium
and for uraninite.

Radio-thorium.—It is important to note that radio-thorium
gave values of λ almost identical with those for radium and
thorium, but initially the rays from radio-thorium appear to be
slightly more penetrating, possibly because the self-absorption
is less. Dr. Hahn has shown that radio-thorium has produced
Th.X and Th. Emanation, and since radio-thorium gives
rays similar to thorium we have another proof, if proof were
needed, of the similarity between radio-thorium and thorium.
Radio-thorium has not yet been obtained in a pure state,
but the eleven milligrams kindly lent me by Dr. Hahn, as
measured by the γ rays, were equivalent to 1570 grams of
thorium nitrate. This result is of the same order as that
found by Hahn, when using γ rays, so that radio-thorium
is 143,000 times more active than thorium nitrate. But it
will be seen later that the latter, thorium nitrate, loses about
15 per cent. of its γ activity by self-absorption, so that
radio-thorium is actually about 12×10^4 more active than
an equal weight of thorium nitrate. Now radium bromide
was found to be about 15×10^6 times as active as thorium
nitrate, when measured by the γ rays; hence the impure
radio-thorium had an activity about one-hundredth part of
that of pure radium bromide.

It will be convenient to state here the actual figures obtained for the substances which have been considered up to the present point. The differences in the observed intensities are probably within the limits of experimental error, as it is difficult to place such different substances under precisely similar conditions with respect to the electroscope.

γ *rays.*

Total thickness of lead in cm.	Intensities.			
	Radium.	Uraninite.	Thorium nitrate.	Radio-thorium.
·64	100	100	100	100
1·21	72	73	72	75
1·79	52	53	52	55
2·36	40	41	40	42
3·00	29	30	31	31

Uranium.—The γ ray effect of uranium was found to be surprisingly small. A kilogram of thorium nitrate gave ten times the effect of a kilogram of uranium nitrate, measured through ·64 cm. of lead. The γ rays of uranium were so readily absorbed, that it was not possible to measure λ over a wide range, or with great accuracy. The mean of repeated observations between ·28 and ·57 cm., and also between ·64 and ·92 cm., gave $\lambda = 1\cdot4$; so that the γ rays, like the β rays of this substance, appear to be homogeneous. A specimen of pure uranium gave $\lambda = 1\cdot6$ approximately. In the case of uranium nitrate the loss of γ activity by self-absorption must be large.

The ratio of the activities of uranium nitrate and thorium nitrate was determined by measurement of the α and β ray effect from 12 grams of each substance.

α ray 0·6,

β ray 6·0,

whilst for a kilogram of each,

γ ray 0·1.

It will thus be seen that these methods offer no certain test for comparing the total activities of uranium and thorium or of uranium and radium, and the same difficulty will be found in the case of actinium. On the other hand, the

equality of the absorption of the γ rays of thorium and radium makes the γ radiation an accurate comparative measure of the joint quantities of these radioactive substances in two given bodies.

Actinium.—Dr. Godlewski has published in the Philosophical Magazine for September 1905 an account of some penetrating rays of actinium, with a coefficient of absorption equal to 4·5. He experimented with a preparation of Giesel's actinium, activity about 300, and he was able to take measurements for thicknesses ranging up to about 3 mms. of lead. Using a stronger preparation of Debierne's actinium, activity about 700, it was possible to carry Godlewski's work further. When the rays passed only through ·45 mm. of the zinc which formed the base of the electroscope, the activity was measured by 2·35 scale-divisions per minute. On adding successively ·15, ·30, ·45 mm. of lead, the observed activity fell by an exponential law, with $\lambda = 10·5$. These rays probably consist partly of homogeneous β rays, and they are not plotted in the diagram (fig. 2). Further

Fig. 2.
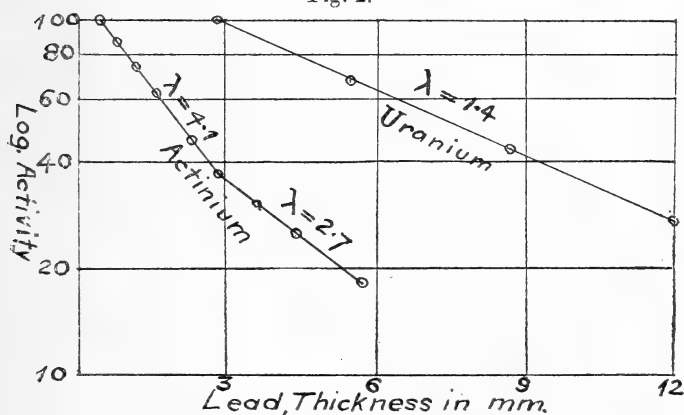
sheets of lead, ·15 mm. thick, were added, until a total thickness of 2·85 mm. was reached. The results plotted on logarithm paper gave a straight line, so that the rays are homogeneous between these limits. This result confirms Dr. Godlewski's work with Giesel's actinium, but he found $\lambda = 4·5$, and in the present case $\lambda = 4·1$. These values are nearly equal, for λ is a sensitive function, and the difference is probably due to experimental conditions. The two preparations of actinium have the same emanation and excited activity, and the same coefficients of absorption are therefore to be expected.

When a thickness of 2·85 mm. of lead was reached a fairly well-marked change occurred, and λ became equal to 2·7, and this value was maintained to a thickness of 4·35 mm., or even to 5·7 mm. Beyond that the effects were small and very difficult to measure, but the mean of repeated observations indicated that λ was equal to about 2 between 5·7 and 8·7 mm.

The results are shown in fig. 2*, but it may be convenient to state them in tabular form also :—

| Thickness in mm. | | λ (cms.)$^{-1}$. |
Zinc.	Lead.	
·45	+ 0	
		10·5
„	+ ·45	
		4·1
„	+ 2·85	
		2·7
„	+ 5·7	
		2·0 ?
„	+ 8·7	

It appears, then, that actinium has two types of β rays and one or two of γ rays. The two types of β rays are both homogeneous and easily, but not equally, absorbed. It is, of course, possible that there is one type of β rays, and that all the rest are γ rays. The question can only be settled by photographs of actinium rays deflected in a magnetic field. This work is under investigation by Dr. Godlewski.

It is noteworthy that both uranium and actinium have homogeneous β rays, and that these substances give rise to easily absorbed γ rays. But thorium and radium have heterogeneous β rays, and the resulting γ rays are much more penetrating. We may fairly conclude that the high velocity β particles give rise to the more penetrating γ rays, and that low velocity β rays generate easily absorbed γ rays. And we have here a striking similarity with the generation of Röntgen rays by the cathode rays. For in a "hard" bulb, with a good vacuum, the cathode rays have a high velocity, and when these are abruptly stopped a penetrating type of Röntgen rays arises. But in a "soft" bulb, the low velocity cathode rays generate easily absorbed Röntgen rays. So also the abrupt expulsion of high velocity β rays causes

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* It was not possible to exhibit all the results of this paper in a single diagram. In fig. 1 radium and thorium are compared with uranium, and in fig. 2 actinium and uranium are contrasted. The horizontal scales in the two diagrams are not the same.

highly penetrating γ rays, and the low velocity β rays produce weak γ rays. In the case of uranium and actinium there appear to be no β rays sufficiently swift to cause the most penetrating type of γ rays, for the β rays are homogeneous, and have a more or less uniform velocity.

Standard of γ Ray Measurement.

The writer suggests that one kilogram of pure thorium nitrate enclosed and tightly sealed in a thin glass vessel, 16 cms. in diameter, would form a convenient standard for testing the amount of radium or thorium in a given ore in bulk. The γ activity could be tested through a centimetre of lead, and the rays of actinium or uranium would then be practically excluded. The results of various experimenters could be readily compared in terms of such a standard.

Concentration and Distance.

If the distance of an active substance from the bottom of the electroscope is varied, the fall of potential of the gold-leaf also alters. In comparing two substances their centres of mass should be made to coincide relatively to the electroscope. The measured activity was found, roughly, to vary inversely as the square of the distance from the active substance to the centre of the electroscope.

In order to see whether the observed effect was dependent on the degree of concentration of the substance under investigation, a few pieces of pitchblende were placed, first at the centre, then at the circumference of a glass dish 16 cms. in diameter. The activity observed was three per cent. stronger when the pitchblende was all near the centre of the dish. If, therefore, the dish were full, the loss owing to the scattering would be about one and a half per cent. This loss is due to the fact that the cone of rays cut by the electroscope is different when an active particle is at the centre, and when it is at the circumference of the glass vessel.

Self-Absorption.

Two methods were employed to form an estimate of the loss of activity due to self-absorption by thorium nitrate. In the first case the electroscope was placed on a platform of lead ·64 cm. thick. A kilogram of thorium nitrate was then placed beneath, and the activity measured 2·3. A small quantity of radium was placed at the centre of the mass, and the total activity was 84·4. Next the activity of the radium

without the thorium was found to be 98·9.   Hence the activity of the radium was reduced from 98·9 to 82·1, by the absorption of half the thickness of the thorium nitrate. But λ for thorium and radium has the same value, and in the thorium nitrate the central layers are absorbed to an average extent.   Hence the actual reading of the activity of thorium nitrate being 100, the corrected reading would be about 122.

A second method of finding the self-absorption was by the measurement of layer after layer, superadded to one another, in a circular dish, 16 cms. in diameter, adjusted so that the centre layer was at a constant distance from the electroscope. The rise of activity was indicated by a straight line when the masses were taken as abscissæ and the activities as ordinates.   The absorption of a thin layer of 200 grams could not be large, and by successive approximations the self-absorption was found to be about 15 per cent. for 1 kilogram of the material placed beneath ·64 cm. of lead.

These two results are in rough agreement.   Moreover, the intensities found for 1 kilogram of thorium nitrate, 1 kilogram of pitchblende, 2 mg. of radium bromide, and 11 mg. of radio-thorium, followed the same curve for various thicknesses of lead.   It is fair to conclude that the loss of activity of a kilogram of pitchblende, due to self-absorption and to lack of concentration, can be fully corrected by an addition of twenty per cent. to the observed value, when ·64 cm. of lead cut off the β rays.

Summary.

(1) Radium, uraninite, thorium, and radio-thorium emit γ rays which are absorbed at the same rate by lead.
(2) For thicknesses of lead between ·64 and 3·0 cms. the values of λ range from ·57 to ·46 for all these substances.
(3) Uranium nitrate is weak in γ rays, and these are readily absorbed ; λ = 1·4 between 2·8 and 12·1 mm. of lead.
(4) Actinium emits four types of rays :—
   1. α rays.
   2. β rays which are homogeneous.
       λ = 163 (Godlewski).
   3. More penetrating rays, either β or γ.
       λ = 4·5 (Godlewski).
       λ = 4·1 (Eve), between ·45 and 2·8 mm.

4. Most penetrating rays, probably γ rays.
   λ=2·7 to 2·0, from 2·8 to 8·7 mm.
(5) A kilogram of thorium nitrate sealed in a thin glass
    vessel 16 cms. in diameter, placed under a layer of
    lead 1 cm. thick, might be adopted as a convenient
    standard for measuring the quantity of radium or
    thorium contained in a given mass of ore.
(6) The self-absorption of the γ rays from 1 kilogram of
    thorium nitrate in a vessel 16 cms. in diameter and
    about 3·4 cms. deep, is such that about 18 per cent.
    should be added to the results actually obtained.
    When used as a standard no such correction should be
    made.

In conclusion I take pleasure in expressing my gratitude
to Professor Rutherford for proposing the experiment, and
for his usual kind interest and ever ready assistance.

McGill University, Montreal,
    December 19, 1905.

L. *The Osmotic Pressures of Alcoholic Solutions.*
By P. S. BARLOW, *B.A., St. John's College, Cambridge*[*].

THE earlier experimental work of this paper dealt with the
use of the ordinary copper-ferrocyanide membrane for
solutions in which the solvent was ethyl alcohol. Tammann [†]
showed that ethyl alcohol could get through this membrane
against the osmotic current, but did not itself seem capable
of setting up an osmotic current. As there did not seem to
be sufficient evidence on this point to justify the assumption
that all copper-ferrocyanide membranes, however prepared,
could not be used with alcoholic solutions, it was considered
necessary to try the cells which had already been in use.
The substance of the membrane is precipitated in the colloidal
state ; and the outward passage of the alcohol would appear
to point to the absorption, under pressure, of the alcohol by
the membrane. A large number of experiments were made.
All the results were negative and in agreement with Tam-
mann's work ; but the work done seemed justified by the limited
knowledge we have of the detailed properties of colloids.
The inevitable small differences of preparation of the mem-
branes might very well have produced some change in the

colloid that would modify its behaviour towards alcohol (as observed by Tammann).   A considerable time was devoted to this work, a full account of which would be needlessly long ; it need only be said here that the results give fresh negative evidence in support of the theory that for the osmotic pressure to show itself, the membrane must be able to dissolve the solvent and have a distinct " attraction of solution " for it. Every opportunity was allowed as regards time, some of the experiments continuing for several months, and all for more than seven days.   The solutions used were of lævulose, lithium chloride, lithium sulphate (anhyd), camphor, shellac, methyl oxalate, ferric chloride (anhyd), glycerine.   In addition to the cylindrical cell of the usual type, cells were also made of thin porous plates in which the membrane was formed.   All the cells worked normally with water.

The copper-ferrocyanide membranes being unserviceable, other membranes were tried.   Of the vegetable and animal parchments, the bladder membrane was the only one to show any pressure.   The cell was made by stretching the prepared bladder across a brass cylinder.   The cylinder consisted of two parts which could be joined by means of flanges and screws.   The bladder was held between the flanges, thus dividing the interior into two compartments.   The side connected to the gauge was filled with the solution ; the other side was connected to a small glass reservoir and was filled with the solvent.   The bladder was backed on the side of the solvent (the outside) by perforated zinc.

The bladder cell (B II.) was used with solutions of lithium chloride, methyl oxalate, and camphor in absolute alcohol. Of these, the lithium chloride and camphor solutions gave osmotic pressures.   The methyl oxalate showed no sign of pressure.   No quantitative values were expected from the experiments with bladder, it being a very imperfect semi-permeable membrane.   The result, however, was satisfactory after the experiments with the ferrocyanide membrane as showing that, under conditions depending on the membrane, these solutions would give pressures with the solvent.

I am not aware that any quantitative results have been published of experiments made with non-aqueous solutions to measure directly the osmotic pressure.   Flusin[*] has made quantitative measurements of the rate of inflow, using vulcanized indiarubber and bladder.   He found that the amount of liquid entering in a given time is proportional to the

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[*] *Comptes Rendus,* cxxxi. p. 1308, Dec. 31, 1900.

quantity of the liquid absorbed per unit area of the membrane. In general, work with non-aqueous solutions seems so far to have been confined to electrical conductivity and the depression of the freezing-point.

I turned to pure indiarubber as a possible membrane when it was clear that the ferrocyanide cells were unserviceable. The pure substance I was unable to obtain, so I made use of guttapercha tissue. Separating alcohol and water, this tissue gave a pressure on the side of the water. I shall say here that I am not aware that guttapercha has been used for the direct measurement of osmotic pressures, though I think it has been used to show that with it, the osmotic current is towards the water. I may also say in passing, that thin and slightly vulcanized indiarubber tissue has been tried with these solutions, but no pressures were obtained with the samples used.

The experiments that remain to be recorded are those in which guttapercha tissue was the membrane. The first series of readings with solutions of lithium chloride was obtained with cell B I. Only three pressures were obtained in three weeks ; that is, including the cleaning of the cell, each reading below took more than a week to obtain. The maximum pressure did not require all that time to be set up, but the cell was left untouched for some time to make certain that a steady value had been reached.

Guttapercha membrane.	Theoretical Pressure.	Observed Pressure.	Temperature.
By Cell B I.	312 mm.	4·4 mm.	11°·3 C.
,, ,, ...	937	12	11°·6
,, ,, ...	1562	37·6	11°·6
,, B II. ...	1875	32·6	11°·6

No LiCl could be detected on the outside after the experiments.

Plotting the observed pressures as ordinates and the theoretical ones (which represent the concentrations) as abscissæ, we get the following curve (p. 598).

The falling away indicated by the pressure given by B II. is unexpected ; it will be referred to later. Here we need only consider the curve as given by the values for cell B I.

The curve produced to the left cuts the *y*-axis at a point

above the origin ; that is, the pure solvent gives an osmotic pressure—which is impossible. According to the curve, the osmotic pressure in the early stages does not increase as rapidly as the concentration ; while later, it increases more rapidly. Hence the curve is below the theoretical one for a water solution.

Fig. 1.—Curve I.
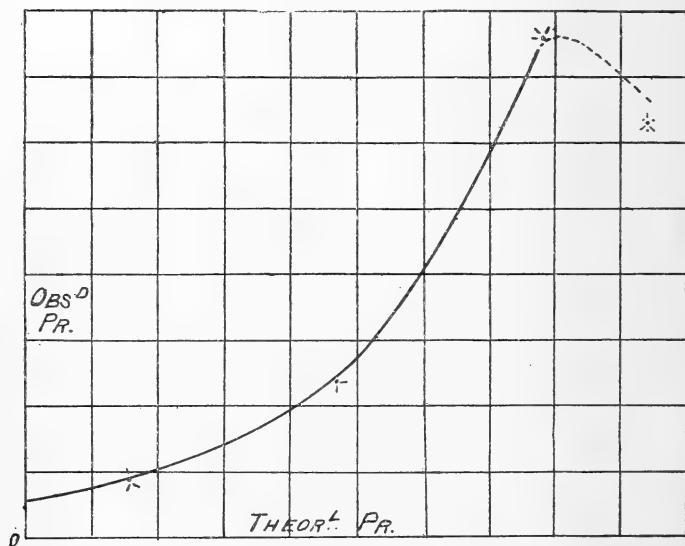
The curve can only be taken as a preliminary one. But one thing seems evident : that there is some substance present which produces an osmotic pressure in addition to that due to the lithium chloride. The suggestion that naturally occurs is that the extremely hygroscopic lithium chloride introduced a little water into the solution ; and previous experiment has shown that, with the guttapercha membrane, water behaves as the solute. If this explanation is legitimate, the zero of the curve for the lithium chloride will be raised and the curve should touch or cut the x-axis on the right of the origin.

In a paper on experiments with alcohols and water (part of which appeared in the Phil. Mag. July 1905) it was suggested that, if the solution of the solvent by the membrane was an essential condition of osmotic phenomena, then the tendency of the membrane to retain the solvent would oppose the dilution of the solution and so cut down the osmotic pressure. The effect of such depression would be most

marked for dilute solutions*. On this theory a curve, obtained as the above one, should cut the x-axis at that point which would denote the strength of the solution whose attraction for the solvent would just be balanced by the tendency of the membrane to retain it.

A further remark may be made here about the change in form of the curve to which theory would point. Since the attraction of the membrane plays a more important part in cutting down the osmotic pressure of the weaker solutions, the curve (plotted as above) should be convex towards the y-axis. The above curve may therefore be raised in its lower portion by disturbing influences. Even the assumption of considerable dissociation at this dilution would not account for the slow fall of the curve.

These considerations on the above curve showed the necessity for a more careful examination of the osmotic pressures of such solutions. This was undertaken as follows :

" Absolute " alcohol was redistilled after having stood over fresh and very good quicklime for three weeks and then over dehydrated copper sulphate for another week. It was distilled on a water-bath directly from the latter. The first and last portions of the liquid were neglected ; the receiving flask was only open to the air through a calcium-chloride tube. The redistilled alcohol was kept in a flask under a desiccator or with the neck of the flask covered tightly with a guttapercha cap. Every precaution was taken to avoid absorption of moisture, all measurements, &c., being made as quickly as possible where it was necessary to expose the alcohol to the air. The specific gravity of the alcohol after distilling was ·7973 at 12° C., compared with water at the same temperature. That of the alcohol before distillation was ·7980 at the same temperature. This shows that the alcohol, as supplied to the laboratory, was really very good. I point this out here, because in the last of the next series of experiments I was compelled to use " absolute alcohol," owing to having run short unexpectedly of the redistilled liquid.

The lithium chloride used was the salt supplied to the laboratory as " pure ; " the salt used in these experiments was kept for some weeks previously in a weighing tube over phosphorus pentoxide in a small desiccator. Care was taken to expose the salt as little as possible.

Arrangements were made for carrying out the experiments

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* Prof. Kahlenberg, by simultaneous work not yet published, has arrived at the same result (Phil. Mag. Feb. 1905, p. 228).

at 0° C. An ice-chest was made of tin in the form of a double box, the inner box in which the cells were to be placed being about three inches from the walls of the outer on all sides. The top was closed by a deep tray filled with ice. The inner box could thus be surrounded by broken ice. The outside was covered with three or four layers of cotton-wool. By means of a deep groove which was surrounded by the ice, the inner box was connected to the surface of the outer one, and in this groove lay the horizontal glass tube which connected the gauge to the cell. The gauge could thus be outside the case.

The ice-chest was made so that two experiments could be carried on together, but it was only possible to get cell B III. to work. Several days were spent in attempts with B I. and B II., but without success ; and their use for the purposes of this paper had to be discarded. It was unfortunate to be thus reduced to one cell, as the time at my disposal for the work was limited ; and each reading obtained (for a steady value) required a week's time. Cell B III. is the newest type of the brass cells used, and has worked the best. The following is a diagram. The section at right angles to the paper is circular.

Fig. 2.
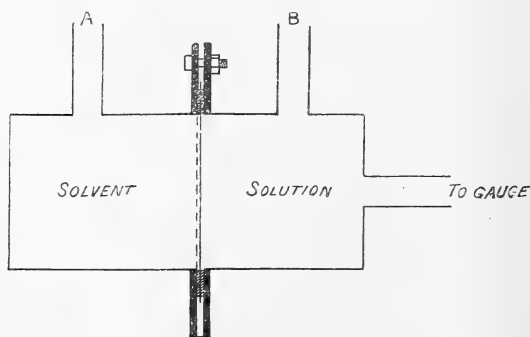
The opening A was connected to an adjustable mercury-gauge, or was open to the air through a large calcium-chloride tube. B was closed by thick rubber tubing. The membrane stretched between the flanges was backed by perforated zinc. In these experiments A was connected by a long piece of tubing to a calcium-chloride tube which was kept outside the ice-chest, the tubing being brought through the groove connecting the two boxes.

A thermometer passing through the ice-tray which formed the lid of the chest was placed so as to be near the osmotic

cell. By keeping the chest packed with broken ice, the temperature inside remained steady at zero. The temperature near the gauge at the outer surface was very steady at about 11°·5 C. There was a very small volume of solution above the mercury in the gauge which was above the inside temperature.

The conductivity of the solutions was found after the steady value for the pressure was obtained. The liquid was withdrawn from the cell and 5·4 c.c. (giving what seemed to be a suitable depth of liquid) were used in a small electrolytic cell. The electrodes were small platinum plates, platinized and then heated to redness. The commutator and galvanometer method of Fitzpatrick and Whetham was used in finding the electrolytic resistance. The cell was standing in broken ice for at least an hour before readings were taken. The resistance of the cell was found; the battery reversed and the resistance again found. The electrolytic cell was then reversed and two observations of its resistance taken as before. The inverse of the mean of these four values was taken as the conductivity, thus eliminating as far as possible polarization effects in the cell, and thermoelectric effects in the commutator. The following table contains all the readings I have been able to get during five weeks by means of cell B III. :—

Strength of Solution.	Theoretical Pressure (van't Hoff) at 0° C.	Observed Pressure at 0° C.	Conductivity $\left(\frac{1}{R}\right)$ at 0° C.
·144 normal.	2440 mm.	195 mm.	·0042
·089	1660	208	·0029
·040	690	131	·0016
·020	345	76	·00087

The last experiment was made using the ordinary absolute alcohol; these values therefore may not altogether agree with previous values, but it does not seem likely that the use of this alcohol will have greatly affected the readings.

The following curves (pp. 602–3) are obtained from the numbers in the above table.

The differences in Curves II. and III. are such as are accounted for by the form of Curve IV. The conductivity falls off rather more quickly than the strength.

Before discussing these curves I must say that in the above experiments, lithium chloride was found in the outer alcohol.

Fig. 3.—Curve II.
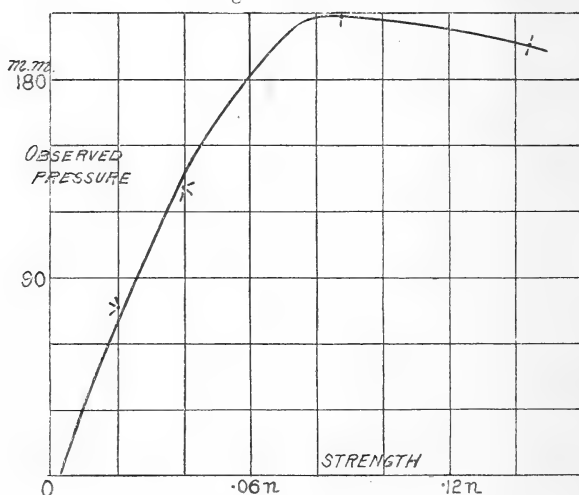
Fig. 4. –Curve III.
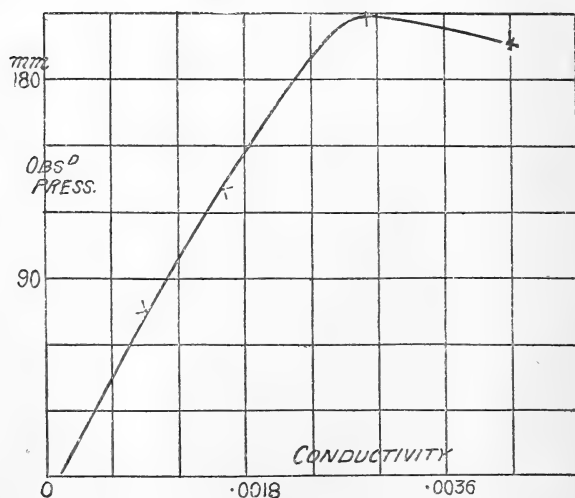
Its amount was distinctly less for the weaker solutions. "It will be remembered that in the experiments of the first table, no salt was found on the outside : there the pressures set up did not exceed 40 mm. All the pressures in the last table are far below some pressures that have been obtained with

the same tissue, and therefore the presence of lithium chloride outside is not due to a leak in the membrane caused by over-strain. Obviously above a certain pressure the guttapercha

Fig. 5.—Curve IV.
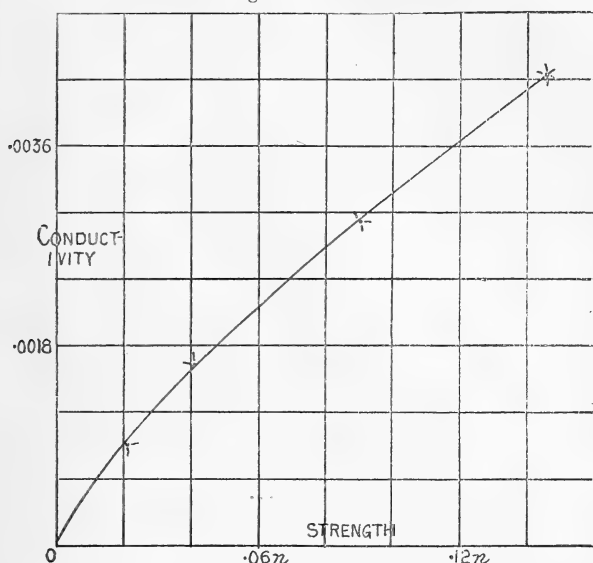
membrane is only imperfectly semi-permeable for the case of lithium chloride. It may be that this membrane is permeable to chlorides as the copper ferrocyanide one is; and, like the latter, would act more perfectly with sulphates.

The general character of Curves II. and III. agrees with theory : the observed pressure decreases more rapidly than the strength of the solution, owing to the counter attraction exerted by the membrane on the solvent. The fall of the curves towards the right is unexpected. (A similar fall was indicated by Curve I.) The permeability of the membrane to lithium chloride will not account for this : at most it could only make that portion of the curve horizontal. The only explanation that seems reasonable, supposing the upper part of the curve is reliable, is that beyond a certain strength of solution there is polymerization of the solute or a more complicated grouping of the solute and solvent molecules. This is supported by the form of Curve IV.

Curves II. and III. cut the *x*-axis at points denoting very weak solutions ; and therefore indicating that the equivalent strength of the membrane, in its attraction for the solvent,

is very small. The curves, however, do not embody the results of a sufficiently large number of experiments ; so that all that can be said is that, accepting the absorbent action of the membrane, they are in general agreement with theory. The work must at present be left at this point, but the writer hopes it may be of value in having opened up an interesting and profitable line for further research in the theory of solution.

At present our knowledge of non-aqueous solutions consists of such various and disconnected data, so often disagreeing among themselves when arrived at by different methods, that further accumulation of experimental results is required before any generalization can be attempted. It has become increasingly evident that the theories of solution based on work with aqueous solutions cannot satisfactorily cover the data obtained where the solvent has been other than water. The problem becomes more complicated as the influence of the solvent is found to be more marked. At the same time this comparatively recent development in experiment may be expected to afford a means of making an advance towards the truth of the matter.

All we can say now, Raoult said in summary in the last section of his ' Cryoscopie ' :—" ... the molecules of a salt (in solution) can be hydrated, polymerized, ionized at the same time, in proportions varying with temperature and concentration ; and, consequently, produce cryoscopic effects (and among these may be included the parallel osmotic ones) much more complicated than is generally believed."

LI. *On the Constitution of the Atom.*

To the Editors of the Philosophical Magazine.

Gentlemen,—

MAY I put forward a suggestion in connexion with a question discussed by Lord Rayleigh in his paper " On Electrical Vibrations and the Constitution of the Atom " (Phil. Mag. Jan. 1906) ?

Lord Rayleigh states an objection against regarding the atom as a system in steady orbital motion, rather than as one performing small oscillations about a position of statical equilibrium,—namely, that the sharpness of spectral lines indicates a definiteness of structure such as it is difficult to imagine associated with a system of electrons in orbital motion. He goes on to say : " It is possible, however, that

the conditions of stability or of exemption from radiation
may after all really demand this definiteness. . . . The fre-
quencies observed in the spectrum may . . . form an essential
part of the original constitution of the atom as determined
by conditions of stability."

If this were so, these frequencies would depend only on
the constituents of the atom and not on the actual type of
motion taking place in the atom. Thus if we regard the
atom as made up of point-charges influencing one another
according to the usual electrodynamical laws, the frequencies
could depend only on the number, masses, and charges of the
point-charges and on the æther-constant V. What I wish to
point out first is that it is impossible, by combining these
quantities in any way, to obtain a quantity of the physical
dimensions of a frequency.

If to the quantities already mentioned we add another, for
instance the energy of motion of the atom, it may be possible
to obtain frequencies. Here, however, the frequencies will
be functions of the energy. And, as Lord Rayleigh says,
the energy must change in the course of time, whereas the
frequencies, so far as we know, do not.

Or, instead of combining the original quantities with the
energy, we may combine them with a length, in such a way
as to obtain frequencies. Thus Lord Rayleigh obtains fre-
quencies in his analysis, but only in virtue of having introduced
the radius of the imaginary sphere of positive electrification.
If this positive electrification, instead of being limited to an
invariable sphere, were supposed free to expand under its
self-repulsion, Lord Rayleigh's ρ_0 would be indefinite, as
would consequently be the frequencies also.

The situation with regard to linear dimensions is precisely
the same as that with regard to frequencies. It is impossible
to derive a scale of linear dimensions from the quantities
permanently associated with point-charges and æther. Thus
the atom would have no definite size, but would expand and
contract indefinitely under external influences.

It seems, then, that we must somehow introduce new
quantities—electrons must be regarded as something more
complex than point-charges. And when we have once been
driven to surrendering the simplicity of the point-charge
view of the electron, is there any longer any objection to
putting the most obvious interpretation on the line-spectrum,
and regarding its frequencies as those of isochronous vibra-
tions about a position of statical equilibrium? The main
objection felt, as I understand, against this interpretation,
lies in its being inconsistent with the point-charge view of an

electron. I suggest that the objection applies equally to the "orbital motion" interpretation.

Lord Rayleigh's objection to the statical interpretation, that the theoretically calculated spectrum series would be for p^2 and not for p, and that they would apparently be too complex to agree with the observed facts, remains, and seems, it is true, to be *almost* insuperable; but is the case any worse than that which might have been made out against the atomic theory a century ago from the fact of the ratios of specific heats of gases forming the simple series $1 + \dfrac{2}{3+n}$, in spite of the enormous complexity of the atomic conception of matter?

Evidence, which seems to me to have great weight, can be derived from considerations of the partition of energy. The value of Dulong and Petit's constant shows that those parts of the energy which vary with the temperature are fully accounted for by the potential and kinetic energies of the atom regarded as a point; and the same is shown by the fact that γ for monatomic gases has the value $1\frac{2}{3}$. Thus the energies of the degrees of freedom which represent rotation or internal motions of the atom, are either infinitesimal or are incapable of variation under the ordinary interatomic forces.

Thus if electrons are describing orbits, the planes of these orbits must remain almost or completely fixed in space. If we rotate a body in the hand the atoms must maintain their directions in space, as indeed, on this view of the atom, would be shown also by the fact of our not experiencing any gyrostatic couple. But if the body is a crystal, its optical constants and axes do not change by rotation—as they surely would if all the atoms rotated their planes relatively to the body as a whole—and if the body is a permanent magnet, its magnetic properties and constants do not change.

A similar dilemma is soon reached with reference to the *velocity* of rotation. The Zeeman effect shows that this *can* be altered even by the fields available in the laboratory, while the value of γ would show that it is not altered by the much more effective magnetic fields which ought (on the "orbital motion" theory) to be found at collisions of molecules.

Finally, may I mention that in the Phil. Mag. for Nov. 1901, I attacked a problem similar in many respects to that which forms the main substance of Lord Rayleigh's paper? The actual premises upon which I worked were different from those of Lord Rayleigh—his positive sphere being represented, in my work, by a crowd of positive electrons, and the definiteness of structure, which he obtains by regarding

this sphere as rigidly fixed, being obtained in my work by using a law more general than ee'/r^2. Both papers, however, deal with a spherical atom executing small vibrations: in both the principal vibrations are found to correspond to spherical harmonics, the order of the harmonic being the n of the spectrum series. Lord Rayleigh obtains only a single series, falling off from the head according to the law $1/n$, and also a single spectral line. Nothing corresponding to this series appears in my work; but corresponding to the single line I obtained a number of series falling off according to the observed law $1/n^2$. Given the value of e/m for electrons (or electric fluid) the same relation can be obtained, in either case, between the frequencies and the radius of the atom. Unfortunately the numerical calculation given in § 23 of my paper was inaccurate: corresponding to frequencies of the order of those of light the radius must be about 3×10^{-7}, which is not good as regards agreement with facts, but is perhaps as near to the true value as can be expected in a vague calculation of this kind.

My picture of the atom led to a Zeeman effect, consisting of a splitting into three or more sharp and fully polarized lines. Lord Rayleigh's model would, I believe, do the same. If, however, the atom consisted of electrons in orbital motion, the planes of these orbits making all directions with the lines of magnetic force, it appears as if the Zeeman effect could at best consist of a widening into a continuous band, a point which has, I think, already been made by Lord Kelvin.

Princeton, N. J.,   J. H. JEANS.
   Jan. 24th, 1906.

LII. *The Dielectric Strain along the Lines of Force.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN the issue of December 1905 you published a paper by Mr. L. T. More, " On the Dielectric Strain along the Lines of Force," in which our paper " Ueber die Electrostriction des Glases"[*] is criticised. We should be much obliged if you would publish the following reply.

Our experiments have shown that the expansion of the volume of electrified glass tubes and spheres was in some kinds of glass in agreement with the mechanical pressure of the armatures and the elastic constants of the glass. In other kinds of glass the expansion was found to be too small. These results confirm in general those of Quincke, but are

[*] *Ann. d. Physik* (4) ix. p. 1217 (1902).

in opposition to the results of Cantone, who found the expansion greater than the theoretical value. They do not disagree with those of Mr. More, supposing he has used Thuringian or a similar kind of glass.

In the above-mentioned paper Mr. More says:—

(1) That our method was " faulty."

(2) That " we had attempted to show that the real elasticity of glass was greater when obtained from electrical stress than when found by mechanical and acoustical methods."

(3) That we had mistaken the nature of the problem in regarding the effect as being due to a variation of elasticity of electrified and not-electrified glass, while the effect obtained by us was produced partly by the faults of the method employed, partly by the alteration of the inductive capacity of glass by pressure.

As to (1): We have improved the method of Quincke until the observations showed sufficient regularity for the same kind of glass. It seems that Mr. More distrusts the exactness of our method especially for the reason that the results obtained were very different with different kinds of glass. We will show afterwards that, on the contrary, these differences are an indirect proof of the correctness of our statements.

As to (2): Mr. More misunderstood what we called "the coefficient of elasticity by electrostriction." We introduced this coefficient because it enabled us to compare in a simple way our results for different forms of tubes and spheres. He has overlooked that we have made some experiments (p. 1258) expressly for the purpose of proving that there is no real change of the coefficient of elasticity by electrification.

As to (3): Mr. More has overlooked also that we published some months later a second part of our investigation, entitled " Ueber die Änderung der Dielectricitäts-constante des Glases mit dem Druck " *. In this paper we tried to examine if there was an alteration of the inductive capacity with mechanical pressure in the same glass tubes as we had used in the experiments on electrostriction. The result was, that he same kinds of glass, in which the expansion by electrification was found too small, showed an increase of the inductive capacity with pressure. The agreement was not only in sign, but even in value.

We are glad to see that in the principal points Mr. More agrees totally with our views, viz., that there exist no other real forces in electrostriction than the pressure of the armatures. But we think we have done a little more than he. Mr. More has shown that in some cases there was no electrostriction, and suggested, without ascertaining it by experiments, that

* *Ann. d. Physik* (4) xi. p. 619 (1903).

all opposite results were faulty. We have proved experimentally, that in the cases where the expansion was found too great (Cantone) there were errors in the measurements; and in the cases where the expansion was found too small (Quincke and Wüllner & Wien), we proved also by experiments, that the variation of the dielectric power agrees in sign and value with the observed discrepancy. Accordingly there remain no observations leading to the conclusion that there are other forces in electrostriction than the pressure of the charged armatures.

We are, Gentlemen,

Aachen and Danzig,
February 1906.

Yours faithfully,
A. WÜLLNER ; M. WIEN.

LIII. *Note on "Minimum Deviation through a Prism."*
By R. CHARTRES [*].
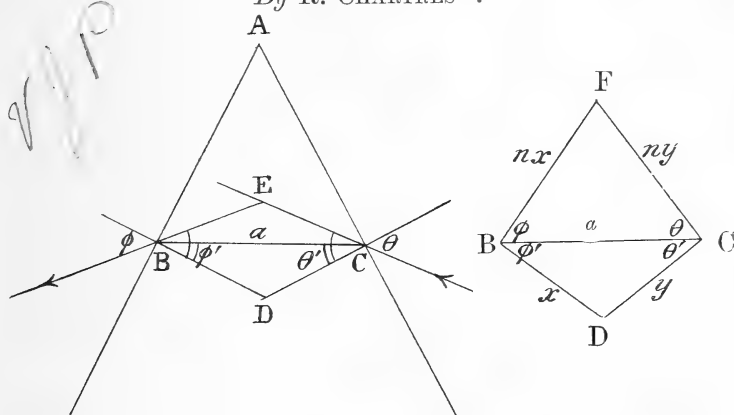
$\theta + \phi =$ minimum. $\theta' + \phi' =$ constant. $\sin \theta = \mu \sin \theta'$. $\sin \phi = \mu \sin \phi'$.

Make $F C B = \theta$, $F B C = \phi$,

then F is to be a maximum, while D is constant.

(1) $\therefore$ $\cos F - \cos D = \dfrac{a^2}{2xy}\left(1 - \dfrac{1}{n^2}\right)$, will be a minimum.

But since $n = \dfrac{\mu \sin (\theta' + \phi')}{\sin (\theta + \phi)}$,

$\therefore$ n, as also $\left(1 - \dfrac{1}{n^2}\right)$, will be a maximum when $\theta + \phi$ is a minimum.

$\therefore$ that (1) should be a minimum xy must be a maximum;

$\therefore$ $x = y$, $\theta' = \phi'$, and $\theta = \phi$.

* Communicated by the Author, being reprinted, with additions, from the Phil. Mag. vol. vi. p. 529 (1903).

LIV. *Notices respecting New Books.*

The Theory of Experimental Electricity. By W. C. D. WHETHAM,
   M.A., F.R.S., Fellow of Trinity College, Cambridge. Cambridge:
   at the University Press. 1905. Pp. xi + 334.

ALTHOUGH there are countless books in existence dealing with
   the subject-matter of the present volume, there is no doubt
that this text-book fills a distinct gap, and that it will strongly appeal
to all students who look for something more in a text-book than a
mere exposition of the leading facts and principles of the science,
and who are capable of feeling something of that spirit of enthusiasm
which has inspired numerous recent investigators in their epoch-
making discoveries. It is owing to these discoveries that a
considerable remodelling of our electrical text-books has become
not only desirable, but imperative. For, to quote the author's
remarks in his preface, "the great shift in the chief points of
interest of experimental electricity, due to recent development in
physical science, has changed the proportion of the various
branches of the subject, and has put out of date many of the
older standard text-books." We therefore cordially welcome the
author's effort at presenting the subject from a thoroughly modern
standpoint and in correct perspective, and congratulate him on the
success with which he has accomplished his task.

   As compared with the older text-books, the work under review
is characterized by the relatively large amount of space devoted
to electromagnetic waves, electrolysis, the conduction of electricity
through gases, and radio-activity. These subjects occupy fully
one-third of the volume, and the author's account of them is both
clear and fascinating. The book seems remarkably free from
minor blemishes, although we must protest against the author's use
of the expression "the electromotive force *between two points*,"
and against the statement, on p. 111, that "in the science of
current-electricity, it is usual to call a difference of potential an
electromotive force"—a statement which is somewhat surprising
in a modern text-book on the subject.

La Théorie Moderne des Phénomènes Physiques: *Radioactivité, Ions,
   Électrons. Par* A. RIGHI, Professeur à l'Université de Bologne.
   Traduction libre et notes additionnelles par E. NÉCULCÉA. Préface
   de G. LIPPMANN. Paris: "L'Éclairage Électrique," 1906.
   Pp. 125.

WHEN reviewing the first Italian edition of Prof. Righi's little
   treatise some two years ago, we had occasion to refer to it as one
of the most lucid and delightful expositions of a subject of ab-
sorbing interest. That the work in question has met with the
appreciation which it deserves in other countries is evidenced by
the appearance of the present French translation of the second
Italian edition. The translator—M. Néculcéa—has supplied some
additional bibliographical and other notes, and a very interesting

preface is contributed by Professor Lippmann. Little need be said in praise of the French edition ; the translation is well done, and we can only express the hope that it will serve to make this charming book better known to the English-speaking public.

Chemie der Alicyklischen Verbindungen. Von OSSIAN ASCHAN, A.O. Professor an der Universität Helsingfors. Mit vier eingedruckten Abbildungen. Braunschweig: F. Vieweg & Sohn. 1905. Pp. xlv + 1163.

THE enormous growth of organic chemistry during the last two decades has necessitated the subdivision of the subject into a number of branches, of which one—the chemistry of the so-called *alicyclic* compounds (a term due to E. Bamberger)—forms the subject of the present exhaustive treatise. This class of organic compounds includes a large number of substances of considerable technical importance, and is of special interest to students of pharmaceutical chemistry. The mere bulk of the volume before us bears ample testimony to the immense amount of material accumulated in this department of organic chemistry ; there are over 5000 references to original sources of information. The necessity of some sort of systematic account of the large amount of work done in this direction is therefore amply evident, and it is fortunate that this extremely laborious task has been undertaken by an author whose scientific activity has contributed to the advancement of this particular branch of organic chemistry. The present volume fills a gap and will earn for its author the gratitude of all advanced students of organic chemistry.

The work is divided into two parts, Part I. being general, and Part II. special. Part I. consists of an introduction in which the author defines the limits of his subject, and deals with homology, isomerism and nomenclature ; and of four sections, dealing respectively with the theoretical development of the chemistry of alicyclic compounds, the effect of the ring structure on the chemical nature of such compounds, its effect on some of their physical characteristics, and the stereo-chemistry of alicyclic compounds. Considering next the special Part II., we have in Section I. a systematic account of the modes of formation and preparation of alicyclic compounds, in Section II. an account of monocyclic, in Section III. of bicyclic, and in Section IV. of tri- and poly-cyclic compounds. A copious index closes this most valuable and exhaustive work of reference.

Spectroscopy. By E. C. C. BALY, *F.I.C.*, Lecturer on Spectroscopy and Assistant-Professor of Chemistry, University College, London. With 163 illustrations. London : Longmans, Green & Co. 1905. Pp. xi + 568.

THE enormous development of spectroscopy and spectroscopic methods has for some time past rendered the appearance of a comprehensive and up-to-date text-book on this subject highly desirable.

This want is now supplied by Mr. Baly's excellent work, which forms one of the series of *Text-Books of Physical Chemistry* now being issued by Messrs. Longmans, under the general editorship of Sir W. Ramsay. After an historical introduction devoted to an account of the rise and progress of spectroscopy, the author deals with the slit, prisms, and lenses, and their combination to form a complete prism-spectroscope. The practical use of this instrument is then fully dealt with. The diffraction-grating and its practical use are next considered. Then follow chapters dealing with measurements in the infra-red and ultra-violet regions, with the application of interference methods to spectroscopy, the efficiency of the spectroscope, the photography of the spectrum, the production and nature of spectra, the Zeeman effect, series of lines in spectra, and change of wave-length by pressure and by motion in the line of sight. An interesting feature of the book is the detailed and fully illustrated description of Rowland's grating ruling-engines given in the concluding chapter. The Appendix contains recipes for silvering-solutions for glass mirrors. An index of names and subjects is provided at the end of the book. In spite of the vast amount of matter dealt with, most of the explanations given are characterized by clearness, and the book contains many valuable practical hints which are the result of the author's own experience, and for which the reader will feel duly grateful. There are a number of obvious misprints —which is almost inevitable—and occasionally there are lapses from that clearness which characterizes the book as a whole. We would particularly direct the author's attention to the entirely inadequate if not actually misleading account of the principle of Langley's bolometer given on p. 229, which should be thoroughly revised in the next edition.

The Science Year-Book. With Astronomical, Physical & Chemical Tables, Summaries of Progress in Science, Directory and Diary for 1906. *Edited by* Major B. F. S. BADEN-POWELL. London: King, Sell & Olding, Limited. 1906. Pp. 208 + 365 + 27 + vi.

THE scope of this most useful year-book, which is in its second annual issue, is sufficiently described by its title. Compressed within the first 208 pages is a vast amount of information, including brief summaries of scientific progress during the past year. Then follows a diary, with a whole page for every day of the year. A few additional pages for memoranda &c. follow, and finally an index is provided.

It was hardly to be expected that a publication of this kind should be quite free from error. We note, for example, that Prof. Unwin's name is still given as that of the Professor of Engineering at the Central Technical College; a number of misprints are almost unavoidable—but these are small blemishes, and do not seriously mar the extreme usefulness of the book as a whole.

Der Bau des Fixsternsystems, mit besonderer Berücksichtigung der photometrischen Resultate. *Von* Dr. Hermann Kobold, A. O. Professor an der Universität und Observator der Sternwarte in Kiel. Mit 19 eingedruckten Abbildungen und 3 Tafeln. Braunschweig: F. Vieweg und Sohn. 1906. Pp. xii + 256. ["Die Wissenschaft," Heft 11.]

The problems which are considered in the present monograph have long formed the subject of much speculation on the part of astronomers. It is only, however, with the development of the recent powerful methods of research in this field that definite knowledge has taken the place of vague conjecture, and that trustworthy conclusions have been arrived at. The author divides his admirable account of the present state of the subject into three sections. Section I. deals in a general way with types of instruments and methods of observation. Section II. deals with the results obtained up to the present; while Section III. contains an account of the conclusions which may be drawn from them. Bibliographical references are given in an appendix. It may be interesting to quote the following sentences from the conclusion, in which the author thus sums up our knowledge regarding the structure of the universe: "Throughout a finite space of spherical form are scattered bodies of greatly varying mass and physical state. Besides gaseous nebulas of very low temperature there occur bodies in the state of greatest condensation and highest incandescence. The arrangement of the individual masses is no random or uniform one, but they are collected in heaps around single concentration centres, which, however, are loosely connected with each other and are arranged in the form of a large, many-branched spiral. In the more remote parts of this spiral there predominate the hotter and gaseous stars, while those more closely related to the sun—which is not far removed from the centre of the spiral—also resemble it, on the whole, in their physical state. The sun has a motion of translation—shared by the majority of the nearer stars—towards a point in the milky way, which forms the principal plane of the entire spiral."

Die Psychischen Massmethoden. *Von* Dr. G. F. Lipps, Privatdozent der Philosophie an der Universität Leipzig. Mit 6 eingedruckten Abbildungen. Braunschweig: F. Vieweg und Sohn. 1906. Pp. vii + 151.

It is but seldom that physicists devote much attention to the study of the psychical processes involved in the making of observations. True, they are generally introduced to Fechner's Law in connection with photometry, but there their psychical studies may be said to begin and end. To those with a metaphysical turn of mind, the monograph under review, which forms No. 10 of the series known as "Die Wissenschaft," may be recommended as affording a detailed account of the subject and methods of experimental psychology.

Terrestrial Magnetism and its Causes. *By* F. A. BLACK. London :
  Gall & Inglis. 1905. Pp. 226.

ACCORDING to the author's views, the "electric waves, or electric
displacements in the ether" which are projected from the sun,
"dash against the surface of the earth, or at least the enveloping
atmosphere, and thus come under the influence of the earth's
movements" (p. 15). "An electric sheet is thus wound around the
earth from apex to apex in the course of the diurnal rotation.....
The sheet may be compared to a great series of electric wires
wound around the earth from east to west. It is suggested that
this would furnish a simple explanation of the magnetization of the
earth "(p. 20). Having quoted the above extracts, we feel that
it is unnecessary to add anything further, as they are quite sufficient
to enable our readers to form an idea of what sort of book it is that
we are dealing with.

Physics. *By* CHARLES RIBORG MANN, the University of Chicago,
  and GEORGE RANSOM TWISS, the Central High School, Cleveland.
  Chicago : Scott, Foresman & Co. 1905. Pp. x + 453.

THE writing of a thoroughly satisfactory elementary book on
physics is by no means an easy task, and the authors of the
present volume are to be congratulated on having achieved a very
considerable amount of success in this difficult undertaking. The
book is just of the type that will appeal to youthful minds : it is
profusely illustrated, and at every point the principle under
discussion is exemplified by reference to some real thing which is
already within the sphere of the reader's experience. The style is
easy, and the mode of treatment full of suggestion and likely to
stimulate the imagination and awaken the interest of the young
student. The summaries, questions, problems and suggestions to
students at the ends of the chapters are also admirable features of
the book, and must have involved a good deal of time and trouble
in their preparation. There is a striking absence of pedantry and
dogmatism about the book. We have noticed but few misprints or
errors, but would suggest the substitution of "rate of change of
motion " for " change of motion " in the last line on p. 47.

LV. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

January 24th, 1906.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

THE following communications were read :—

  1. ' On the Igneous and Associated Sedimentary Rocks of Llan-
gynog (Caermarthenshire).' By T. Crosbee Cantrill, B.Sc., and
Herbert Henry Thomas, M.A., F.G.S.

  The sedimentary rocks associated with the various igneous
masses comprise the following :—

LOWER OLD RED SANDSTONE.	Red marls and sandstones, with cornstones and conglomerates at the base.
ORDOVICIAN (ARENIG).	*Didymograptus-bifidus* Beds. Blue-black shales, with one or more thick bands of grit towards the base. *Tetragraptus*-Beds. Black and buff shales with thin grit-bands : thick bands of ashy grits and conglomerates towards the base.

These rocks are described in detail. They occur in two main anticlines, overfolded, and complicated by thrusts which cut out a great part of the intervening syncline. They are covered unconformably by the lower beds of the Old Red Sandstone. The igneous rocks occur in three well-defined areas, which belong to the same petrographical province, near Coomb, at Capel Bethesda, and at Lambstone. Both interbedded and intrusive rocks are represented, and full petrographical descriptions of all types are given in the paper. The latter include diabases, and the large porphyry-mass of Lambstone. The extrusive rocks have been determined to occur in the following order :—(1) augite-andesites ; (2) rhyolites ; and (3) augite-andesites, with some hornblende-andesite. The extrusive rocks are interbedded with fluxion-breccias and with tuffs ; they are associated with the lower members of the *Tetragraptus*-Beds, and are consequently of Lower Arenig age; while the intrusive rocks have been injected into the extrusive rocks, and have also affected the *Tetragraptus*-Beds, but at what date exactly it is impossible to say, except that it antedates the Old Red Sandstone. Much of the folding and faulting was accomplished before the Lower Old Red Sandstone was deposited, but certain faults involve this formation, and make it clear that there was an important later movement.

2. 'The Buttermere and Ennerdale Granophyre.' By Robert Heron Rastall, B.A., F.G.S. (Christ's College, Cambridge).

This paper embodies the results of field-mapping and microscopical study of the large mass of igneous rocks known, collectively, as the Buttermere and Ennerdale Granophyre. From the facts put forward it is concluded that the intrusion is an example of an acid-magma, which has crystallized under the peculiar set of conditions that gives rise to a very perfect development of granophyric structure. These conditions are probably, to a certain extent, intermediate between those of plutonic and true hypabyssal rocks. The masses appear to be of the 'cedar-tree' laccolite-type intrusive about the junction of the Skiddaw Slates and the Borrowdale rocks, but penetrating into the higher rocks. Besides the normal acidic rock, which comprises the bulk of the intrusions, there are some marginal patches of more basic character, showing obvious genetic relationship, and slightly earlier in point of time than the intrusion of the acidic rock. These basic fore-runners afford evidence of differentiation of the magma before intrusion—an example of Prof. Brögger's deep-magmatic differentiation. Considered as a whole, the character of the magma shows closer affinity to the tonalite-group than to the true granites, although it is somewhat more acid than the majority of tonalites. The more basic types include dolerites, quartz-dolerites, and a rock-type intermediate between

quartz-dolerites and granophyres, for which no satisfactory name seems to exist. There is also a development of peculiar rock-types as the result of the re-mixing of previously-differentiated partial magmas of an acid and a basic character respectively. A study of the distribution of different types of granophyric structure shows a certain regularity of arrangement, and an attempt is made to reconcile these with known physical laws, especially with reference to eutectics ; and it is concluded that the structure is the result of crystallization under conditions intermediate between those which produce typical plutonic and hypabyssal rocks.

February 7th.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

The following communications were read :—

1. 'The Carboniferous Limestone (Avonian) of the Mendip Area (Somerset), with especial reference to the Palæontological Sequence.' By Thomas Franklin Sibly, B.Sc., F.G.S.

2. 'The Igneous Rocks of the Eastern Mendips.' By Prof. Sidney Hugh Reynolds, M.A., F.G.S.

The igneous rocks associated with the Old Red Sandstone of the Mendips are exposed along the crest of the range from Beacon Hill on the west to near Downhead on the east, a distance of rather more than two miles. Hitherto they have always been regarded as intrusive, but the opening of some new excavations has shown that they are associated with a considerable thickness of tuffs, and are in all probability contemporaneous lava-flows.

The exposures show a division into three sections—those of Beacon Hill, Moon's Hill, and Downhead ; and a large quarry has been opened in the trap in each section. The trap, which can be traced fairly continuously from one end of the area to the other, is very uniform in character, consisting (as already noted by Dr. Teall) of a non-amygdaloidal pyroxene-andesite, which usually contains augite in addition to enstatite. A fine section of tuff some 100 feet thick is seen lying with perfect conformity below the trap in the New Quarry near Stoke Lane ; and an interesting little exposure of tuff, remarkable for the numerous rounded blocks of trap present, is seen in the excavation for the rifle-butts on Beacon Hill. The tuff here dips under the Old Red Sandstone to the north. Although the tuff is seen *in situ* only at the above two points, loose pieces have been met with at a number of other spots all along the southern outcrop of the trap, and point clearly to the occurrence of a continuous band underlying it.

Though no sedimentary rocks are seen in direct contact with those of the igneous series, outcrops of Old Red Sandstone completely surround the exposures of trap and tuff, and occur in such close relation to them as to leave little room for doubt that the igneous series is of Old-Red-Sandstone age. On the other hand, Silurian fossils were met with below the igneous series at a point to the west of Downhead, and render it possible that the igneous rocks may be of Silurian age, and the equivalents of those which are exposed at Tortworth.

LVI. *On the Ionization of Various Gases by the α Particles of Radium.* By W. H. BRAGG, *M.A., Elder Professor of Mathematics and Physics in the University of Adelaide* [*].

IN a paper "On the Recombination of Ions in Air and other Gases" (Phil. Mag. April 1906, p. 466), Mr. Kleeman and I have described the preliminary steps of an enquiry into the total ionization produced in different gases by the α particle of radium, and the influence of physical conditions thereon. With the assistance of Mr. J. P. V. Madsen, B.Sc., I have made many experiments in continuation of this enquiry. It is necessarily a lengthy one, and in some respects difficult, so that on many of the points involved no definite conclusions are yet within reach. On others, results have been obtained which are, I think, of some interest and importance. In this paper I propose to describe the work which has been done; and, in addition, to make some reference to (*a*) the magnetic deflexion of the α particle, (*b*) its acquirement of a positive charge.

As described in the paper referred to, the total ionization of a gas can be measured in terms of the product of the co-ordinates of a certain point on the ionization curve. The true measure is, of course, the area between the curve and the axes of coordinates. But experiment shows that all ionization curves due to radium in radioactive equilibrium are of the same form, and differ from each other only in the

[*] Communicated by the Author.

application of some factor to all their ordinates or all their abscissæ. Thus the product of the coordinates of some standard point is proportional to the area of the whole curve, and may be taken as a relative measure of the total ionization.

In all the experiments to which I am about to refer, the α particles cross at right angles a shallow ionization chamber whose upper electrode is a brass plate and the lower a brass gauze ; the distance between the electrodes is about 3 mm. An electromotive force of 300 volts is usually applied to the gauze, giving an electric force of 1000 volts per cm. This is generally sufficient to ensure saturation ; that is to say, to avoid all errors due to diffusion, general recombination, and initial recombination. When it is insufficient, the proper correction is found and applied. The ionization chamber is enclosed within a vessel which is satisfactorily air-tight except at higher temperatures, and this again within an electric oven. The gas under observation can thus be subjected to various pressures and temperatures. In the case of such substances as benzene and carbon tetrachloride, a temperature of from 60° to 90° is necessary to ensure a convenient gas-density. There is, however, this drawback to the use of high temperatures, that the insulators begin to lose their efficiency, and the joints cease to be air-tight. I find it necessary to use glass as an insulator instead of sulphur, for the latter cracks under the unequal expansions due to alteration of temperature. In the case of vapours, a certain quantity of air usually finds its way into the apparatus, for, as just mentioned, the joints leak somewhat at the higher temperatures. The amount so entering is sometimes determined by opening a communication between the vessel and an evacuated bulb, and weighing the quantity of mixture drawn off. The bulb and connexions are placed within the oven, and communication is made by opening a pinch-cock worked by a key projecting outside the oven. In this way, condensation in cold tubes is avoided. This method is not always employed, for as soon as the stopping-power of the gas is sufficiently well known, the proportion of the mixture is much more easily, and I think at least as accurately, determined by observation of the range of the α particles therein.

The insulation leak is determined by measuring the deflexion of the electrometer first for ten seconds, and then for twenty. With no leak the latter should be double the former ; this is never quite the case, and the correction factor can be obtained from a comparison of the two values.

The factor to be applied to a ten-second leak varies from about 1·03 at 40° C. to about 1·10 at 70° C. ; at 90° C. it is much higher.

The total ionization is measured in terms of the product R I. The ordinate R is the range of the α particle, due to that product of radium whose speed is next to that of Ra C. In air at 760 mm. pressure and 20° C., R = 4·83 cm. very nearly. The abscissa I is the ionization produced in the chamber described when the radium layer is at a distance of 4·83 cm. from the middle of the chamber : or, more correctly, it is proportional to the ionization that would be produced in a very shallow chamber at that distance. The effect is wholly due to the particles from Ra C, the chamber being out of range of all the others.

These two quantities R and I differ materially from each other in two respects. To take the less important consideration first, the former quantity lends itself readily to exact measurement, the latter does not. The range of the α particle in a gas can be measured to an accuracy of one or two per cent. by a few minutes' observation, and to a much higher degree with greater care : it is perhaps the easiest of the measurements made in these experiments. By far the greatest difficulties which I find in the determination of the stopping-power of a gas lie in the purification and analysis of the gas.

On the other hand, the abscissa I is much more difficult to measure. It is affected by variation in the sensitiveness of the electrometer, by leakage through the insulators, by variation of the dimensions of the apparatus, and its true value is not given unless enough electric force is applied. None of these things affects the range. But it is not merely in the details of measurement that these two quantities differ. They appear as physical constants to be in distinct categories: so far, that is to say, as can be observed at present. The stopping-power of an atom is a constant of the atom, unaffected by its association with other atoms in molecular structure, independent of pressure and temperature. In a paper by Mr. Kleeman and myself (Phil. Mag. Sept. 1905), we gave a list of the stopping-powers of various substances, and since then we have made many other experiments in the same direction. In no case have we found a departure from the additive law which was not within the errors of experiment. That is to say, the range of the α particle in a given gas can always be predicted from the composition of the gas molecule. Not only so, but the stopping-powers of the various atoms are very nearly proportional to the square roots of their

weights, so that a simple, if approximate, law covers all the phenomena.   It even seems justifiable now to go one step further.   If the list in the paper quoted be examined, or the more comprehensive list in Table A, it will be found that the

TABLE A.

Table of Stopping-powers of various Gases.

Gas.	Experimental value. Air = 1.	Proportional to $\sqrt{w}$. Air = 1.	Proportional to w. Air = 1.	$\cdot118 \sqrt{w} + \cdot003 \times w.$
H_2	·243	·264	·0695	·242
O_2	1·055	1·054	1·11	1·04
N_2O	1·46	1·52	1·53	1·49
CO_2	1·47	1·51	1·53	1·48
CS_2	2·18	1·95	2·71	1·96
C_2H_2	1·11	1·17	·905	1·13
C_2H_4	1·35	1·44	·975	1·37
C_6H_6	3·37	3·53	2·71	3·39
C_5H_{12}	3·59	3·86	2·50	3·66
CH_3Br ...	2·09	2·03	3·28	2·11
CH_3I	2·58	2·35	4·90	2·52
C_2H_5Cl ...	2·36	2·31	2·23	2·30
C_2H_5I......	3·13	3·06	5·40	3·20
$CHCl_3$ ...	3·12	2·95	3·81	3·00
$C_4H_{10}O$ ...	3·40	3·67	2·57	3·51
CCl_4	4·02	3·59	5·41	3·68

TABLE B.

Table of Stopping-powers of various Metals.

Metal.	Experimental value. Air = 1.	Proportional to $\sqrt{w}$. Air = 1.	Ratio of two preceding columns.
Al	1·45	1·37	1·06
Fe	2·26	1·97	1·15
Ni	2·46	2·20	1·12
Cu	2·43	2·10	1·16
Ag	3·17	2·74	1·16
Sn	3·37	2·88	1·17
Pt	4·16	3·68	1·13
Au	4·45	3·70	1·20
Pb	4·27	3·78	1·13

The last column of Table A shows the application of the formula $a \sqrt{w} + bw$: the agreement with the second column is very close, considering that only two constants are used, and one of these is of little importance except in the case of the heavier atoms.   But the formula does not appear to apply to the metals where the stopping-power varies very nearly as the square root of the atomic weight.   This is certainly a difficulty.

stopping-powers are systematic even in their slight departure from the square-root law. For, whilst dependent mainly on the square roots of the weights, they have a leaning towards the weights themselves. We did not call attention at the time to this fact, for we thought it might be a spurious effect. But it has appeared so regularly in all further determinations that it seems right to note it, and to attempt an explanation of its physical meaning.

If we assume the correctness of the explanation already given of the square-root law, viz., that the α particle spends energy for the most part on tearing away electrons from their attachment at the edges of the atom disks, then the natural complement to this is the further assumption that electrons in all parts of the atom disk may be disturbed to vibration by the passage of the particle, which latter therefore spends a small amount of energy in simple proportion to the weight of the atom. If w be the atomic weight, the stopping-power of an atom should therefore be capable of expression by the formula

$$a\sqrt{w} + bw;$$

where the former term is usually by far the most important. The close agreement of the figures in the second and fifth columns of Table A shows that this is very nearly the case.

As regards pressure and temperature, I have not yet found any effect produced by variation of these conditions. The quantity RP/T appears to be a constant, P being the pressure and T the absolute temperature. This implies that the stopping-power of an atom or molecule is independent of P and T. Examples of the fact that RP is constant whilst T is constant are given in the paper " On the Recombination of Ions in Air and other Gases." The following experimental result will serve as an illustration of the fact that R is proportional to T when P is constant. The ionization vessel filled with air was raised to a temperature of 90° C., the pressure being 763 mm. R was then found to be 5·98. Now when $p = 760$ mm., and T = 20° C., R = 4·83. And

$$\frac{4 \cdot 83 \times 363 \times 760}{5 \cdot 98 \times 293 \times 763} = 1 \cdot 005.$$

It has of course been pointed out by several observers that the ionization effects of radium are largely independent of pressure and temperature and of physical and chemical conditions generally.

This, however, does not cover the present statement, which refers to the stopping-power of the atom, a quantity

which has not previously been the subject of measurement so far as I am aware.

To sum up, the range of the α particle in a given gas is, in the first place easily measured, and in the second place is simply related to the constitution of the gas and independent of its state. It is a delightful contrast to some other radio-active quantities, and often gives a welcome foothold in difficult places.

The quantity I is in quite a different class. It is much more difficult to measure accurately, as I have already described. But there appears to be a more important difference in that the total ionization of a gas is not simply dependent on the weights of the atoms of which it is composed. Molecular structure counts for something. Perhaps, also, the various atoms do not yield ions in simple proportion to the energy spent on them, but this point is not yet sufficiently clear.

An example of this want of uniformity has already been given in the paper to which reference has been made. It was shown that RI in ethyl chloride is much greater than RI in air. The difference must be yet a little greater than that shown, as no allowance was made for the small quantity of air mixed with the heavy gas. Again, RI in standard pentane (mostly C_5H_{12}) is nearly half as much again as in air, and the same is almost certainly true of benzene (C_6H_6), but this vapour is harder to treat than pentane, since a high temperature is necessary. Generally speaking, the more complex gases yield the greater number of ions. But the yield does not depend only on the number of atoms in the molecule. Acetylene (C_2H_2) yields 25 per cent. more than air, yet CO_2 with only one atom less yields but 5 per cent. more ; and ethylene (C_2H_4) yields the same as acetylene, though it has two atoms more. Of course in the last case the atoms added are very light; and H_2 itself has, according to my measurements, a slightly lower value (for RI) than air. Rutherford also found this to be the case.

On the other hand, the influence of complexity can be illustrated by the cases of acetylene and ethylene as compared with benzene and pentane.

In order to bring out the significance of these comparisons, it should be pointed out that the α particle spends exactly the same amount of energy in every gas (Bragg, Phil. Mag. Nov. 1905).

Thus in different gases different numbers of ions are

produced for the same expenditure of energy. It is quite clear, however, that this does not imply that the α particle finds it easier to produce ions in some gases than others. For, if so, there would be some influence on the stopping-power of atoms dependent on the number of ions produced. But the stopping-power is connected with the atomic weight by a simple law: the number of ions produced is not. Plainly the energy spent by an α particle in an atom and the resulting ionization are not directly connected: there is an intervening link.

Either the ions made by the α particle produce others in some cases, or some of the ions made never emerge from the atoms. There is something which prevents the simplicity of the law governing the expenditure of energy by the α particle from repeating itself in the amount of ionization produced. I think it is increasingly clear from our experiments that there is a secondary ionization within the molecule itself. The ions first made, or possibly X-ray pulses accompanying ionization, have, in some cases, enough energy to make fresh ions before leaving the molecule. Thus, for example, one molecule of C_6H_6 is found to rob the α particle of just as much energy as three molecules of C_2H_2. But more ions are made out of the one C_6H_6 than out of the group of three acetylene molecules. This may be explained on the ground that the twelve atoms are crowded together, so that an ion projected under ionization from one of the atoms strikes one of the others with an energy undiminished by motion through the field of the positive from which it was originally separated, and so capable of detaching a second electron. In further consequence the ions emerging from a C_6H_6 molecule move more slowly than those from a C_2H_2, and are more liable to initial recombination. This is in agreement with experiment: it is far harder to saturate benzene than acetylene.

The secondary ionization would appear to take place within rather than without the molecule, because the amount of it does not depend upon the distance of the molecules from one another. The total ionization is independent of the pressure. It is certainly not due to the electric field, for if it were there would be no saturation value of the current.

I subjoin the details of two of the many experiments which Mr. Madsen and I have made. We hope to give a fuller description at some future time.

Determination of Stopping-power, and of RI *in Pentane.*

Electrodes 3 mm. apart (nearly).   Volts applied $= 300$.
Temperature of apparatus $= 35°$ C.
Apparatus charged with vapour from standard pentane.

Distance from Ra $\leftrightarrows$ to middle of ionization chamber.	Leak in ten seconds.	Pressure inside apparatus.
2·8	1982	
2·9	1431	41·15 cm.
3·0	1192	
3·1	1171	
3·2	1193	
3·3	1227	41·15 cm.
Thin Cu foil over Ra	108	

These being plotted, it is found that $R = 2·95$, $I = 1044$, the copper leak having been deducted.

Thus $R = 2·95$ in this mixture of pentane and air, at a pressure of 41·15 cm., and a temperature (observed) of 308° (absolute).   But at a pressure 760 cm., and 293° absolute, R in air is 4·83.

Hence the mixture stops $\dfrac{4·83}{2·95} \cdot \dfrac{7600}{4115} \cdot \dfrac{308}{293} = 3·14$ times as much as air.

A special set of readings at 3·2 cm. is now taken, three for 10 seconds and three for 20 seconds.   The means are 1196 and 2325 respectively.   Comparing these, it is found that the 10-second reading should be multiplied by 1·03 in order to allow for leakage by the insulators.

Again, a set is taken with 600 volts between the plates, and it is found that the mean reading when the copper leak is deducted is 1134.   At the same time the reading for 300 volts, copper leak being deducted, is 1088.   Thus saturation is nearly complete.

A quantity of the gas is now drawn over into an exhausted bulb, whose temperature (that of the oven) is 311 (absolute): the pressure is observed to be 34·5.   The weight of this gas is ·2536 gr.   It is then calculated from a knowledge of the capacity of the bulb that the mixture weighs 2·22 times as much as air.   From this it is found that to every molecule of pentane there are ·23 molecules of air, assuming the pentane molecule to weigh 2·5 times as much as air.   If $s =$ stopping-power of pentane, we have therefore

$$\frac{·23 + s}{1·23} = 3·14 \; ;$$

$$\therefore \qquad s = 3·59.$$

Again
$$RI = 2\cdot95 \times 104\cdot4,$$
$$= 308, \text{ uncorrected.}$$

Correcting for want of saturation,
$$RI = 308 \times \frac{1134}{1088},$$
$$= 321.$$

On the same day and under the same conditions RI for air $= 231$. The leakage correction is found to be the same for both, and need not be applied. Now, as far as consumption of energy is concerned, $\cdot23$ molecules of air are equivalent to $\cdot23/3\cdot59$ molecules of pentane, $= \cdot065$. Hence if all the energy had been spent as pentane molecules, the value for RI would have been

$$1\cdot065 \times 321 - \cdot065 \times 231,$$
$$= 342 - 15,$$
$$= 327.$$

Finally

$$\frac{\text{Total ionization in pentane}}{\text{Total ionization in air}} = \frac{327}{231} = 1\cdot41.$$

It is quite possible that the low density of the mixture ($2\cdot22$ instead of the $2\cdot5$ of pure pentane) is due to an admixture of lighter and more volatile hydrocarbons. If it were so, the result as to the value of RI would not be affected in any essential way.

Determination of Stopping-power and RI of Acetylene (C_2H_2).

Same conditions as previous experiment. Apparatus charged to atmospheric pressure with gas; when tested, gas was found to contain less than 2 per cent. of impurities. Temperature of apparatus $= 37°\cdot5$ C. Barometer $= 763$ mm.

Distance from Ra to middle of ionization chamber.	Leak in ten seconds.
4·2	1430
4·3	1276
4·4	1024
4·5	818
4·6	698
4·7	688
4·8	701
4·9	698
Cu over Ra.	46
At 5·2 for 300 volts net leak	= 688
and „ 600 „ „ „	= 706

Plotting these values it is found that $R = 4 \cdot 57$, I (less copper leak) $= 635$. Hence RI, corrected for want of saturation, $= 298$.

Hence,

$$\frac{\text{Total ionization for } C_2H_2}{\text{Total ionization for air}} = \frac{298}{231} = 1 \cdot 29.$$

Also stopping-power

$$= \frac{483}{457} \cdot \frac{760}{763} \cdot \frac{3105}{2930} = 1 \cdot 11.$$

In the paper by Mr. Kleeman and myself, to which I have already referred, it was pointed out that Rutherford had found it more easy to obtain the saturation current from a gas when it was removed from the influence of the ionizing agent. We observed that this could be easily explained by supposing initial recombination to be completed before the gas was subjected to the electric field. It is nevertheless no essential feature of the initial recombination hypothesis that the act of recombination should take place within any set time. The one important point is that the recombination takes place between two ions originally forming parts of one molecule. It is quite conceivable that for a certain time the positive and negative may remain "semi-detached," their recombination in suspense until precipitated by some change of conditions. Curiously enough Mr. Madsen, working in this laboratory, has not yet been able to repeat Prof. Rutherford's experiment; and his results point to a prolonged existence of these pairs. He finds it hard to saturate a mixture of air and ether vapour which has been ionized by radium and then drawn away into a separate ionization chamber. It is not easy to reconcile this result with that of Prof. Rutherford : and it will be necessary to repeat the experiment under varying conditions. The point is one of considerable interest, for the existence of these pairs would help to explain much of the mechanism of phosphorescence. They would appear to be connected with the clusters of J. B. B. Burke, which were produced by ionization, gave rise to phosphorescent glow, contained energy, yet were not electrified. It is of interest in this connexion that the photograph which

Sir William and Lady Huggins made of the phosphorescent glow of radium showed the bands of the gas in which the salt was imbedded. Rutherford also has shown that the α particle can no longer cause phosphorescence when it has lost its power of ionization.

The Magnetic Deflexion of the α Rays.

In the *Physikalische Zeitschrift* for Oct. 15th is a paper by M. Becquerel, "Ueber einige Eigenschaften der α Strahlen des Radiums." The author discusses the theory that the α rays gradually lose their velocity as they spend their energy on the ionization of the media through which they pass, a theory which I put forward about two years ago [*], and which has the support of much experimental evidence accumulated by Prof. Rutherford [†], and by Mr. Kleeman and myself [‡].

He maintains that the theory is unsuccessful in explaining the experiments which he has himself performed, and in particular he describes one experiment which he has devised as a crucial test and which he considers to show that the theory is incorrect.

It is as follows (*loc. cit.* p. 688) :—

The rays from a small quantity of radium salt are allowed to stream upwards through a narrow slit and fall upon a photographic plate. A powerful magnetic field deflects them slightly to one side. The field is reversed when the experiment is halfway through, and as a result two images of the slit appear, slightly separated, upon the plate. Now, M. Becquerel covers half the slit with a thin sheet of aluminium; and according to the theory which I have advanced, the α rays which pass through the sheet are thereby retarded. Consequently, M. Becquerel argues, these α rays should be more bent to one side than those which have not passed through the aluminium, and the images on the plate should show a break, the lines being more widely separated in one half of the picture than in the other.

But M. Becquerel is under a misapprehension on this point. Paradoxical as it may appear at first sight, no such

* Australasian Association for the Advancement of Science Report, Dunedin, January 1904.
† Phil. Mag. July 1905.
‡ Phil. Mag. Dec. 1904 and Sept. 1905.

break ought to appear, and the photographic
result is quite in accordance with the theory
that the particles lose speed as they pass
through matter.

In order that this may be clear, it is
necessary first to consider the order of the
deflexions of the α rays in the magnetic field,
on the various theories that have been proposed.

Suppose that an α particle is projected from
O in the direction ON with velocity v_0, and
that the action of a field H causes it to describe
the curved path OA.

In the first place, let the velocity be constant
throughout, and the path be therefore circular
as M. Becquerel supposes. Then, since the
curvature is small, $AN = a^2/2\rho$, where $ON = a$,
and ρ is the radius of curvature.

$$\therefore \quad AN = \frac{He}{mv_0} \cdot \frac{a^2}{2}.$$

In the second place let the velocity diminish as the
distance from O increases : and let us take the extreme case,
where the velocity vanishes at a distance a from O.   Let
the path in this case be OA'. It does not make very much
difference what law of diminution of velocity we adopt : let
us suppose, as my experimental results seem to indicate, that
the particle spends its energy at a rate which is inversely
proportional to the square of its speed.   In this case,

$$\tfrac{1}{2}m\frac{dv^2}{ds} = kv^{-2},$$

s being measured from O, and therefore,

$$v^4 \propto (a-s),$$

so that

$$\frac{v^4}{v_0^4} = \frac{a-s}{a}.$$

Thus

$$\rho = \frac{mv_0}{He}\left(1-\frac{s}{a}\right)^{\frac{1}{4}},$$

and we obtain easily that, if $\rho = ds/d\psi$,

$$\psi = \frac{4a}{3}\left\{1-\left(1-\frac{s}{a}\right)^{3/4}\right\}\frac{He}{mv_0}.$$

Now, provided that $\int \psi \, ds$ is small, this quantity is very nearly equal to A'N, the total deflexion of the ray.

But this integral

$$= \int_0^a \frac{4a}{3} \left\{ 1 - \left(1 - \frac{s}{a} \right)^{3/4} \right\} ds \cdot \frac{He}{mv_0}$$

$$= \frac{He}{mv_0} \cdot \frac{4a^2}{7} ;$$

and this quantity is very small, since it is only slightly greater than AN.

Finally then we have that

$$\text{A'N/AN} = 8/7.$$

It is easily found that if we had supposed the particle to spend its energy uniformly along its path, we should have obtained the result A'N/AN = 4/3.

It will thus be clear that, on any reasonable hypothesis as to the particular law of diminution of velocity, the actual path of the particle differs very little from a circle. In the extreme case which I have considered, the small deviation therefrom at the end of the path is small compared with the widths of the images in M. Becquerel's photograph. If the particle ceases to ionize whilst its velocity is still great, as has been shown by Prof. Rutherford, the variation is still less.

Let us now consider the circumstances of M. Becquerel's experiment.

As a first approximation, suppose the widths of the groove containing the radium salt and of the slit to be negligible.

If no magnetic field is acting, all the α particles move in the vertical line ON. The range of the particles from Ra C is very nearly 7·0 cm.; from which it follows that the number which pass any given point P is proportional to the defect of OP from 7·0 cm., or, in other words, that the number n which end their flight on any unit of length of ON is a constant. The other three groups of particles have, as their furthest distances of penetration, 4·8, 4·2, and 3·5 cm. respectively. Thus between 4·8 and 4·2, $2n$ particles end their flight on each unit of length, between 4·2 and 3·5 the number is $3n$, and from that point up to the radium $4n$. The radium salt is supposed deep enough to supply all these, *i. e.* its depth is taken to be at least ·001 cm. Suppose now a powerful magnetic field to

be brought into play, the direction of the lines of force being normal to the plane of the paper. The paths of the particles are curved to one side, and the curvature is greater the nearer the particle is to the end of its course. Let OA and OQ represent two such paths. Their separation from each other is considerably exaggerated in the figure. If all the paths were drawn the locus of Q would be seen to be a curve, whose curvature in contrast to that of the path of any one particle would be greater the further the distance from A. This is in agreement with M. Becquerel's experiments, as I have previously pointed out[*].

Fig. 2.

The width of the trace upon the paper of all the paths of the particles is very small, and is almost too fine to be shown on a diagram.

It is perhaps well to point out that there is no break in this trace at the critical points 4·8, 4·2, and 3·5. It is quite smooth from end to end. These points mark the extreme distance to which various bundles of rays penetrate; but the deflexion of an α ray which ends its course at a given point is independent of the particular radioactive material from which it has come: the only varying characteristic of an α particle is its velocity.

We must now take into account that the widths of the slit and the groove are not negligible, as is clearly to be seen from the photograph under consideration. There is consequently, so to speak, a large penumbra. Thus the trace upon the plane of the paper of all the α rays is such as is represented in fig. 3, the deflexions being all exaggerated so as to be capable of depiction.

Now suppose an aluminium plate is placed, as in M. Becquerel's experiment, over the slit so that the α

Fig. 3.

Q

Photographic plate

Slit

Ra.
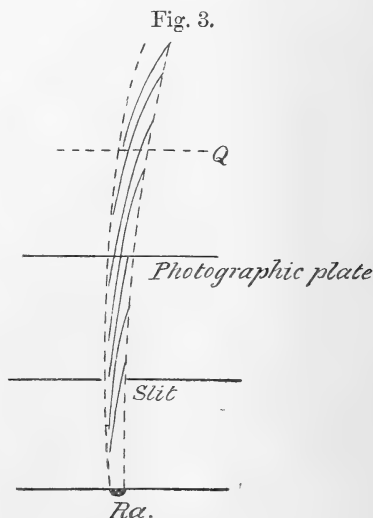
* Phil. Mag. Dec. 1904, p. 737 ; *Jahrbuch d. Rad. u. Elektr.* 1905, p. 14.

particles have to pass through it on the way to the photographic plate. M. Becquerel supposes that there ought therefore to be an increased displacement of the photographic image. But this is not so. The path of any one α particle is slightly deflected, but the whole trace is not appreciably disturbed. The aluminium diminishes the range of every α particle by the same amount, but the only result is to cut off all the rays which would have gone past a certain point, say Q, and to cause them to take the places of those rays which fell short of Q ; these latter being further shortened. This does not in the least affect the position of the outer edge of the trace upon the photographic plate; and though there must be a slight movement of the inner edge, so that the trace is somewhat narrower, the change is so small that it could not possibly be detected, as a glance at the photograph will show. Magnetic dispersion of the α rays does exist : it has been directly shown by Rutherford, and, as I think, indirectly by M. Becquerel's own experiments, in the peculiarities of the curvature of his photographic traces. But it could not be shown in the manner of the experiment which M. Becquerel now describes. That would be analogous to the search for evidence of the motion of the stars in the line of sight in the displacement of the visible spectrum as a whole ; whereas the measurement to be made is of the displacement of the Fraunhofer lines in the spectrum, *i. e.* of one set of waves which can be isolated for consideration. It is here that Rutherford's experiment is differentiated from that of M. Becquerel. The former employed as a source of rays a wire coated with a thin layer of Ra C emitting α particles of uniform velocity, which is analogous to confining one's attention in the star problem to waves of one length. Moreover, Rutherford passed his α particles for some considerable distance through a vacuum whilst yet under the influence of the magnetic field. Thus the evidence of the increase of curvature in their paths, originally caused by the loss of velocity in penetrating matter, was accumulated. But if, as in M. Becquerel's experiment, the path is in the air, then any appreciable increase of curvature closely precedes the cessation of all evidence of motion, and the result must be in any case almost beyond detection.

M. Becquerel remarks that there is no evidence in his photographs of the greater precision of the outer line of the trace, which I had anticipated. But the photograph which he now publishes show that there is too much penumbra for such an effect to be visible.

The Positive Charge of the α Particle.

Considerable discussion has recently taken place as to the mode in which the α particle acquires its positive charge. It has been pointed out more that once that it may be explained as the result of ionization by collision (Rutherford, Address to Congress at St. Louis, 1904 ; Bragg, Phil. Mag. Dec. 1904), and that the same hypothesis will explain the deposit of the radium emanation on the negative electrode (Bragg and Kleeman, Phil. Mag. Dec. 1904). In the case of the emanation an explanation, which I understand to be similar, has been carefully worked out by Makower (Phil. Mag. Nov. 1905).

Rutherford has shown that the α particle is charged at the moment of leaving the radium salt. But I do not think that the result is in any way prejudicial to the collision theory. He evaporated a very weak solution of radium on a plate, and supposed that as a result he had an excessively thin layer, so that the particle made very few collisions before emergence. But when such deposits are examined under a microscope, it is seen that the salt is gathered in little heaps, and there is no true layer at all. The bulk of the α particles pass through hundreds of atoms before emergence, and there is ample opportunity for ionization by collision.

We find that the α particle spends energy in causing the expulsion of electrons from the atoms of any gas which it traverses. We find also (see Tables A and B, above) that the expenditure of energy by the α particle follows the same law when the atoms are massed together in a solid. The solid must therefore be ionized, just as the gas is, and we should expect slow-moving electrons to be projected from Ra itself, and from both sides of any solid screen through which the particles pass. Surely this is the effect lately observed by **J. J.** Thomson, Rutherford, and others. This has been suggested by Soddy ('Nature,' March 1905).

LVII. *Note on the Deflexions caused by a Break in an Over-head Wire carried on Poles. By* W. Hawthorne, *B.A., B.E., and* W. B. Morton, *M.A.*[*]

IN the modern conditions of electrical power-transmission along overhead wires, it is a matter of practical importance to know what will happen in case a wire breaks. The relaxing of tension at one point will throw a one-sided pull on the range of poles on either side of the break; and it is essential for safety that the maximum deflexion thus produced should be below a definite limit.

* Communicated by the Authors.

The theoretical investigation, given below, presents no special difficulty and leads to results of some interest. It is followed by a numerical evaluation of the deflexions for a particular case, in which we have used values for the constants derived from a case occurring in actual engineering practice.

We shall suppose the wire to be fastened at its ends to massive "anchor-poles" whose yielding is negligible, and to be supported by equally-spaced flexible poles. The relation between force and deflexion for a horizontal pull applied to the top of a pole is supposed known. If this be expressed by the equation

$$T = \phi(x),$$

then in the region of safe deflexions the function is linear, and ϕ may be regarded as a numerical constant.

The other relation which enters into the problem is that giving the alteration in the horizontal tension of a flat catenary for a small horizontal displacement of its ends.

The arc of a catenary is connected with the abscissa of its end and the parameter by the equation

$$s = c \sinh \frac{x}{c}.$$

Differentiating this, keeping s constant,

$$dc \left(\sinh \frac{x}{c} - \frac{x}{c} \cosh \frac{x}{c} \right) + dx \cosh \frac{x}{c} = 0,$$

$$dx = dc \left(\frac{x}{c} - \tanh \frac{x}{c} \right).$$

If ϕ is the inclination of the tangent at the end of the catenary,

$$\frac{x}{c} = \log \left(\sec \phi + \tan \phi \right);$$

so that if ϕ is small, $\dfrac{x}{c}$ is small of the same order, and

$$\frac{x}{c} - \tanh \frac{x}{c} = \frac{x^3}{3c^3};$$

$$\therefore \quad dc = \frac{3c^3}{x^3} dx.$$

If w is the weight of unit length of wire and T the horizontal tension,

$$\mathrm{T} = wc \ ;$$

$$\therefore \quad d\mathrm{T} = \frac{3\mathrm{T}^3}{2w^2x^3} \cdot 2dx = \frac{3\mathrm{T}^3}{2w^2v^3} \times \text{(increase of horizontal span)}.$$

We shall write f for $\dfrac{3\mathrm{T}^3}{2w^2x^3}$.

Let the flexible poles be numbered from an anchor-pole. Let x_r be the deflexion of the top of the rth pole, and let the break occur beyond the nth pole. We shall take first the simpler case where there is only one wire suspended, or where, if there are several wires, these all break together.

Denote the horizontal tensions in successive intervals by $\mathrm{T}_{01}, \mathrm{T}_{12}, \ldots \mathrm{T}_{(n-1)n}$, and the common value before the break occurs by T. Then we have

$$\mathrm{T}_{01} \qquad = \mathrm{T} - fx_1,$$
$$\mathrm{T}_{12} \qquad = \mathrm{T} - f(x_2 - x_1),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\mathrm{T}_{(n-1)n} = \mathrm{T} - f(x_n - x_{n-1}).$$

$$\mathrm{T}_{01} - \mathrm{T}_{12} = \phi x_1,$$
$$\mathrm{T}_{12} - \mathrm{T}_{23} = \phi x_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\mathrm{T}_{n-2, n-1} - \mathrm{T}_{n-1, n} = \phi x_{n-1},$$
$$\mathrm{T}_{n-1, n} = \phi x_n.$$

On eliminating the tensions $\mathrm{T}_{01}, \mathrm{T}_{12}$, &c., we get the following equations connecting consecutive deflexions, in which for shortness we write

$$2 + \frac{\phi}{f} = a,$$
$$\frac{\mathrm{T}}{f} = b,$$

$$x_2 - ax_1 \qquad\qquad = 0,$$
$$x_3 - ax_2 + x_1 \qquad = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_n - ax_{n-1} + x_{n-2} = 0,$$
$$(a-1)x_n - x_{n-1} \quad = b.$$

The first $(n-1)$ equations determine the ratios of the n deflexions, in terms of the single constant a. It will be seen that the *relative* magnitudes of the yieldings in the successive poles do not depend on the position of the break. The latter fixes the absolute scale, in accordance with the last equation.

The x's form a recurring series, whose " generating function " is

$$\frac{x_1}{1 - ay + y^2};$$

i. e., x_r is the coefficient of y^r in the expansion of this fraction in ascending powers. Expressed explicitly we have

$$\frac{x_r}{x_1} = a^{r-1} - \frac{r-2}{1} a^{r-3} + \frac{(r-4)(r-3)}{1 \cdot 2} a^{r-5}$$
$$- \frac{(r-6)(r-5)(r-4)}{1 \cdot 2 \cdot 3} a^{r-7} + \dots.$$

Consider now the ratios of *consecutive* deflexions. We have

$$\frac{x_2}{x_1} = a,$$

$$\frac{x_3}{x_2} = a - \frac{x_1}{x_2} = a - \frac{1}{a},$$

$$\frac{x_4}{x_3} = a - \frac{x_2}{x_3} = a - \frac{1}{a - \frac{1}{a}},$$

$$\frac{x_r}{x_{r-1}} = a - \frac{1}{a - \frac{1}{a -}} \dots (r-1) \, a\text{'s}.$$

Now, from the nature of the case, $a > 2$; and it can be shown that the continued fraction then approaches a finite limit as r increases [*]. In fact, if z represents this limiting value, we have

$$z = a - \frac{1}{z};$$

or z satisfies

$$z^2 - az + 1 = 0.$$

The roots of this are $\frac{1}{2}\{a \pm \sqrt{a^2 - 4}\}$, which are real if $a > 2$.

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[*] *Cf.* Chrystal's 'Algebra,' vol. ii. p. 482.

Since $\dfrac{x_r}{x_r-1} > 1$, we take the upper sign and so get

$$\operatorname*{Lt}_{r=\infty} \frac{x_r}{x_r-1} = \frac{a+\sqrt{a^2-4}}{2}.$$

Thus at a distance from an anchor-pole the successive deflexions approach a geometric progression. It will appear in the subsequent part of the work that in our actual case this state of affairs is closely approached after seven or eight poles are passed.

The existence of this limiting ratio, taken in conjunction with the last equation of the set, leads to the consequence that the *maximum deflexion*, which of course occurs in the pole next to the break, and which becomes greater as this pole is further removed from the anchor-pole, *approaches a limiting value*. If this limit is below the allowable safe deflexion, it does not matter how widely the anchor-poles are distributed.

To find this limiting deflexion we have

$$x_n \left\{ (a-1) - \frac{x_n-1}{x_n} \right\} = b,$$

$$x_n = \frac{b}{(a-1) - \dfrac{x_n-1}{x_n}}.$$

Giving the ratio in the denominator its limiting value, we find that the deflexion cannot exceed

$$\frac{2b}{a-2+\sqrt{a^2-4}}.$$

If the *original* horizontal tension acted on one side of the pole next the break, it would produce a deflexion $\dfrac{\mathrm{T}}{\phi} = \dfrac{b}{a-2}.$ It will be seen that the actual maximum deflexion is less than this, on account of the increased sag of the remaining wires, in the ratio

$$2 : 1 + \sqrt{\frac{a+2}{a-2}}.$$

It is easy to extend the analysis to the case where some, only, of a number of similar wires are broken. We shall give the equations arrived at for the deflexions.

Let the break occur between the nth pole counting from one anchor-pole, and the mth counting in the opposite direction

from the next anchor-pole. Call the deflexions on one side of the break $x_1 \ldots x_n$, and on the other $y_1 \ldots y_m$, x_n y_m being on the two sides of the break, and x's and y's being measured away from the break on both sides. Let s represent the fraction $\dfrac{\text{number of wires remaining}}{\text{number originally on}}$.

The equations giving the $(n-1)$ ratios of x's and $(m-1)$ ratios of y's are the same as those already got. The connecting equations which determine the actual magnitudes are

$$(a-1+s)x_n - x_{n-1} + sy_m = b(1-s)$$
$$= (a-1+s)y_m - y_{m-1} + sx_n.$$

The quantities a and b have their former meanings, T and w which occur in them referring to the whole set of wires and not to a single wire.

We have
$$(a-1)x_n - x_{n-1} = (a-1)y_m - y_{m-1}.$$

So that the ratios of the deflexions on the opposite sides of the gap are independent of the number of wires broken.

If the break is sufficiently distant from both ends of the row so that we may use the limiting value for $\dfrac{x_{n-1}}{x_n}$ and $\dfrac{y_{m-1}}{y_m}$, we get

$$x_n = y_m = \frac{2b(1-s)}{a-2+4s+\sqrt{a^2-4}}.$$

Coming now to the application of our results to the actual case referred to, we have the following data from which to calculate the constants a and b.

Number of wires 6.
Tension of each wire 1000 lbs.
Weight of each wire ·42 lb. per foot.
Distance between consecutive poles 100 yards.

A test of the stiffness of a pole showed that a deflexion of 3 ins. was caused by a force of 800 lbs. applied to the top in a horizontal direction.

From these values we calculate

$$a = 2\cdot21,$$
$$b = 0\cdot40,$$

the units being lbs. and feet.

The following table gives the calculated deflexions in feet

when the break occurs after the 1st, 2nd, ... 7th pole, counting from an anchor-pole.

Deflexion of..	1st	2nd	3rd	4th	5th	6th	7th	Pole.
When $n=1$	·33							
2	·24	·53						
3	·16	·36	·62					
4	·11	·23	·41	·67				
5	·07	·15	·26	·43	·68			
6	·04	·09	·17	·27	·44	·69		
7	·03	·06	·11	·17	·28	·44	·694	

The limit of the ratio of successive deflexions is 1·58, and the limiting value approached by the maximum deflexion is ·696 foot. It will be seen that this is almost reached at seven poles' distance from the fixed end.

Fig. 1.
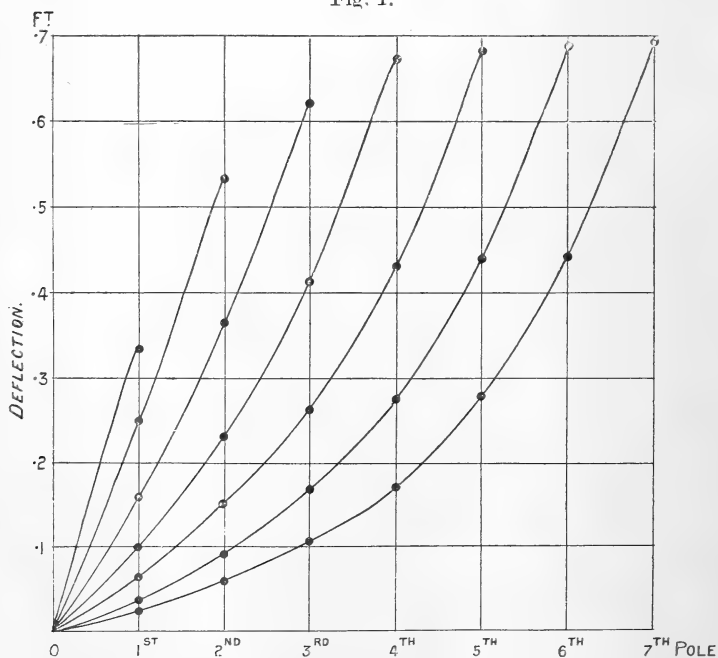
These results are shown graphically in fig. 1. The points corresponding to each position of the break are strung

together on a curve, although of course the intermediate points on the curves have no physical meaning. The asymptotic approach of the extreme points of the curves to the limiting value is very obvious.

If the original horizontal tension acted on one side of a pole, it would cause a deflexion of 1·9 feet, supposing the linear relation to hold up to this range.

We have calculated also the *maximum* deflexions liable to be caused by breakage of 1, 2, ... 6 wires, the point of break being distant from both fixed ends.

No. of Wires broken.	Deflexion.
1	·03 ft.
2	·07
3	·127
4	·215
5	·367
6	·696

Fig. 2.
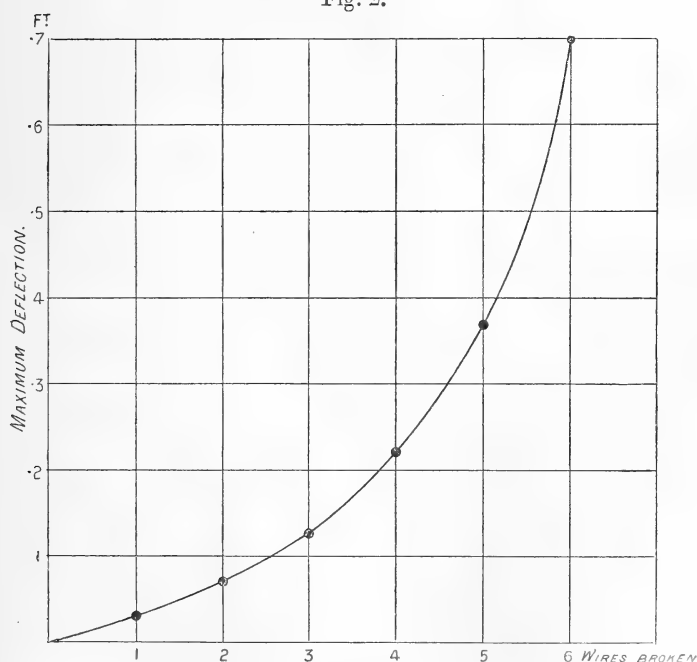
These numbers are plotted in fig. 2. The points lie on an hyperbola, as is shown by the equation obtained above to connect x_n and s.

LVIII. *The Isothermal Distillation of Nitrogen and Oxygen and of Argon and Oxygen.* By J. K. H. INGLIS [*].

MANY investigations have been made in order to find the relation connecting the composition of the vapour with the composition of the liquid, when a mixture of two liquids is distilled isothermally (see Young's 'Fractional Distillation,' Macmillan & Co., where a full summary of the literature on this subject may be found); and it has been shown that in some cases the relation takes the simple form $r_v = k \cdot r_l$, where r_v is the ratio of the two substances in the vapour, r_l the corresponding ratio in the liquid, and k a constant. In most cases, however, k is not an absolute constant but varies slightly with the molecular composition of the liquid; and we thus get mixtures (*a*) which have a maximum vapour-pressure, (*b*) which have a minimum pressure, and (*c*) which although having neither maximum or minimum vapour-pressure do not satisfy the above relation. These different cases have been very fully investigated by Zawidzki and others at ordinary temperatures; but the only paper dealing with distillation at low temperatures is one by Mr. E. C. C. Baly (Phil. Mag. vol. xlix. p. 517, 1900), who carried out a series of distillations of liquid air under a pressure of one atmosphere. In isobaric distillations, however, the conditions are not so simple as they are when the temperature is constant; so at Mr. Baly's suggestion I decided to complete his work by making a series of isothermal distillations at low temperatures, and I received considerable help from him in the early experiments when we hoped to make the research a joint one.

Many forms of apparatus have been devised for carrying out distillations isothermally, but none of them were suitable for work at low temperatures. Moreover, most of the forms used are open to the objection that care is not taken to ensure that the vapour is in complete equilibrium with the liquid; and in addition considerable error may be introduced by back condensation, &c. The apparatus used by Zawidzki (*Zeit. f. phys. Chemie*, vol. xxxv. p. 129) avoids most of these sources of error; but even in this case, since the sample of distillate is obtained by distilling over a small quantity of the liquid, the composition of the liquid may change during this operation, and so introduce error. Such an error would be of considerably greater importance in dealing with

liquefied gases; for in this case one cannot work with such large quantities of liquid as Zawidzki used. But this error and many others may be eliminated by circulating the vapour through and through the liquid until no further change in either takes place ; and then collecting and analysing a portion of the vapour which has in this way been brought into equilibrium with the liquid.

With such an arrangement it was necessary to keep the temperature of the distillation-bulb very constant for a considerable time, and an accurate means of measuring the temperature was also required. The distillation-bulb was therefore immersed in a long cylindrical vacuum-vessel containing about 600 c.c. of liquid air, and the pressure under which this air was boiling was varied by means of a Fleuss pump so as to keep the temperature constant. To measure the temperature, a bulb containing liquid oxygen and connected to a manometer was immersed in the liquid air so that the manometer registered the vapour-pressure of pure oxygen at the temperature of the distillation-bulb. With this arrangement the Fleuss pump was worked at such a rate that the manometer showed a constant pressure. In the different experiments this pressure amounted to either 100, 200, or 300 mm., and the variation of pressure seldom exceeded 0·5 mm. This variation corresponds to a temperature difference of 0°·03 C., 0°·02 C., and 0°·01 C. in the three different cases. These pressures, according to Travers (Phil. Trans. 200. p. 105) correspond to 74°·7, 79°·07, and 82°·09 Absolute measured on the hydrogen scale.

The apparatus used for carrying out the distillations is shown diagrammatically in fig. 1, the liquid oxygen bulb and the corresponding manometer being omitted. The rubber cork A, which fits the mouth of the vacuum vessel containing the liquid air, has passing through it the liquid oxygen bulb and several tubes of which three, 1, 2, 3, are joined to the distillation-bulb P, and another, not shown in the figure, leads to the Fleuss pump. In an experiment, B is half filled with the mixture to be distilled, and the vapour, after being thoroughly cooled by the spiral 4, is blown through the liquid in B, the tube 3 then carrying the vapour on into the circuit. The tube 1 is joined to the bulb B quite close to the bottom, and for a considerable part of its length consists of an extremely fine capillary drawn out of an ordinary piece of capillary tubing. This tube is also connected to a gas-holder C, so that when the reservoir of C is lowered liquid is sucked up the fine capillary to a warm part above the liquid air, and there boils as a whole and is collected in the gasholder.

The circulation of the vapour is carried on by means of the mercury circulator DEFG of the pattern described by Collie (Journ. Chem. Soc. 1889, p. 110). Mercury flows from the reservoir D through a rubber tube furnished with a screw-clip and then falls in drops down the tube EF. As each

Fig. 1.
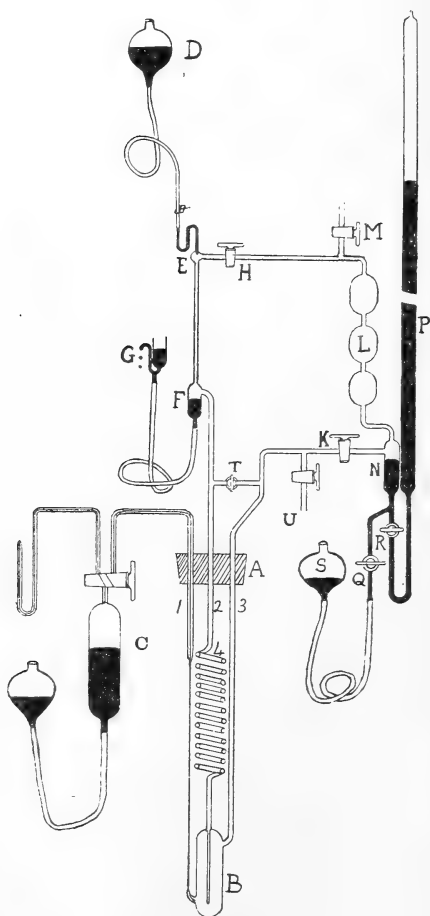
drop falls it drives on the gas contained in the tube EF and pumps gas in the direction KLHE. In this way the circulator pumps the vapour up the tube 3 and blows it down 2,

and thus sends it round and round the circuit. The mercury after it falls down the circulator is collected at the exit G, the height of which above or below F is adjusted so as to keep the surface of the mercury just below the side tube leading from F.

The sample of vapour is collected in the bulbs L through which the vapour is circulated; by closing the taps HK this sample of vapour is shut off from communication with the liquid in P, and can be collected by opening the tap M which leads to a Töpler pump. The pressure under which the liquid is boiling can be measured in two different ways. It can either be calculated from the difference in height of the mercury surfaces in F and G and the height of the barometer; or it may be measured directly by means of the closed manometer NP. The shorter open limb N of this manometer can be shut off from the closed limb P by means of the tap R. By means of the taps QR and the reservoir S the amount of mercury in the manometer can be so regulated that when R is open the mercury surface in N is close below the side tube coming from the tap H. In this way dead space is as far as possible avoided. This manometer could be used for pressures up to 770 mm., and the manometer GF was used only for pressures greater than this. Readings of the manometer NP were made during the circulation, so that one could see when the reading became constant; but the final readings were taken by means of a telescope after the circulator had been stopped and after the tap T connecting the tubes 2, 3 had been opened. It was found convenient to close the tap R a moment after T had been opened and before the readings were taken.

The gases were stored over water in ordinary glass gas-holders—the argon alone being kept over mercury; and after passing through soda-lime and phosphorus-pentoxide tubes, were admitted to the apparatus by means of the tap U. As regards the gases themselves, the oxygen was in all cases obtained by the decomposition of potassium per-manganate. The nitrogen was prepared by heating a mixture of solutions of ammonium sulphate and potassium nitrite, and was fractionated by means of liquid air before being passed into the gasholders. A small quantity of oxygen was then added and the gas left standing over water. In this way, any traces of nitric oxide were turned into nitrous or nitric acids which dissolved in the water, and the small quantity of oxygen (0·5 per cent.) remaining did not matter, as mixtures of nitrogen and oxygen were to be used. The

argon used, which was kindly supplied to me by Sir Wm. Ramsay, was purified by means of a hot mixture of quick-lime and magnesium, and was fractionated with liquid air to remove traces of helium and neon. Its spectrum showed that it was extremely pure.

The analyses in each case were carried out by measuring off 10–13 c.c. of the gas, and then removing the oxygen by means of a pellet of yellow phosphorus. A measurement of the volume of the residual gas then gave the molecular composition. Tests showed that analyses carried out in this way did not have a greater error than 0·1 per cent., which was sufficiently accurate.

Two parts of the apparatus needed exhaustive testing before one could be sure of the results. The method of taking the sample of the liquid was based on the assumption that by taking a fine enough capillary, the liquid would evaporate as a whole and would not fractionate itself. The first capillaries employed were found to give very variable, and therefore untrustworthy samples; but by inserting a fine drawn-out capillary, as drawn in fig. 1, concordant results were obtained. These tests were carried out as follows:—About 5 litres of dry air were condensed in the bulb B, and samples of the liquid, which half-filled the bulb, were then taken and analysed. The percentages of nitrogen found were 78·77, 78·52, 78·74, 78·66, and 78·84, which results, omitting the second one, give 78·75 per cent. as a mean, the deviation of the second one from this mean being 0·2 per cent. But all the results are lower than the true percentage of nitrogen in the air, which was found to be 79·06, 78·99 per cent. in two consecutive experiments. This difference can be explained as follows:—When the air was condensed, the spiral 4 and the part of the bulb B above the liquid were filled with vapour which was in equilibrium with the liquid, and which therefore did not have the same composition as the air. The vapour, in fact, contained an excess of nitrogen, so that the liquid contained too high a proportion of oxygen. The vapour also being at a temperature of less than 80° Abs., the dead space contained nearly 4 times as much gas as the same space would at ordinary temperatures. Thus 25 c.c. of dead space represented 100 c.c. of vapour measured at ordinary temperatures. This vapour contained about 5·5 per cent. of oxygen, so that the dead space contained $94·5 - 4 \times 5·5 = 72·5$ c.c. of nitrogen above the proper amount for the oxygen. This amount of nitrogen is to be deducted from 5 litres of air; and this brings down the true

percentage in the liquid to 78·69 per cent., which is sufficiently near the mean value 78·75 per cent. found by experiment. It is certainly surprising that the amount of dead space can make so great a difference in the composition of the liquid; but it is of course due to the temperature of the dead space being so low and to the great difference in the vapour-pressures of nitrogen and oxygen. These tests show therefore that the sample of the liquid can be taken pretty accurately, the error being certainly not more than 0·3 per cent. and usually not more than 0·1 per cent.

The sampling of the vapour offered no possibility of error, but it was necessary to ascertain how long the circulation must be carried on in order to obtain true equilibrium between the vapour and the liquid. About 5 litres of a mixture of nitrogen and oxygen containing about 75 per cent. of nitrogen were condensed in the bulb B, and after ten minutes circulation (the circulator pumped the gas at the rate of about 50 c.c. a minute) samples of the liquid and vapour were taken. Circulation was then carried on for a further fifteen minutes and fresh samples taken, and again for a further twenty minutes and a third pair of samples taken. The tubes K, L, H were then pumped out, and while some more of the mixture was being taken into B through the tap U the tap K was opened, thus filling the bulbs L with the mixture; a second series of experiments was then made exactly as at first. The results are given in Table I.

TABLE I.

Time of circulation......	10 min.	15 min.	20 min.	10 min.	15 min.	20 min.
Per cent. of Nitrogen in liquid	74·9	75·0	74·4	74·8	74·3	74·4
Per cent. of Nitrogen in vapour	93·1	93·0	92·7	93·0	92·8	93·0

These experiments show that the liquid sampling was not very reliable; but the experiments were made before the method of liquid sampling had been perfected. The vapour samples show slight variations, and equilibrium is apparently reached after ten minutes circulation; so a *minimum* of twelve minutes was fixed upon as being sufficient to ensure equilibrium. It should be remembered that the percentage of nitrogen in the liquid sample must steadily fall; for the vapour is richer in nitrogen than the liquid, and some of the vapour is removed in taking each sample of the vapour. This to a certain extent explains the lower value obtained for the

third sample, *i. e.* after twenty minutes circulation in each case.

The apparatus having thus shown itself accurate and convenient for working, two series of distillations were undertaken. In the first of these, the results of which are given in Table II., twenty-three mixtures of nitrogen and oxygen were distilled at that temperature (74°·7 Abs.) at which the vapour-pressure

TABLE II.

$$P_{N_2}/P_{O_2} = 5·31. \qquad C = 6·60 - 0·028 \ m.$$

Temp. 74°·7 Abs. Pure oxygen has vapour-press. = 100 mm.

No.	Molec. percent. N_2 in liquid. Obs.	Molecular percentage Nitrogen in Vapour:			Total Pressure: mm.			Part. Pressure Nitrogen.	Part. Pressure Oxygen.
		obs.	calc.	diff.	obs.	smoothed	diff.		
							%	mm.	mm.
1...	0·0	0·0	0	...	100·0	100·0	...	0·0	100·0
2...	5·3	25·5	26·5	−1·0	130·6	130·0	+·5	34·5	95·5
3...	7·3	34·3	33·6	+·7	140·7	141·0	−·2	47·5	93·5
4...	11·4	44·5	44·7	−·2	162·9	162·7	+·1	72·7	90·0
5...	16·6	55·5	55·0	+·5	190·2	190·0	+·1	104·5	85·5
6...	21·0	61·4	61·5	−·1	211·2	210·5	+·3	129·5	81·0
7...	25·6	66·5	66·9	−·4	232·2	232·7	+·1	155·7	77·0
8...	30·5	71·2	71·6	−·4	254·5	255·0	−·2	182·5	72·5
9...	33·4	73·6	73·9	−·3	267·1	267·8	−·2	197·9	69·9
10...	37·5	76·8	76·7	+·1	284·8	285·0	−·1	218·6	66·4
11...	41·8	80·0	79·7	+·3	304·2	303·7	+·2	242·0	61·7
12...	44·5	81·5	81·1	+·4	315·6	315·0	+·2	255·4	59·6
13...	48·4	83·0	83·1	−·1	330·1	330·5	−·1	274·6	55·9
14...	52·2	84·6	84·6	0	346·4	346·7	−·1	293·3	53·4
15...	56·4	86·4	86·6	−·2	361·1	363·0	−·5	314·3	48·7
16...	57·3	87·2	87·0	+·2	366·0	366·5	−·1	318·8	47·7
17...	63·1	89·2	89·2	0	388·7	390·0	−·3	347·9	42·1
18...	68·6	91·0	91·1	−·1	408·5	411·2	−·7	374·6	36·6
19...	74·8	93·0	93·0	0	432·5	435·5	−·7	405·0	30·5
20...	79·2	94·2	94·3	−·1	453·3	453·0	+·1	427·2	25·8
21...	84·0	95·9	95·7	−·2	467·5	471·0	−·8	450·7	20·3
22...	88·9	97·0	97·1	−·1	489·7	490·0	−·1	475·8	14·2
23...	93·9	98·4	98·4	0	509·8	509·0	+·2	500·8	8·2
24...	99·4	99·9	99·9	0	530·0	529·0	+·2	528·5	0·5
25...	(100·0)	(100·0)	(100)	...	(531·0)	(531·0)	...	(531)	

of pure oxygen is 100 mm.; and the total pressure, the composition of the vapour, and the composition of the liquid were determined. In the second series, which was carried out at 79°·07 Abs., where the vapour-pressure of pure oxygen is 200 mm., only eleven mixtures were taken; the results are given in Table III.

<div style="text-align:center">

TABLE III.

$P_{N_2}/P_{O_2} = 4 \cdot 655.$ $C = 5 \cdot 48 - 0 \cdot 0207\, m.$

Temp. $79°\cdot07$ Abs. Pure oxygen has vapour-press. $= 200$ mm.

</div>

No.	Molec. percent. N_2 in liquid. Obs.	Molecular percentage Nitrogen in Vapour:			Total Pressure: mm.			Part. Pressure Nitrogen.	Part. Pressure Oxygen.
		obs.	calc.	diff.	obs.	smoothed	diff.	mm.	mm.
							%	mm.	mm.
1...	0	0	0	...	200	200	...	0	200
2...	3·4	16·7	16·0	+·7	233·2	231·5	+·7	37·0	194·5
3...	10·7	39·2	38·7	+·5	297·8	295·5	+·8	114·3	181·2
4...	19·7	55·5	55·4	+·1	371·8	371·5	+·1	205·8	165·7
5...	27·8	65·4	65·4	0	435·7	435·5	+·0	284·8	150·7
6...	40·5	75·8	75·9	−·1	529·7	530·0	−·0	402·2	127·8
7...	51·0	82·1	82·1	0	(599)*	604·0	−·8	495·5	108·5
8...	60·4	86 4	86·6	−·2	660·4	667·0	−1·0	577 6	89·4
9...	72·8	91·5	91·3	+·2	746·0	750·0	−·5	684·7	65·3
10...	82·8	94·8	94·8	0	815·5	816·0	−·0	773·6	42·4
11...	90·8	97·6	97·2	+·4	872·5	870·0	+·3	845·6	24·4
12...	99·7	99·7	99·95	−·15	931·0	928·5	+·2	928·0	0·5
13...	(100·0)	(100·0)	...	...	...	931·0	...	931·0	0

* In the case of Exp. 7, the total pressure was by a slip read only during circulation and is therefore certainly too low.

Discussion of Results.

If the ratio of nitrogen to oxygen in the vapour be compared with the same ratio for the liquid, it is found that the quotient of these two ratios is a linear function of the molecular composition of the liquid. Thus taking the results of Table II. we find that the experimental values, *i.e.* those values given in the second and third columns, approximately satisfy the relation

$$\frac{\text{Ratio } N_2 : O_2 \text{ in vapour}}{\text{Ratio } N_2 : O_2 \text{ in liquid}} = 6 \cdot 60 - 0 \cdot 028\, m = C,$$

where m is the molecular percentage of nitrogen in the liquid. Similarly in Table III. the value of this same quotient is $5 \cdot 48 - 0 \cdot 0207\, m$. Now by means of these formulæ we can calculate the composition of the vapour from the composition of the liquid, and these results are given in the fourth column of the tables under the heading " nitrogen in vapour calculated." It will be seen that the differences from the observed values are slight except when the value of m is small. Now in this case an error of $0 \cdot 1$ per cent. in the liquid corresponds

to an error of 0·5 per cent. in the calculation of the percentage of the vapour; so that the apparently large differences between the calculated and observed values correspond to only small errors in the analysis of the liquid. The formulæ given can therefore be fairly used to smooth the experimental results.

As regards the total pressure, since this could only be smoothed graphically, the results were plotted on squared paper on which 2 mm. corresponded to 1 per cent. in the liquid, the pressures being plotted full size. A steady curve was then drawn through the experimental points and the "smoothed" values of the total pressure taken from this curve. Comparison of the "smoothed" and "observed" values shows a fairly good agreement considering how rapidly the vapour-pressure changes with the temperature. From these "smoothed" values of the total pressure and the composition of the vapour, the partial pressures of the nitrogen and the oxygen may be calculated, and these results are given in the last columns of the tables.

When now these partial pressures and the total pressure are plotted in the usual way against the molecular composition of the liquid (see fig. 2 for the results of Table II.), it is found that the curves obtained, though they have only slight curvature, are certainly not straight lines—a result which was to be expected from the fact that the relation $r_v = kr_l$ did not hold good. This relation is the mathematical expression of the property " the ratio of the concentrations in the vapour is proportional to the ratio of the concentrations in the liquid." Hence we might expect to find the concentration of either substance in the liquid proportional to its partial pressure as vapour, this being the relation known as Henry's law. Now the vapour is usually plotted against the molecular percentage, and a straight line for the vapour-pressure would therefore indicate a proportionality between the vapour-pressure and molecular percentage. But since the volume of a gramme-molecule of a liquid is a quantity which is not the same for different liquids, the molecular percentages are not true concentrations. Hence a straight line in the usual method of plotting would indicate a deviation from Henry's law; and it is therefore evident that the correct method is to plot the partial pressures against the true concentration.

To obtain the true concentration one needs to know the density of each mixture used. But a close approximation to the actual value of the density may be obtained by calculating the density of the mixture from the densities of the pure components, *assuming that no contraction takes place.* Any

contraction would of course make this calculated value wrong; but the error thus introduced would be of a higher order than the differences in density of different mixtures of the same two liquids. Measurements of the density of liquid

Fig. 2.

Vapour-pressures at 74°·70 Abs
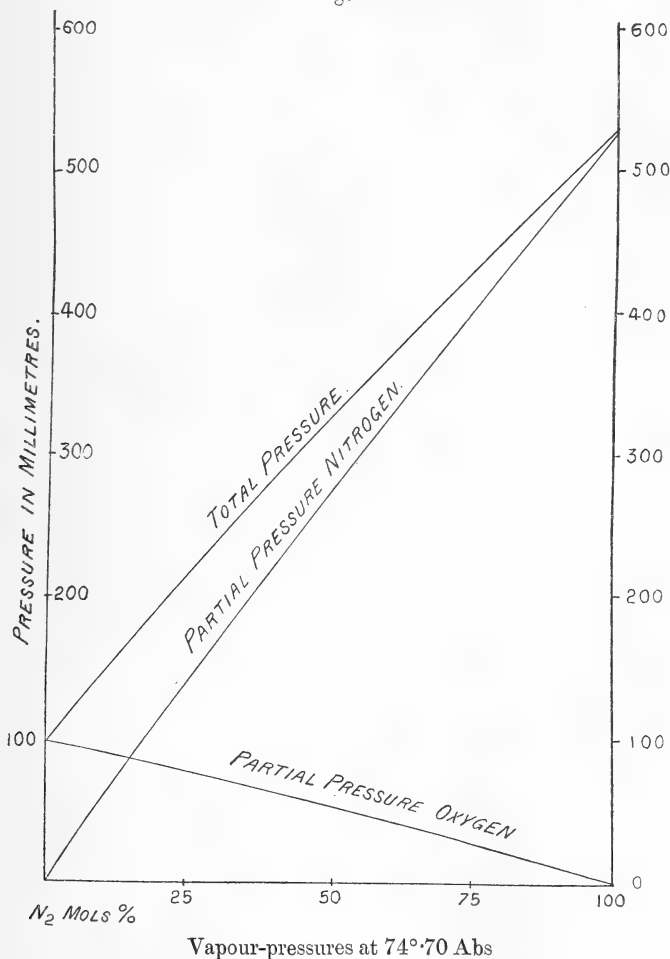
nitrogen and oxygen have been made by Baly and Donnan (Trans. Chem. Soc. 1902, p. 907); and they found the values 0·8225 and 1·2160 respectively at that temperature at which Baly's curve for the vapour-pressure of oxygen gives the

vapour as 100 mm.; and the values 0·8022 and 1·1947 respectively at the temperature corresponding to an oxygen vapourpressure of 200 mm. According to Travers (*loc. cit.*), Baly's measurements of the temperature are erroneous; but the thermometer used being the same in the two cases the densities given are probably correct.

Hence for the results of Table II. the volume of one hundred gramme-molecules of a mixture containing m molecules per cent. of nitrogen will be

$$\frac{28m}{0·8225} + \frac{32(100-m)}{1·2160} \text{ c.c.,}$$

and the molecular concentrations will be obtained from the molecular percentages by dividing each molecular percentage by the corresponding value of this expression. Similarly for the results given in Table III. the factor is

$$\frac{28m}{0·8022} + \frac{32(100-m)}{1·1947} \text{ c.c.}$$

For simplicity, however, the concentrations can be better stated as grams of the corresponding substance per 100 c.c. of the mixture; and the results of Tables II. and III. are given in this way in Tables IV. and V. If Henry's law holds, the quotient obtained by dividing the concentration by the partial pressure should be a constant, so the values of this quotient are also given in the Tables.

These two tables show that the solubility of nitrogen in the oxygen obeys Henry's law quite rigidly up to a molecular percentage of nearly 70 per cent., but that oxygen does not obey this law. The value of the quotient $\dfrac{\text{concentration}}{\text{pressure}}$ for oxygen varies in such a way as to point to association of oxygen molecules when dissolved in nitrogen. Now the surface-tensions of liquid oxygen, nitrogen, argon, and carbon monoxide were determined by Baly and Donnan (*loc. cit.*), and they concluded that the pure liquids showed no association. But looking at their results more closely, we see that the value of the temperature coefficient of the molecular surface-energy of oxygen, viz. 1·917, is not the same as that found for the other three gases, which all have a coefficient nearly equal to 2·003. Baly and Donnan concluded that the probable value of this coefficient at low temperatures was too uncertain for any conclusion to be drawn as to association; but the fact that argon, nitrogen, and carbon monoxide all have the same coefficient points to the value

TABLE IV.—Temperature 74°·7 Abs.

No.	Concentration of Nitrogen.	Partial Pressure of Nitrogen.	Concen. Press.	Concentration of Oxygen.	Partial Pressure of Oxygen.	Concen. Press.	Conc. O_2 $(Press. O_2)^{1·09}$
		mm.			mm.		
1 ...	0·0	0·0	...	121·6 (pure)	100·0	1·216	
2 ...	5·6	34·5	·1624	113·4	95·5	1·187	
3 ...	7·6	47·5	·1600	110·4	93·5	1·180	
4 ...	11·7	72·7	·1602	104·2	90·0	1·158	
5 ...	16·8	104·5	·1608	96·7	85·5	1·131	
6 ...	21·0	129·5	·1622	90·5	81·0	1·117	
7 ...	25·3	155·7	·1625	84·1	77·0	1·093	
8 ...	29·8	182·5	·1633	77·6	72·5	1·070	
9 ...	32·3	197·9	·1630	73·7	69·9	1·055	
10 ...	36·0	218·6	·1647	68·4	66·4	1·031	
11 ...	39·6	242·0	·1637	63·0	61·7	1·022	
12 ...	41·9	255·4	·1641	59·7	59·6	1·001	
13 ...	45·1	274·6	·1642	54·9	55·9	·982	
14 ...	48·2	293·3	·1643	50·4	53·4	·944	
15 ...	51·5	314·3	·1639	45·5	48·7	·934	
16 ...	52·2	318·8	·1637	44·4	47·7	·932	
17 ...	56·6	347·9	·1626	37·8	42·1	·899	
18 ...	60·8	374·6	·1623	31·8	36·6	·868	·630
19 ...	65·2	405·0	·1610	25·1	30·5	·824	·607
20 ...	68·3	427·2	·1599	20·5	25·8	·795	·581
21 ...	71·7	450·7	·1591	15·6	20·3	·769	·586
22 ...	75·0	475·8	·1576	10·7	14·2	·753	·593
23 ...	78·3	500·8	·1563	5·8	8·2	·709	·586
24 ...	81·9	528·5	·1550	(0·5)	(0·5)	(1·0)	
25 ...	82·3 (pure)	(531·0)	·1550	0	0	...	

TABLE V.—Temperature 79·07 Abs.

No.	Concentration of Nitrogen.	Partial Pressure of Nitrogen.	Concen. Press.	Concentration of Oxygen.	Partial Pressure of Oxygen.	Concen. Press.	Concen. $(Press.)^{1·15}$.
		mm.			mm.		
1 ...	0	0	...	119·5	200	·597	
2 ...	3·5	37·0	·0951	114·2	194·5	·587	
3 ...	10·8	114·3	·0948	103·4	181·2	·571	
4 ...	19·4	205·8	·0944	90·5	165·7	·546	
5 ...	26·8	284·8	·0941	79·5	150·7	·528	
6 ...	37·7	402·2	·0938	63·3	127·8	·496	
7 ...	46·2	495·5	·0931	50·7	108·5	·467	
8 ...	53·4	577·6	·0924	40·0	89·4	·447	·228
9 ...	62·3	684·7	·0910	26·6	65·3	·408	·218
10 ...	69·2	773·6	·0894	16·4	42·4	·387	·221
11 ...	74·4	845·6	·0880	8·6	24·4	·353	·219
12 ...	80·0	928·0	·0863	(0·3)	(0·5)		
13 ...	80·2	931	·0861	0	0		

of that coefficient being the normal one, so that oxygen must be slightly associated. The association calculated from these figures would be $\left(\dfrac{2\cdot003}{1\cdot917}\right)^{\frac{3}{2}} = 1\cdot068$. Now if oxygen is associated so that n molecules in the vapour become *one* molecule in the liquid, we should have, according to Henry's law, concentration $=$ const. $\times$ (pressure)^{n}.

But this equation takes no account of any relation between the association and the concentration, and therefore at its best can only approximately represent the facts when the concentration is small. In the equation, n and the constant are both unknown, but may be determined very easily by plotting the logarithm of the concentration against the logarithm of the pressure. If, now, this be done for experiments 23–20 of Table IV. (for all of which the concentration of the oxygen is low), it is found that the four points obtained lie close to a straight line the slope of which indicates that the pressure and concentration satisfy the relation, $-$ concentration $=$ const. $\times$ (press.)$^{1\cdot09}$, thus indicating an association factor $=1\cdot09$. This is of the same order as Baly and Donnan's factor $1\cdot068$, so that the agreement is satisfactory. Similarly the results of Table V. point to an association factor $1\cdot15$, but a slight error in the composition of the vapour in experiment 11 in that table would be sufficient to explain the increase from $1\cdot09$ to $1\cdot15$. Values of the expressions $\dfrac{\text{concentration O}_2}{(\text{pressure})^{1\cdot09}}$ and $\dfrac{\text{concentration O}_2}{(\text{pressure})^{1\cdot15}}$ are given in Tables IV. and V. (for low concentrations of oxygen), and the values obtained show how the relation is satisfied. Since the size of the associated oxygen molecule is not known, one cannot introduce the corresponding modifications in the formula deduced from Henry's law, so that the formula used cannot be expected to give good results. In addition to this, Richardson (Phil. Mag. [6] vii. p. 266) has shown that one must consider separately the solubilities of the simple and associated molecules.

The results point, therefore, to nitrogen obeying Henry's law and to oxygen only obeying it when we allow for association. Now in the results of the many isothermal distillations which have been carried out at ordinary temperatures (see Zawidzki, *loc. cit.*) agreement with Henry's law has not been looked for, as investigators have only looked for a linear relation between the partial pressure and the molecular percentage. The same difficulty that we have experienced in the calculation of the densities of the mixtures arises here also in calculating the concentrations; but it may be over-

come in the same way as before. A few of Zawidzki's results have therefore been recalculated as concentrations, and the results showing the relations between concentration and partial pressure are given in the Tables VI. to X.

TABLE VI.

Propylene Bromide and Ethylene Bromide.   85°·05 C.

No.	Concentration of Propylene Bromide.	Pressure of Propylene Bromide.	Concen. Press.	Concentration of Ethylene Bromide.	Pressure of Ethylene Bromide.	Concen. Press.
1 ...	0	0	...	203·4	172·6	1·178
2 ...	4·4	3·2	1·38	198·6	167·8	1·183
3 ...	15·5	10·2	1·52	186·3	158·6	1·175
4 ...	31·3	19·9	1·57	169·0	145·1	1·165
5 ...	46·7	29·4	1·59	152·6	132·2	1·152
6 ...	60·7	37·3	1·63	136·9	121·4	1·128
7 ...	63·2	38·1	1·66	134·1	120·8	1·110
8 ...	82·8	52·9	1·56	112·7	101·7	1·108
9 ...	85·1	52·9	1·61	110·2	100·5	1·097
10 ...	105·1	67·7	1·55	88·2	81·9	1·077
11 ...	122·1	79 3	1·54	69·6	64·0	1·088
12 ...	139·5	92·5	1·51	50·5	48·0	1·052
13 ...	153·1	102·5	1·49	35·6	34·3	1·038
14 ...	163·1	110·4	1·48	24·7	23·5	1·051
15 ...	172·0	117·1	1·47	14·9	13·8	1·077
16 ...	175·1	120·1	1·46	11·3	10·1	1·123
17 ...	179·9	123·8	1·45	6·2	4·6	1·359
18 ...	182·7	126·5	1·45			
19 ...	185·6	127·2	1·45			

TABLE VII.

Benzene and Ethylene Chloride.   49°·99 C.

No.	Concentration of Ethylene Chloride.	Pressure.	Concen. Press.	Concentration of Benzene.	Pressure.	Concen. Press.
1 ...	0	0	...	84·5	268	·315
2 ...	16·3	32·0	·511	73·1	231·5	·316
3 ...	32·4	69·2	·469	61·7	189·8	·325
4 ...	33·0	70·5	·468	61·3	188·5	·325
5 ...	46·7	98·9	·472	51·7	156·0	·332
6 ...	59·3	123 6	·480	42·9	127·9	·335
7 ...	75·7	154·9	·489	31·3	92·5	·339
8 ...	88·1	178·1	·495	22·7	65·9	·344
9 ...	109·6	216·9	·505	7·5	21·7	·347
10 ...	120·4	236·2	·510	0	0	...

TABLE VIII.

Carbon Tetrachloride and Ethyl Iodide.   49°·99 C.

No.	Concentration of Iodide.	Pressure.	Concen. / Press.	Concentration of Chloride.	Pressure.	Concen. / Press.
1	0	0	...	153·4	306·3	·501
2	5·8	15·3	·381	148·6	295·6	·503
3	14·3	38·4	·373	141·7	280·8	·505
4	31·8	80·6	·394	127·3	249·4	·510
5	46·3	110·9	·417	115·4	227·2	·508
6	67·0	153·1	·437	97·4	193·0	·505
7	84·1	183·4	·459	84·3	167·5	·503
8	186·7	354·0	·527	0	0	...

TABLE IX.

Carbon Tetrachloride and Benzene.   49°·99 C.

No.	Concentration of Chloride.	Pressure.	Concen. / Press.	Concentration of Benzene.	Pressure.	Concen. / Press.
1 ...	0	0	...	84·5	268	·315
2 ...	8·4	18·5	·455	79·9	253·4	·315
3 ...	19·3	40·5	·477	73·9	237·1	·312
4 ...	28·9	59·7	·484	68·6	221·8	·309
5 ...	41·0	82·9	·494	62·0	202·5	·306
6 ...	48·0	97·0	·495	58·2	191·3	·304
7 ...	63·7	128·7	·495	49·5	165·8	·299
8 ...	89·0	176·4	·505	35·6	124·6	·286
9 ...	106·5	211·8	·503	26·0	93·4	·279
10 ...	119·8	238·5	·502	18·7	68·3	·273
11 ...	153·3	306·3	·501	0	0	...

In two cases, viz. propylene and ethylene bromide and benzene and ethylene chloride, Zawidzki found that the partial pressure of either component of the mixture was proportional to its molecular percentage.   Now, as will be seen in Tables VI. and VII., the change from molecular percentage to concentration upsets this relation; but the deviation from Henry's law is not very great, and any association would be very slight.   In the case of mixtures of carbon tetrachloride with ethyl iodide, Zawidzki obtained slightly curved partial pressure-lines; but it will be seen from Table VIII. that the carbon tetrachloride obeys Henry's law very closely, while the agreement for ethyl iodide is not so good.   Similarly in Table IX., carbon tetrachloride and benzene show a fair agreement with Henry's law.   It must be remembered that in all these cases the concentration has

TABLE X.

Benzene and Acetic Acid.   49°·99 C.

No.	Concentration of Benzene.	Pressure.	Concen. / Press.	Molec. Weight of Acetic Acid in Vapour.	Concentration of Acetic Acid.	Pressure.	Concen. Press.
1 ...	84·5	(268·9)	(·314)	...	0	0	
2 ...	83·3	262·3	·318	83·0	1·44	3·63	·397
3 ...	81·6	258·7	·316	89·1	3·51	6·53	·539
4 ...	81·0	257·2	·315	90·1	4·30	7·25	·593
5 ...	77·4	249·6	·310	94·7	8·56	11·5	·745
6 ...	74·6	244·8	·305	97·1	11·92	14·2	·839
7 ...	69·3	231·8	·299	99·3	18·3	18·4	·994
8 ...	65·9	224·7	·294	100·4	22·4	20·5	1·092
9 ...	57·9	211·2	·274	102·0	32·1	24·8	1·29
10 ...	53·3	200·9	·265	102·5	37·6	27·1	1·39
11 ...	51·4	195·6	·263	103·1	39·8	28·7	1·39
12 ...	33·0	153·2	·215	104·8	62·1	36·3	1·71
13 ...	30·6	147·2	·208	105·0	65·0	36·8	1·77
14 ...	26·6	135·1	·197	105·4	69·8	40·2	1·74
15 ...	11·8	75·3	·157	107·0	87·5	50·7	1·73
16 ...	1·5	13·3	·114	107·9	99·9	54·7	1·83
17 ...	0·5	3·5	·148	107·9	101·1	54·7	1·85
18 ...	0	0	...	...	101·7	(56·5)	(1·80)

been calculated from the densities of the two components *assuming that no contraction takes place on mixing.* Until the densities of these mixtures have been determined, a close relation between the calculated concentrations and the pressures cannot be looked for. It is possible that this explains the fact that in Tables IX. and X. the quotient for benzene rises with the concentration, whereas in Table VII. the quotient falls.

But the case of the distillation of acetic acid and benzene, the results of which are given in Table X., is particularly interesting from the point of view of agreement with Henry's law. Acetic-acid vapour has a molecular weight which shows that the vapour consists of a mixture of simple and double molecules. Now when acetic acid is dissolved in benzene, even in the most dilute solutions, the acetic acid consists entirely of double molecules. Hence, when acetic-acid vapour dissolves in benzene, association takes place, and this ought to be shown if the partial pressure be plotted against the concentration. From Table X. it will be seen that neither the partial pressure of the acetic acid nor that of the benzene is proportional to the concentration, and that the association factor in the case of acetic acid is considerable.

The association varies with the concentration, so that calculations from the partial-pressure curve are not very accurate. If the log of concentration be plotted against the log of the partial pressure as before, the slope of the curve at each point may be taken as a measure of the association for that concentration. The result obtained is an association factor 1·50, when the concentration of acetic acid is 3·51 gr. per 100 c.c. According to the molecular weight the factor should be 1·35. It will be seen, therefore, that again the agreement is not close ; but since the method of calculation does not take account of the separate solubilities of the single and double molecules, as is really necessary, a better agreement can hardly be looked for.

Zawidzki's experiments, therefore, to a certain extent support the view that the relation between the partial pressure and the concentration can be obtained by means of Henry's law, so that it may be concluded that oxygen is associated when dissolved in nitrogen and also in the pure state.

Distillation of Argon and Oxygen.

In the separation of two gases from one another by means of fractionation at low temperatures, it often happens that at the temperature used the one substance is below its melting-point. In this case the relations which hold during distillation are modified by the fact that the total pressure of the saturated solution of the one substance in the other may be greater than the vapour-pressure of either pure substance. This happens in the case of argon and oxygen. The melting-point of argon is only a little below its boiling-point; and at the temperature of fairly fresh liquid air, argon is a solid with a vapour-pressure of over 400 mm. In order to see how a mixture of argon and oxygen behaved when distilled isothermally, a few experiments were made at 82·09 Abs. with the apparatus already described. As, however, the quantity of argon available was only 550 c.c., the greatest percentage of argon that could be used was 13·6 per cent. by volume. The results of these experiments are given in Table XI. The vapour-pressure of pure oxygen at 82·09 Abs. is 300·0 mm., and this value, together with the experiments 2, 3, 4, 5, gives five points on the vapour-pressure diagram. Experiment 6 was carried out in quite a different way. A quantity of pure argon was prepared and its vapour-pressure at 82°·09 Abs. was found to be 411·0 mm. On adding a small quantity of oxygen the vapour-pressure rose to 420 mm., and it remained equal to this in spite of

continued addition of oxygen so long as any of the argon remained solid. Hence, 420 mm. is the total pressure, above a saturated solution of argon in oxygen at 82°·09 Abs. Since the solution is saturated, the partial pressure of the argon must be equal to the vapour-pressure of solid argon at the same temperature. Hence the partial pressure of the oxygen is $(420-411)$ mm. $= 9\cdot0$ mm. and the percentage of argon in the vapour is $\dfrac{411}{420} \times 100 = 97\cdot8$ per cent. Several samples of the saturated solution were collected in the way used earlier for liquid samples, and it was found that the composition of the liquid remained constant, however the amount of oxygen added might vary, so long as there was solid argon present. The analysis of the samples showed that the liquid contained 92·7 per cent. by vol. of argon. Hence we see that the liquid containing 92·7 per cent. of argon gave a vapour containing 97·8 per cent. of argon and exerted a vapour-pressure $= 420$ mm.

TABLE XI.

Argon and Oxygen.　Temp. 82°·09 Abs.

No.	Molec. percent. Argon in Liquid. Obs.	Molec. percent. Argon in Vapour.		Total, obs.	Pressure, smoothed	Partial Pressure, Oxygen.	Partial Pressure, Argon.
		Obs.	Calc.				
						mm.	mm.
1......	0	0	0	300·0	300·0	300·0	0·0
2......	3·30	5·76	...	308·2	307·5	290·5	17·0
3......	5·6	9·15	...	310·8	312·0	283·5	28·5
4......	10·2	16·0	...	320·1	321·2	269·8	51·4
5......	13·6	20·6	...	328·7	327·5	260·0	67·5
6......	92·7 Saturated	...	97·8	420·0	420·0	9·0	411·0

Since the densities of mixtures of argon and oxygen at 82°·09 Abs. are not known, and the density of liquid argon at that temperature cannot be measured, one cannot plot the pressures against the concentrations. In default of this, the pressures may be plotted against the molecular percentages, and this has been done in fig. 3. The broken lines in this figure are drawn through the points representing the partial pressures for low concentrations of argon; and it will be noticed that they deviate considerably from the full

lines which are drawn to show what the partial pressures
would be if they were proportional to the molecular per-
centage. It will be noticed also that the partial pressure of
oxygen above the saturated solution is much less than would
be expected from analogy with the nitrogen-oxygen curve.

Fig. 3.

Argon and Oxygen at 82°·09 Abs.
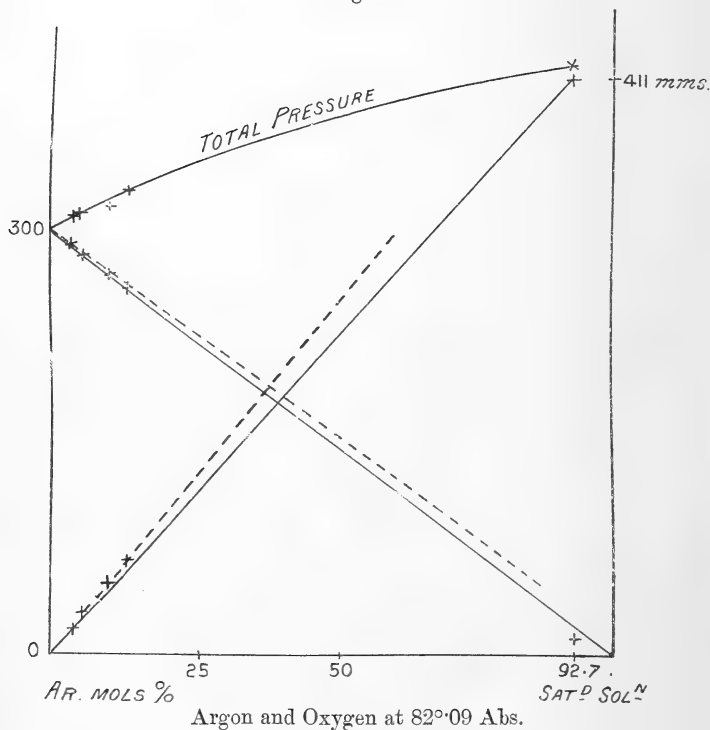
This shows, therefore, that the separation of a solid from a
liquid in which it is soluble is a much more complicated dis-
tillation-process than the separation of two liquids. A
simple separation can be obtained, however, if one can lower
the temperature so far that the vapour-pressure of the pure
solid is negligible compared with the vapour-pressure of the
other substance. In this case a single distillation effects a
complete separation, and therefore in practice one always
endeavours to obtain this condition of affairs.

In conclusion, I wish to express my thanks to Sir William
Ramsay for the kind interest he took in this research.

University College,
    London, W.C.

LIX. *The Construction and Use of Oscillation Valves for Rectifying High-Frequency Electric Currents.* By J. A. FLEMING, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London* *.

ATTENTION was directed by the author in 1890 to the fact that if two carbon filaments are sealed into a single vacuous glass bulb so as to make an incandescent lamp with two separate carbon loops, the resistance between these filaments, though infinite when the carbon is cold, becomes quite small as soon as the loops are made incandescent †. Moreover, if a metal plate is sealed into an incandescent lamp it was shown that the space between the metal plate and the incandescent carbon filament possesses a unilateral conductivity, negative electricity being able to pass freely from the hot carbon to the plate, but not in the opposite direction ‡. More recently the author discovered that such an arrangement may be used as a valve to permit the passage of one constituent current only of a high-frequency current or to rectify an electric oscillation §. The reason for this action is now recognized to be the copious emission of negative ions or electrons from the incandescent carbon. This operation has been studied quantitatively by the present writer and many other observers.

For the purpose of rectifying electrical oscillations and thus be able to detect them by an ordinary galvanometer, these *oscillation valves* are now made as follows :—A carbon-filament glow-lamp is constructed, the carbon loop of which is upheld in the centre of a highly exhausted glass bulb (see fig. 1). Around the loop is fixed a small cylinder of nickel, C, which is connected to a platinum wire sealed through the side of the bulb. The valve is used as follows:— The carbon loop is made incandescent by a suitable battery of secondary cells, a sliding rheostat being added to adjust the voltage on the terminals of the lamp. The circuit in which oscillations are to be detected is joined in series with a dead-beat mirror-galvanometer, and the valve connected with the circuit by wires joined respectively to the terminal of the nickel cylinder and the negative terminal of the carbon

* Communicated by the Physical Society : read March 23, 1906.

† See J. A. Fleming, " On Electric Discharge between Electrodes at Different Temperatures in Air and High Vacua," Proc. Roy. Soc. Lond. vol. xlvii. p. 122 (1890) ; also " Problems on the Physics of an Electric Lamp," Proc. Royal Institution, vol. xiii. part 34, p. 45 (1890).

‡ See J. A. Fleming, "On a Further Examination of the Edison Effect in Glow-Lamps," Phil. Mag. July 1896.

§ See also Proc. Roy. Soc. Lond. vol. lxxiv. p. 476, 1905, " On the Conversion of Electrical Oscillations into Continuous Currents by means of a Vacuum Valve."

loop. The oscillation valve is most conveniently mounted for this purpose on a special form of stand (see fig. 1). In using the valve the carbon filament must be brought to bright

Fig. 1.
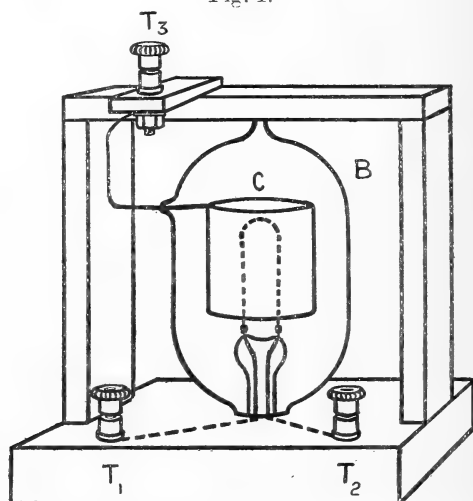
B, Exhausted glass bulb.   C, Nickel cylinder. T_1 T_2, Carbon filament terminals. T_3, Insulated cylinder terminal.

incandescence, about equal to that which in a carbon glow-lamp would correspond to a so-called " efficiency " of 3 watts per candle. So used, the valve enables us to employ a sensitive

Fig. 2.
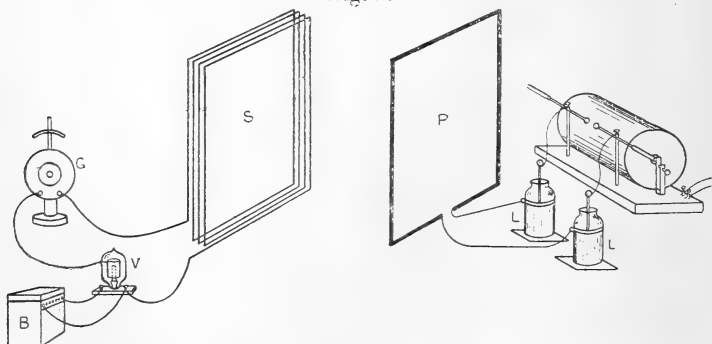
P, Primary oscillation circuit.   S, Secondary oscillation circuit. G, Galvanometer.   V, Valve.   B, 12-volt battery for incandescing filament of valve.

mirror-galvanometer of the ordinary type to detect the presence of electric oscillations in a circuit and to institute comparative measurements.

Thus, for instance, we form an oscillatory circuit (see fig. 2) by connecting a Leyden jar in series with a square coil of wire of a few turns P, and join the condenser and inductance across a spark-ball discharger connected to the secondary terminals of an induction-coil. At a certain distance we place another square coil of wire S in series with a galvanometer G and oscillation valve V. We then find that when oscillations are set up in the primary circuit, we obtain a steady deflexion of the galvanometer indicating that its coils are being traversed by a series of discharges in the same direction, all those in the opposite direction being practically stopped.

The author has already described the methods by which the amount of rectification produced by the valve can be ascertained (see Proc. Roy. Soc. vol. lxxiv. p. 484, 1905). Perfect rectification does not exist, but, as shown, the number expressing the percentage which the actual unilateral electric flow is of that which would flow if the unilateral conductivity were perfect, can be ascertained by sending the current which passes through the vacuous space of the valve through a calibrated galvanometer and electrodynamometer placed in series with each other. In valves as described this rectification may amount to 90 per cent.

We may use the above arrangements to investigate the effect of different kinds of discharge-balls and different lengths of spark. If we employ a fairly long spark in the primary condenser-circuit we may find that we obtain a very small effect on the galvanometer in the secondary circuit, but if we shorten the spark-gap until the spark at the balls is hardly visible, the galvanometer deflexion is generally increased. The reason for this is partly because the oscillations are damped out much more by the long spark than by the short one, and partly because with a short spark the condenser discharges occur more frequently. Hence, although in the latter case the condenser is charged to a less voltage owing to the lower discharge potential, the decreased damping and greater charge frequency causes the galvanometer to be traversed by a larger quantity of electricity per second, and therefore to give a greater deflexion.

In the same manner, we can exhibit the difference in the damping due to variations in the material of the spark-balls. Thus, using iron, brass, and zinc spark-balls of the same diameter and a spark-length of 0·1 mm., the galvanometer deflexions in one case were respectively 40, 57, and 70 scale-divisions, thus showing the smaller damping of zinc spark-balls. The writer has found by this means that carbon in the state used for arc-lamp carbons presents many advantages

as a discharge surface. All who have experimented much with Hertzian oscillators know how the state of the polish of the surface of the metal balls (generally brass) affects the electric wave-producing power. It can be shown by the use of an oscillation valve that for quantitative work a discharger made of carbon rods, as follows, presents many advantages:— A row of arc-lamp carbons C, C, C (see fig. 3) are fixed like posts in a piece of ebonite and another row of slightly conical carbon rods B, B are inserted transversely between them, the distances between the rods being fixed so that very small

Fig. 3.

TO INDUCTION COIL
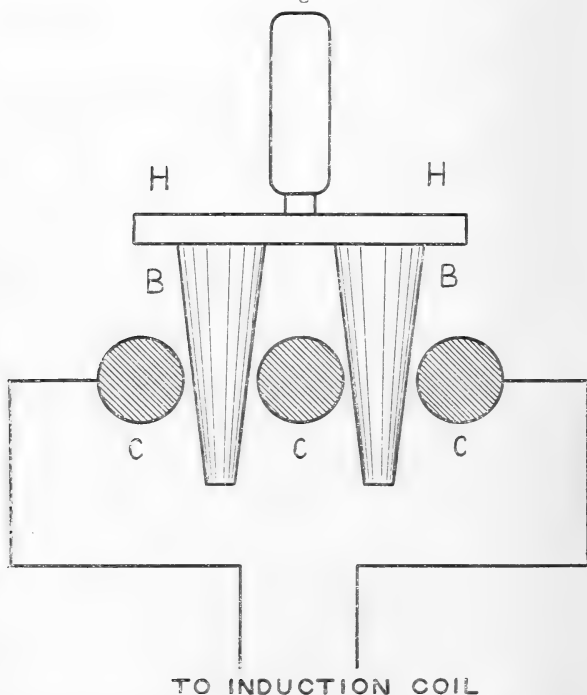
air-gaps are left between carbon and carbon. We thus construct a multiple spark-gap of carbon surfaces which has small damping and great constancy. By enclosing the rods in a non-oxidizing atmosphere we can prevent the rods burning away. Another advantage of the arrangement is the ease with which new surfaces can be brought into use.

We can also investigate by the same means whether the use of spark-balls immersed in oil presents any advantages. Also the same arrangements may be used to exhibit the screening action of conductors for high frequency magnetic

fields. For if we interpose between the primary and secondary circuits a sheet of tinfoil or zinc, we see a notable decrease in the galvanometer deflexion, thus making the screening action of the metal very evident. Employed in this manner, it enables us to show strikingly the rapid rate at which the field due to a current in a square or circular circuit decreases with distance from the circuit, and therefore to illustrate one of the disadvantages under which wireless telegraphy by electromagnetic induction labours when compared with space telegraphy by electric waves. When using the valve to detect the oscillations in an antenna produced by the impact of Hertzian waves, an oscillation transformer is inserted in the circuit of the receiving antenna, and its secondary circuit is connected through a valve with a dead-beat mirror-galvanometer. We are thus able to receive signals over short distances by the direct effect of the rectified oscillations themselves on the galvanometer.

The action of other substances besides incandescent carbon as a cathode in a vacuum-value has also been studied. It has been found by G. Owen[*] and by A. Wehnelt[†] that glowing metallic oxides, including the rare oxides employed in the manufacture of the Nernst lamp-glowers, copiously emit negative ions when incandescent both at atmospheric and at reduced pressures. Wehnelt found that the incandescent oxides of calcium, barium, and strontium produce an abnormally powerful electronic discharge, and, following the recommendations of the author, he has proposed to employ vacuum-tubes with one electrode covered with such oxides and heated, as rectifying valves for alternating currents.

As far back as 1890 the writer showed in a lecture experiment at the Royal Institution that the so-called Edison effect, that is the passage of negative electricity across space from an incandescent carbon filament to a metallic plate near it, could take place at atmospheric pressure if the plate was very near the filament. It is easy to show a similar experiment with a Nernst electric glow-lamp. If a Nernst lamp is supported with the bare glower horizontal and placed within a few millimetres of a vertical insulated metal tube kept cold by being filled with water, it is found that negative electricity will pass freely across the glower to the cold metal, but not in the opposite direction. Hence if the glower and metal tube are inserted as a gap in an electric circuit containing a sensitive galvanometer, and if secondary

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* See G. Owen, Phil. Mag. vol. viii. p. 230 (1904). "On the Discharge of Electricity from a Nernst Filament."

† See A. Wehnelt, Phil. Mag. vol. x. p. 80 (1905). "On the Discharge of Negative Ions by Glowing Metallic Oxides and Allied Phenomena."

oscillations are created by induction in this circuit, we find that the galvanometer gives a steady deflexion showing the passage of a continuous current through it, and therefore of the unilateral conductivity of the space between the glower and the metal tube.

The distance over which this transference of electricity can take place depends very much upon the temperature of the glower, and the amount of rectification of the alternating current obtained upon success in keeping down the temperature of the metallic electrode. This is best achieved by circulating water through it.

It follows as a consequence from the above facts, that there is a considerable emission of negative ions or electrons from the incandescent portion of the lime cylinder used with the oxy-coal gas-burner to produce the lime-light, and that the space near the incandescent portion of the lime cylinder as well as the space near the Nernst lamp-glower is highly conductive by reason of the presence there of negative ions emitted from the oxide surface.

In the practical construction of oscillation valves, the advantage of placing the heated and non-heated electrodes in a vacuum is that the plate which acts as an anode can be placed at a greater distance from the incandescent surface and thereby kept cool, since the electrons ejected from the heated surface are projected to a much greater distance when the atmospheric pressure is reduced. Although platinum coated with calcium or barium oxides undoubtedly emits a much larger electronic current per square millimetre than carbon at the same temperature and under the same surrounding conditions as to gas pressure, I find that for rectifying electric oscillations the carbon-filament oscillation-valve as I have designed it, affords more conveniently all that is required. There are some advantages in employing a thick carbon filament and constructing it to be worked at about 12 volts and take a fairly large current of 2 or 3 amperes. For one thing, the filament is much less likely to be destroyed by over-heating in working, and hence the valve lasts longer. In some cases an advantage may ensue from working valves in parallel, that is joining up a number of such carbon-filament valves with their carbon filaments in parallel on the same heating battery, connecting together the insulated metal cylinders contained in each bulb together, and then using the multiple arrangement as if it were a single valve.

When used, however, to rectify such oscillations as are employed in the receiving circuits of wireless telegraph apparatus, a single valve will do all that is required because

the quantity of electricity which has to be carried is small and the electronic emission from even a 4-volt 1-ampere carbon filament is amply sufficient to carry the negative component of the feeble oscillations used across the vacuous space.

It should be noted that such oscillation-valves as are here described have quite a different range of use from other rectifying arrangements such as the Cooper-Hewitt mercury-vapour tube, and the electrolytic aluminic-carbon valve of Nodon and others.

The electrolytic valves produce no rectifying effects with high-frequency alternating currents, because the time element enters into the formation of the aluminic hydroxide film on which their action depends. On the other hand, the mercury-vapour tubes which have been proposed for use with high-tension alternating currents will not operate below a certain minimum potential-difference between the electrodes. The vacuum-valve as here described, however, will pass current unilaterally with a fraction of a volt difference of potential between the incandescent and the cold electrode, and there is no minimum potential difference below which they will not act; hence their use is conditioned solely by the sensitiveness of the galvanometer employed with them.

By its simplicity and ease of use the carbon-filament vacuum-valve recommends itself as a useful addition to our resources for experimental work in connexion with electric oscillations and electric-wave telegraphy.

LX. *On the Use of the Cymometer for the Determination of Resonance-Curves. By* G. B. Dyke, *B.Sc.*[*]

D R. FLEMING has shown in his recent Cantor Lectures before the Society of Arts [†], that by the introduction of a hot-wire ammeter into the circuit of his direct-reading cymometer, the effective or root-mean-square value of the oscillation current set up in the cymometer circuit can be measured. This instrument was originally designed for the determination of the wave-lengths used in wireless telegraphy by the direct inspection of a scale, and also for the measurement of capacities and inductances; but it has been found that a small addition renders the instrument also available for the determination of resonance-curves, and therefore of

the decrement of oscillation-trains, and of oscillatory spark-resistances.

A direct-reading cymometer was used constructed as described by Dr. Fleming in a paper read before this Society on March 24th, 1905 [*]. The instrument consists essentially of a closed circuit containing a condenser and an inductance, the distinctive feature being the fact that the capacity and inductance are so arranged as to be varied simultaneously and in the same proportion by one movement of a handle. A portion of the closed circuit consists of a straight copper rod, which is placed in the neighbourhood of, and parallel to, the circuit on which the measurements are being made. Then, as Dr. Fleming has shown in the paper referred to above, resonance will take place between the two circuits when the cymometer is so adjusted that its oscillation constant, that is the square-root of the product of the capacity and inductance, has the same value as that of the circuit under test. The position of resonance is detected by the illumination of a Neon vacuum-tube connected between the inner and outer coatings of the condenser. The Neon vacuum-tube, although an excellent detector of the position of resonance, gives but little indication of the relative value of the current in any other position of the cymometer, and is of the nature of an indicator rather than a measuring instrument.

For the determination of the logarithmic decrement, it is requisite to know the relative values of the current in the cymometer for points in the neighbourhood of resonance, and hence a quantitative current-meter must be employed. The instrument should fulfil the following conditions :—

(i.) Its capacity and inductance should be negligible compared with that of the cymometer, otherwise the disturbance of the scale-readings would lead to erroneous results.

(ii.) Its damping factor should be small and should be capable of calculation.

(iii.) It should be fairly rapid in its action.

The above requirements are fulfilled by an addition to the cymometer, made recently by Dr. Fleming, and shown in his Cantor Lectures referred to above.

The ammeter adopted is of the hot-wire type and is inserted into a gap made by cutting the bar of the cymometer ; the current passing through the heated wire being measured by a thermojunction in contact with it.
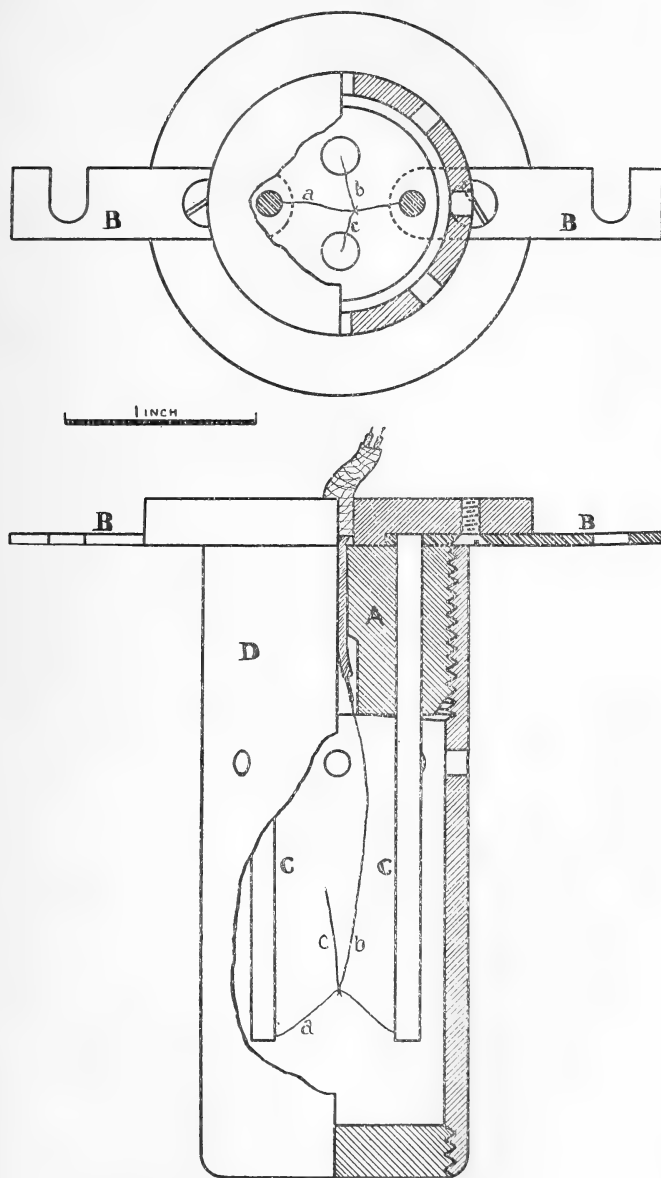
Fig. 1.

Thermoelectric junction employed with Fleming Cymometer to determine mean-square value of the induced current.

2 X 2

The instrument consists of an ebonite block A (fig. 1, p. 667), from the sides of which project lugs BB which are used for connexion to the cymometer. To these lugs are soldered brass rods CC, each about 3 inches long, and between the extremities of these rods is stretched a fine platinoid wire a (diameter ·05 mm.) having a resistance of 3·5 ohms. To the middle of this wire the thermojunction is soldered. This junction consists of two very fine wires, b, c, one of pure iron (diameter ·20 mm.) and the other of bismuth (diameter ·17 mm.); these are attached to the platinoid with special solder of low fusing-point, the contact area being made as small as possible. The other ends of the wires are connected to the galvanometer by a flexible cord. The junction is shielded by the ebonite cap, D, which screws over the plug A. The electromotive force of the couple is observed by means of a low-resistance Paul single-pivot galvanometer, having a resistance of 4·88 ohms and a figure of merit of 19·5 microvolts per division.

The method of calibrating the junction is as follows :— For wires so fine as the platinoid used, the high-frequency resistance is the same as that for low-frequency or continuous current. Hence it is only necessary to connect up the hot-wire ammeter in series with an adjustable resistance and a secondary cell, and to pass currents of known strength through it, observing the corresponding deflexions of the galvanometer.

These observations enable a curve to be plotted from which the root-mean-square (or equiheating) value of the current in the cymometer for any deflexion can be readily read off. For the instrument described the calibration curve is such that the deflexion varies as the 1·9th power of the current.

An auxiliary resistance is also required and is constructed similarly to the hot-wire ammeter just described, except that the thermojunction is omitted and the resistance of the fine platinoid wire is 7·2 ohms. This resistance is arranged so that when required it can be put into a second gap cut in the cymometer bar, a short-circuiting strip being used to complete the circuit when it is not in use. It will be seen that if the cymometer bar is placed in the proximity of a circuit in which oscillations are taking place, the value of the R.M.S. current induced in it can be determined by means of the hot-wire ammeter described above for any position of the cymometer-handle; that is, for any oscillation constant or any frequency of the oscillations in the cymometer within the range of the instrument.

From observations of this R.M.S. current and frequency, it is possible to deduce the logarithmic decrements of both primary and secondary circuits. The logarithmic decrement of

an oscillation per semiperiod is here defined to be the Naperian logarithm of the ratio of two successive maximum oscillations in *opposite* directions. In this connexion it is to be noted that most German physicists have defined the decrement as the logarithm of the ratio of two successive maximum oscillations in the *same* direction.

The problem of the oscillation transformer has been attacked more particularly by Oberbeck, Bjerknes, Drude and Wien; and Bjerknes and Drude have given solutions for obtaining the decrements, and although the proof is very long the final equations are simple and easily interpreted. For the complete proof we must refer to the original papers[*]; the essential parts, however, have been translated into English nomenclature by Dr. Fleming, and are given in his book on "The Principles of Electric Wave Telegraphy" (Longmans, Green & Co.). In this note we can do little more than state the final result of their work. We shall use the following symbols :—

$\delta_1 =$ logarithmic decrement per semi-period of the oscillation in the condenser circuit.

$\delta_2 =$ logarithmic decrement per semi-period in the secondary circuit inductively coupled with the primary.

$n_1 =$ frequency of oscillation in condenser circuit.

$n_2 =$ frequency in secondary circuit.

$J =$ value of R.M.S. current in the secondary circuit corresponding with the frequency n_2.

$J_r =$ maximum value of R.M.S. current in secondary circuit, *i. e.* the "resonance current."

Then Bjerknes shows that the following equation holds:

$$n_1\delta_1 + n_2\delta_2 = \pi(n_1 - n_2)\frac{J}{\sqrt{J_r^2 - J^2}};$$

or, if n_2 is nearly equal to n_1, that is the secondary is very nearly resonant to the primary, this becomes

$$\delta_1 + \delta_2 = \pi\left(1 - \frac{n_2}{n_1}\right)\frac{J}{\sqrt{J_r^2 - J^2}}.$$

Writing

$$x = 1 - \frac{n_2}{n_1},$$

$$y = \left(\frac{J}{J_r}\right)^2,$$

this becomes

$$\delta_1 + \delta_2 = \pi x \sqrt{\frac{y}{1-y}}.$$

* V. Bjerknes: *Annalen der Physik*, vol. lv. (1895) p. 121; and vol. xliv. (1891) p. 74. P. Drude: *Annalen der Physik*, vol. xiii. (1904) p. 512.

This equation gives us the sum of the decrements in the two coupled circuits. In order to obtain the values of δ_1 and δ_2 separately we require some other relation between them. This relation has been given by Drude. He shows that the resonance current in the secondary circuit can be calculated from a formula equivalent to

$$J_r^2 = V_1^2 \frac{C_1 C_2}{8} \frac{\pi^4 n_1 k^2}{\delta_1 \delta_2 (\delta_1 + \delta_2)},$$

where $V_1 =$ maximum value of primary condenser potential-difference.

C_1 and $C_2 =$ capacities in primary and secondary circuits respectively.

$k =$ coefficient of coupling of the two circuits

$$= \frac{M}{\sqrt{L_1 L_2}},$$

where $M =$ mutual inductance between the circuits;

L_1 and $L_2 =$ self-inductances of the primary and secondary circuits respectively.

Passing now to the delineation of the resonance curve, we proceed as follows :—Insert the hot-wire ammeter into the cymometer circuit and place the cymometer parallel to some straight portion of the primary·circuit and at such a distance from it that when the cymometer is adjusted to resonance the current is not too large to be measured on the galvanometer. Then move the cymometer handle slowly from one end of the scale until a readable deflexion appears on the galvanometer. From this point move the handle in small steps, noting at each step the current J in the cymometer as given by the calibration curve of the galvanometer, and the frequency n_2 as read on the cymometer-scale. Proceed in this way until the maximum deflexion is passed and the current has again fallen to a small value. Plot the results thus obtained as a curve having ordinates proportional to J, and abscissæ proportional to n_2. Take off from this curve the maximum value of J. This will be the resonance current J_r; and the corresponding frequency will be the resonance frequency, that is, the frequency n_1 of the primary circuit. Now repeat the observations inserting the auxiliary resistance into the cymometer circuit in addition to the ammeter, taking care not to alter the relative positions of the circuits in so doing. Plot the results as before and obtain the value of the resonance current J_r'. The resonance frequency should of course remain unaltered. The two

curves should now be redrawn, taking the ratio $\dfrac{J}{J_r}\left(\text{or }\dfrac{J'}{J_r'}\right)$ as ordinate, and the ratio $\dfrac{n_2}{n_1}$ as abscissa.

We are now in the position to determine the decrements for the two circuits. Take out from the curve corresponding values of $\dfrac{J}{J_r}$ and $1-\dfrac{n_2}{n_1}$, taking the mean for the two sides of the curve and noting that $\dfrac{n_2}{n_1}$ lies between 0·95 and 1·05, or that $\left(1-\dfrac{n_2}{n_1}\right)$ is not greater than 0·05.

Then, using Bjerknes' formula

$$\delta_1+\delta_2=\pi x \sqrt{\dfrac{\frac{\prime\prime}{}}{1-y}},$$

obtain a series of values for $(\delta_1+\delta_2)$.

Let the mean value of $(\delta_1+\delta_2)=X$.

In like manner, if δ_2' is the logarithmic decrement of the auxiliary resistance, obtain a series of values for $\delta_1+\delta_2+\delta_2'$ by taking out from the second curve corresponding values of $\left(\dfrac{J'}{J_r'}\right)$ and $\left(1-\dfrac{n_2'}{n_1}\right)$ and applying Bjerknes' formula.

Let the mean value of $(\delta_1+\delta_2+\delta_2')=X'$.

On writing out Drude's formula for the two cases we get: when the ammeter only is in circuit

$$J_r^2=V_1^2\dfrac{C_1C_2}{8}\dfrac{\pi^4 n_1 k^2}{\delta_1\delta_2(\delta_1+\delta_2)}\,;$$

and when the auxiliary resistance is inserted

$$J_r'^2=V_1^2\dfrac{C_1C_2}{8}\dfrac{\pi^4 n_1 k^2}{\delta_1(\delta_2+\delta_2')(\delta_1+\delta_2+\delta_2')}.$$

Hence we have the relation

$$J_r^2\delta_2(\delta_1+\delta_2)=J_r'^2(\delta_2+\delta_2')(\delta_1+\delta_2+\delta_2'),$$

or $\qquad J_r^2\delta_2 X=J_r'^2(\delta_2+\delta_2')X'.$

Hence $\qquad \delta_2=\dfrac{\delta_2'X'}{\left(\dfrac{J_r}{J_r'}\right)^2 X-X'}.$

The value of δ_2' may either be taken as equal to $(X'-X)$ or may be calculated from its resistance and the frequency

and inductance of the circuit, for we have

$$\delta_2' = \frac{R \times 10^9}{4n_1 L_1} \; ;$$

where R = resistance of auxiliary wire in ohms,
L_1 = inductance of cymometer in resonance position in cms.
n_1 = resonance frequency.
And as δ_2' is known from either of these equations, δ_2 becomes known, and $\delta_1 = X - \delta_2$;

therefore δ_1 is known.

Hence the decrements of the two circuits are determined.

The resistance of the primary spark can be deduced from the value of δ_1 in the following manner.
Let R' = high frequency resistance of the wire of the primary circuit in ohms.
r = resistance of spark in ohms.
L = inductance of primary circuit in cms.
n_1 = resonance frequency.
Then we know

$$\delta_1 = \frac{(R' + r)10^9}{4n_1 L} \, .$$

$$\therefore \qquad R' + r = \frac{4n_1 L \delta_1}{10^9} \, .$$

$$\therefore \qquad r = \frac{4n_1 L \delta_1}{10^9} - R' \, .$$

The following numerical example of the deduction of the decrements from the resonance curves may be useful as illustrating the method of arranging the work. The primary oscillation circuit consisted of a rectangle of wire (diameter ·162 cm.) inductance = 5000 cms., a condenser of capacity = 5560 micro-microfarads, and a 2 mm. spark-gap between 1·25 inch iron balls. The oscillations were excited by a high tension transformer. The cymometer was set up parallel to one side of the rectangular inductance and about 6 inches away from it.
We will suppose that the resonance curves have been drawn as described above, and the result to be as shown in fig. 2 (α and β), where the full-line curve shows the result obtained with the ammeter (resistance $3\cdot5\,\varpi$) only in circuit, and the dotted curve the result with the extra resistance ($7\cdot2\,\varpi$) also connected.

Fig. 2.
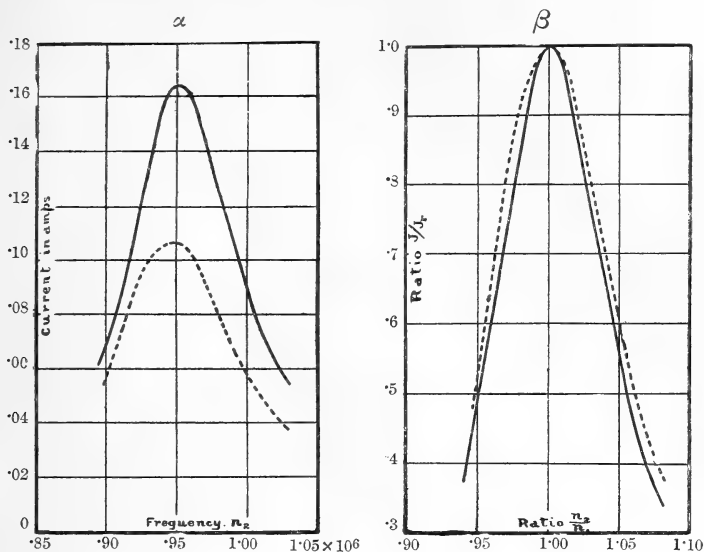
We will now form a table showing the relative values of $\left(\dfrac{J}{J_r}\right)$ and $\left(1-\dfrac{n_2}{n_1}\right)$, and the calculated values of $(\delta_1+\delta_2)$ and $(\delta_1+\delta_2+\delta_2')$.

$\dfrac{J}{J_r}$.	$1-\dfrac{n_2}{n_1}$.	$\delta_1+\delta_2$.	$1-\dfrac{n_2'}{n_1}$	$\delta_1+\delta_2+\delta_2'$.
·95	·0120	·115	·0125	·120
·90	·0165	·112	·0210	·138
·85	·0205	·104	·0255	·130
·80	·0255	·107	·0300	·125
·75	·0293	·105	·0345	·124
·70	·0335	·103	·0385	·119

Taking out the means we find

$$X = ·108,$$
$$X' = ·126.$$
$$\therefore \qquad X'-X = ·018 = \delta_2'.$$

If δ_2' is calculated from the formula

$$\delta_2' = \frac{R' \times 10^9}{4 n_1 L_1},$$

$$R' = 7 \cdot 2 \varpi, \quad L_1 = 768000 \text{ cms.}, \quad n_1 = \cdot 95 \times 10^6,$$

we get $\qquad \delta_2' = \cdot 025.$

Hence
$$\text{mean } \delta_2' = \cdot 022.$$

From fig. 2 (α) we see

$$J_r = \cdot 164 \text{ amps.}$$
$$J_{r'} = \cdot 104 \text{ amps.}$$
$$\therefore \qquad \frac{J}{J_{r'}} = 1 \cdot 53.$$

Calculating δ_2 from the formula

$$\delta_2 = \frac{\delta_2' X'}{\left(\dfrac{J_r}{J_{r'}}\right)^2 X - X'}$$

we get $\qquad \delta_2 = \cdot 022.$

Hence

mean value of $\delta_1 = \frac{1}{2}\{ (X - \delta_2) + (X' - \delta_2 - \delta_2') \}$
$$= \cdot 077.$$

Now the decrement δ of the ammeter *per se* is given by the formula

$$\delta = \frac{R \times 10^9}{4 n_1 L_1},$$

where $R = 3 \cdot 5 \varpi, \quad L_1 = 708000 \text{ cms.}, \quad n_1 = \cdot 95 \times 10^6.$

$$\therefore \qquad \delta = \cdot 012.$$

Hence decrement of cymometer *per se*

$$= \delta_2 - \delta$$
$$= \cdot 022 - \cdot 012 = \cdot 010.$$

Passing to the primary circuit, we had the formula

$$r = \frac{4 n_1 L \delta_1}{10^9} - R',$$

for the spark resistance r; where

$$L = 5000 \text{ cms.}, \quad \delta_1 = \cdot 077, \quad n_1 = \cdot 95 \times 10^6.$$

According to Lord Rayleigh's formula

$$R' = R \frac{\pi d}{80} \sqrt{n},$$

where R = low frequency resistance, and d = diameter of wire in cms.

In our case

$$R = \cdot 0386 \text{ ohms.,}$$
$$d = \cdot 162 \text{ cms.,}$$
$$n = \cdot 95 \times 10^6.$$
$$\therefore \qquad R' = \cdot 23 \, \varpi.$$

Putting in these values we get

$$r = 1 \cdot 23 \, \varpi.$$

The above example is chosen rather as as example of the ease and speed with which reasonably good results can be obtained, than as a criterion of the accuracy of the method, as all the observations necessary for drawing the resonance curves were taken in less than half an hour. If more time is taken over the observations much closer agreement between the calculated and observed values of the decrements can be obtained.

The example first worked out is a case of two rather loosely coupled oscillation circuits, and in this case we have seen that the resonance curve has a single peak. If, however, the coupling is at all tight, the resonance curve develops a double hump, the maxima becoming more and more separated as the coupling becomes tighter and tighter, until when the coupling is perfect (*i. e.* when the mutual inductance is the geometric mean of the two self-inductances), one of the maxima has gone off to infinity, and we are again left with a single-hump resonance curve.

Oberbeck has developed the theory of two coupled oscillation circuits, and has given formulæ by means of which the two resonance frequencies may be predetermined. For the general solution we must refer to the original paper [*], but in one particular case the result deserves special mention on account of its importance in wireless telegraphy. If the primary and secondary circuits are tuned, that is, are so adjusted that when far apart they have the same oscillation constant and the same frequency n_0, then, when put near together so that the coupling coefficient has a value k, the following very simple relations hold between the two

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* A. Oberbeck, Wied. *Ann. der Physik*, 1895, vol. lv. p. 623. See also Dr. J. A. Fleming, "Principles of Electric Wave Telegraphy," chap. iii.

frequencies n_1 and n_2 induced in the secondary circuit and the natural frequency n_0 :—

$$n_1 = \frac{n_0}{\sqrt{1+k}},$$

$$n_2 = \frac{n_0}{\sqrt{1-k}},$$

or

$$\frac{n_1}{n_2} = \frac{\sqrt{1-k}}{\sqrt{1+k}}.$$

Now in the case of some oscillation transformers used in wireless telegraphy the coupling may be about $0\cdot5$.

Hence we should then have

$$\frac{n_1}{n_2} = \sqrt{\frac{\cdot5}{1\cdot5}} = \frac{1}{1\cdot732},$$

or, The frequency of one wave is about $1\frac{3}{4}$ times that of the other.

For a coupling of the order of $0\cdot5$ the method above described may be applied to each hump of the double-humped resonance-curve, and will enable a fairly accurate determination of the decrements of the two oscillations to be made; but, as shown by C. Fisher (see *Annalen der Physik*, vol. xix. p. 182, 1906), when the coupling is very loose we cannot consider the resultant resonance-curve to be identical with the sum of the curves due to each oscillation separately.

Fig. 3.
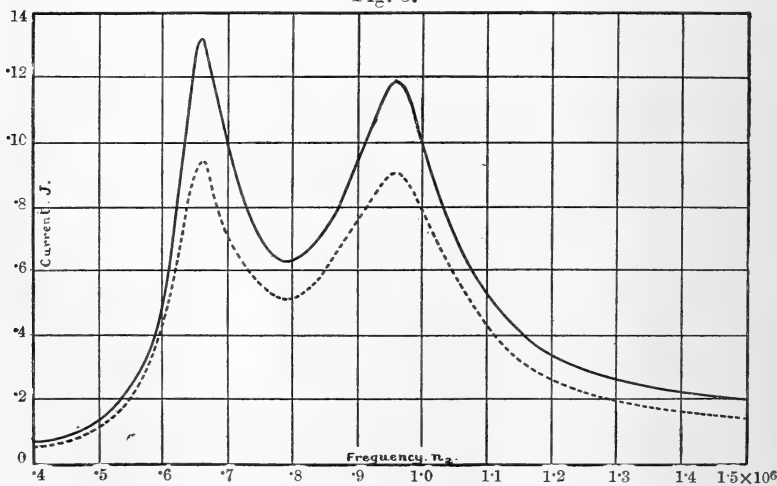
The curve shown in fig. 3 was taken from a closely coupled circuit of this type, and may be taken as indicative of the general type of result to be expected.

The accurate determination of the logarithmic decrement in open oscillation circuits such as these is a matter of considerable practical importance, as most of the damping is due to the radiation of energy from the wire, and not to the resistance as is the case with closed or non-radiative circuits; and as the amount of this radiant energy is not easily estimated mathematically, the value of this important quantity must rest on experimental evidence alone.

The objection may be raised that the type of ammeter described greatly increases the decrement of the cymometer, as its decrement is almost equal to that of the cymometer *per se.* It must be remembered, however, that the part of the instrument in the cymometer circuit is a short piece of very fine wire whose decrement can be easily and accurately calculated, and secondly that the decrement of the cymometer itself does not matter, the primary circuit being generally the one whose decrement is required.

At first sight it would seem that a very obvious variation of the method could be employed which would not suffer from this defect.

Suppose that, instead of cutting the cymometer-bar, a few turns of well-insulated wire are wound round a section of the inductance-coil of the cymometer, and the ends of the coil connected to the fine wire of the ammeter described above, then currents will be induced in this circuit proportional to the currents in the cymometer. This method, however, has more serious drawbacks than the other, as :—

(1) The damping of the cymometer will be just as great if this method is employed, as the same amount of energy has to be supplied to heat the ammeter wire to a definite temperature.

(2) The actual current in the cymometer is not measured but only a current bearing some unknown ratio to it.

(3) The scale of the cymometer is slightly altered when the ammeter is in place, as the arrangement is of the nature of a transformer with a closed secondary circuit, part of the cymometer inductance spiral forming the primary; this will of course annul a portion of the inductance and so alter the scale.

The extent of this alteration of scale may be determined by measurements made at the point of resonance with the Neon tube, first with the secondary open, and second with the secondary closed through the ammeter wire.

Several curves have been obtained using the ammeter in this manner, but as the method shows no points of advantage

over the one first described and has several inherent disadvantages, it was discarded in favour of what might be called the series arrangement.

In conclusion, the author wishes to thank Dr. J. A. Fleming, F.R.S., for his kindness in permiting him to publish the results of these experiments, which were carried out under his direction at the Pender Laboratory, University College, London.

LXI. *Elementary Notes on Thermodynamics: the Plug Experiment.* *By* EDGAR BUCKINGHAM *.

§ 1. THEORETICALLY there are many ways of establishing the thermodynamic scale of temperature, but practically, experiments on the properties of gases offer the only mode of attacking the problem that has been followed with any success. For this purpose, what we need to know is the manner in which the internal energy of the gas used in our standard gas-thermometer varies with the volume, when the temperature is constant ; and for the most precise information on this point, we have, at present, to refer to the porous-plug experiment or something equivalent to it.

The use of reasoning based on the second law of thermodynamics is to be met with very frequently ; and we find, in almost every case, that it involves the assumption that the international hydrogen scale, to which the observations are referred, differs only by an additive constant from Lord Kelvin's thermodynamic scale which appears in the thermodynamic formulæ. It might be expected, therefore, that a clear treatment of the plug experiment would be found in every text-book of thermodynamics, but that expectation is not justified. Some of our best standard works hardly mention the plug experiment, and the general impression one gets from most of the books I have examined, is that the authors consider the subject either too difficult to be discussed or so simple as to deserve only casual notice. I think, however, that a clear and simple treatment of the theory of some method or methods by which the relation of the gas scale to the thermodynamic scale might be found, is distinctly worth while, even if there is nothing new about the discussion except, possibly, the form in which it is put.

§ 2. Beside the original Joule-Thomson form of the plug experiment, in which the flow of gas through the porous

plug is adiabatic *, I shall consider a modified scheme, in which energy is supplied to the gas, during its passage through the plug, at such a rate as to make the flow isothermal †; and, furthermore, I shall consider a simplified form of Gay-Lussac's and Joule's free expansion experiments, in which heat is supposed to be supplied or withdrawn during the free expansion so as to to make it isothermal ‡.

The fundamental equations for these three experiments are very easy to deduce, if one starts by having a clear diagram.
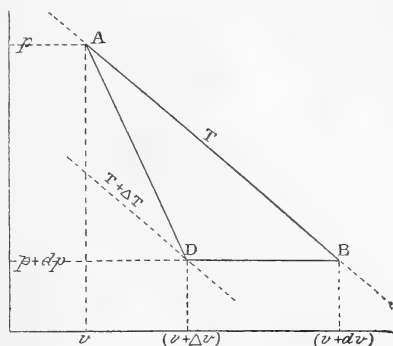
Let A (coordinates: p, v, T) represent the initial state of

* Objection has been made to calling the flow adiabatic, because heat is generated mechanically in the plug. When we consider that this heat is probably mainly, if not entirely, due to viscosity, and if so, is developed *in the gas itself*, and not generated in the solid material of the plug by true friction and then communicated to the gas, the objection does not seem well founded. The term "adiabatic plug experiment" is, at all events, a convenient designation for the Joule-Thomson experiment as distinguished from the proposed isothermal form.

† Phil. Mag. October 1903.

‡ At first sight, it seems doubtful whether we are justified in applying thermodynamic reasoning to an essentially tumultuous process such as the free expansion of a gas; for the use of the two laws in their usual elementary form implies that the system under consideration has a definite temperature, whereas *during* a tumultuous process, the gas cannot be said to have any definite temperature at all. After the tumultuous motions have subsided and the gas has come to a state of internal equilibrium, both mechanical and thermal, it has again a definite temperature. The reasoning by which, from the assumption of the nonexistence of infinite sources and sinks of energy, we establish the existence of the internal energy, ϵ, as a function of the instantaneous coordinates of the system, makes no assumption regarding the nature of the path by which any state of the system may be reached from any other, but assumes only that we may always, *by some means or other,* make the system pass from any possible state to any other. Hence it is entirely legitimate to use the first law—the energy law—in comparing any two states, regardless of the nature of the process by which one has actually been reached from the other, provided it be not *in the nature of things* impossible to make the system pass again by some path or other from the final to the initial state.

unit mass of the gas. Let D (coordinates: $p+dp$, $v+\Delta v$, $T+\Delta T$) represent the state after the irreversible adiabatic flow through the porous plug. Let B (coordinates: $p+dp$, $v+dv$, T) represent the state the gas would have been in, if the same fall of pressure had occurred without change of temperature. The temperature T is to be understood as measured on the scale of a constant-pressure thermometer filled with the gas under investigation.

§ 3. Consider first the Joule-Thomson, or adiabatic, plug experiment. If the state of flow is steady and no exchange of heat with other bodies than the plug is taking place, the change of internal energy of unit mass of gas in passing through the plug is

$$\epsilon_2 - \epsilon_1 = p_1 v_1 - p_2 v_2 :$$

or, if we let the fall of pressure be infinitesimal,

$$\Delta\epsilon = -\Delta(pv) = -p\Delta v - vdp, \quad . \quad . \quad . \quad (1)$$

Δp and dp being identical, as is seen from the figure. Let the gas which has thus passed from the state A to the state D be now brought, at constant pressure, to the state B. If $d\epsilon$ be the excess of the specific internal energy at B over that at A, we now have

$$d\epsilon = \Delta\epsilon + Q + W, \quad . \quad . \quad . \quad . \quad (2)$$

in which Q and W are, respectively, the heat added to and the work done on the unit mass of gas during the isopiestic change DB, which brings it back from $(T+\Delta T)$ to its original temperature T. We suppose, for simplicity, that all quantities of heat are expressed in ergs.

If we let μ be the ratio of the *fall* of temperature to the *fall* of pressure, *i. e.* $\Delta T/dp$, we have $\Delta T = \mu dp$, and the value of Q is evidently *

$$Q = -\mu C_p dp, \quad . \quad . \quad . \quad . \quad (3)$$

C_p being the specific heat at constant pressure. The value of W is given by the equation

$$W = -p(dv - \Delta v). \quad . \quad . \quad . \quad (4)$$

We also have

$$dv - \Delta v = (-\Delta T)\left(\frac{\partial v}{\partial T}\right)_p = -\mu dp\left(\frac{\partial v}{\partial T}\right)_p, \quad . \quad . \quad (5)$$

so that

$$\Delta v = dv + \left(\frac{\partial v}{\partial T}\right)_p \mu dp. \quad . \quad . \quad . \quad (6)$$

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* The figure as drawn corresponds to the ordinary case in which there is a fall of temperature during the flow through the plug. The sign of μ, as here defined, is then positive, while dp is negative, so that the value of Q as given in (3) is in fact positive, as it is evident from the figure that it must be.

By substituting in equation (2) the values given by equations (1), (3), (4), (5), and (6), we get

$$d\epsilon = -p\,dv - v\,dp - \mu C_p\,dp \; ; \quad \cdots \quad (7)$$

and since the differentials which appear in this equation all refer to the isothermal change AB, we may write it in the form *

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = -p - (v + \mu C_p)\left(\frac{\partial p}{\partial v}\right)_\theta. \quad \cdots \quad (A)$$

This equation is, of course, not new, but few writers of text-books on thermodynamics seem to think it worth deducing.

The equations for the other two experiments come out much more directly. In the case of the isothermal plug experiment, in which the gas passes at once from the state A to the state B, let ρ be the ratio of the energy *abstracted* from the gas to the *fall* of pressure dp. Then we have, obviously,

$$d\epsilon = -p\,dv - v\,dp + \rho\,dp, \quad \cdots \quad (8)$$

or by the same considerations as before,

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = -p - (v - \rho)\left(\frac{\partial p}{\partial v}\right)_\theta. \quad \cdots \quad (B)$$

In the irreversible isothermal free expansion from A to B, the external work is zero. If we let λ be the ratio of the energy (heat) which must be *added*, per unit mass of gas, to the *increase* of specific volume dv, in order to make the final temperature of the gas, after its tumultuous motions have subsided, the same as its initial temperature, we have

$$d\epsilon = \lambda\,dv, \quad \cdots \quad (9)$$

or

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \lambda. \quad \cdots \quad (C)$$

§ 4. In deducing equations (A), (B), and (C), no use is made of the second law ; hence they do not involve the thermodynamic temperature θ at all, and cannot, by themselves, tell us anything about it. But suppose, finally, that the gas passes from the state A to the state B by a reversible isothermal expansion. For any reversible change of state of a system having only two degrees of freedom and acted

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* Constancy of temperature is indicated by the subscript θ: if θ is the thermodynamic temperature, T is evidently constant whenever θ is constant.

on by no external forces except a uniform normal pressure, we have

$$d\epsilon = \theta d\eta - p dv, \quad \ldots \quad \ldots \quad (10)$$

which is merely a statement of the two laws of thermodynamics for such a system, η representing the entropy. If the process is isothermal, we may write equation (10) in the form

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \theta \left(\frac{\partial \eta}{\partial v}\right)_\theta - p. \quad \ldots \quad \ldots \quad (11)$$

To give this a physical meaning we have to eliminate the entropy η, by means of the familiar "thermodynamic relation"

$$\left(\frac{\partial \eta}{\partial v}\right)_\theta = \left(\frac{\partial p}{\partial \theta}\right)_v,$$

thus obtaining the equation

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = -p + \theta\left(\frac{\partial p}{\partial \theta}\right)_v. \quad \ldots \quad \ldots \quad (D)$$

§ 5. By comparing equation (D) with (A), (B), and (C) successively, we get three equations which, when slightly transformed by means of the relation

$$\left(\frac{\partial p}{\partial v}\right)_\theta \left(\frac{\partial v}{\partial \theta}\right)_p \left(\frac{\partial \theta}{\partial p}\right)_v = -1,$$

may be written

$$\theta \left(\frac{\partial v}{\partial \theta}\right)_p = v + \mu C_p, \quad \ldots \quad \ldots \quad (E)$$

$$\theta \left(\frac{\partial v}{\partial \theta}\right)_p = v - \rho, \quad \ldots \quad \ldots \quad (F)$$

$$\theta \left(\frac{\partial p}{\partial \theta}\right)_v = p + \lambda, \quad \ldots \quad \ldots \quad (G)$$

and which give relations between θ and the data to be obtained from the three experiments. Expressing these in integral form we have

$$\log \frac{\theta}{\theta_0} = \int_{v_0}^v \frac{dv}{v + \mu C_p} \quad (p = \pi = \text{constant}), \quad \ldots \quad (H)$$

$$\log \frac{\theta}{\theta_0} = \int_{v_0}^v \frac{dv}{v - \rho} \quad (p = \pi = \text{constant}), \quad \ldots \quad (I)$$

$$\log \frac{\theta}{\theta_0} = \int_{p_0}^p \frac{dp}{p + \lambda} \quad (v = \phi = \text{constant}), \quad \ldots \quad (J)$$

Equations (H) and (I) show how we may, if the experimental data are at hand, find the ratio of the numerical values of any two temperatures denoted by θ and θ_0 on the thermodynamic scale, at which the specific volume of the gas has the values v and v_0 at the constant pressure π. Equation (J) gives us similar information in terms of pressure at the constant specific volume ϕ. By using the equation $pv = RT$ which defines the absolute temperature T of the gas-thermometer, when either p or v is constant, it is, of course, easy to replace v or p by the absolute temperature measured by a constant-pressure or constant-volume thermometer filled with the gas in question.

These deductions, which make no pretence of novelty, are all quite simple enough to be put into the most elementary text-book of thermodynamics. In view of the difficulty students have in getting any clear ideas about the thermodynamic scale of temperature from Joule and Lord Kelvin's papers on the plug experiment, I think it would be well if some such simple discussion were more commonly given. For although this treatment is only formal, any one can see from the equations what experimental data are needed and how we must use them if we wish to go on to find actual numerical relations. The main thing is to have the conviction that there *is* a definite way in which the experimental data might be used if we had them, and anyone who has not attained that conviction by the "seeing-is-believing" process must sometimes feel rather at sea in using thermodynamics.

§ 6. In order actually to perform the integrations indicated in equations (H), (I), and (J), we have, in the case of the adiabatic plug experiment, to know the values of μ and C_p as functions of v for the given constant pressure used in our constant-pressure gas-thermometer; and it may be noted that the value of Joule's equivalent is also involved, unless we have the values of C_p measured directly in ergs. For the isothermal plug experiment, we need merely the value of ρ as a function of v at the same constant pressure. By supplying this energy ρ electrically, it may be measured in ergs as exactly as the absolute accuracy of our electrical standards permits. This method would not serve us in the case of hydrogen above $-80°$ C., nor for other gases above their, as yet unknown, inversion temperatures, though it is conceivable that in such cases the Peltier effect might be utilized for electric cooling. For the free expansion experiment the data needed are the values of λ in terms of p at the given specific volume used in the constant-volume gas-thermometer. It may be remarked, in this connexion, that

some degree of uncertainty must always attach to the substitution of experimental values of μ, ρ, and λ for differential coefficients : we can only hope that the extrapolation from finite to infinitesimal differences of pressure is allowable.

The quantity λ, or $(\partial\epsilon/\partial v)_\theta$, is useful for expressing the deviations of the behaviour of any real gas from the simple Boyle-GayLussac law, and it would be convenient to have values of λ tabulated for the various gases. Such a tabulation has been given by Amagat [*] for carbonic acid and ethylene up to 1000 atmospheres, for several temperatures between $0°$ and $200°$ C.; also for oxygen, nitrogen, air, and hydrogen for the mean temperature of $50°$ up to 1000 atmospheres, and for the mean temperature of $25°$ from 1000 to 4000 atmospheres. The computations were made by means of equation (G) from Amagat's measurements of p and $(\partial p/\partial\theta)_v$, on the assumption that θ and T are identical. These tables are interesting and valuable, but for the particular purpose of finding the relation of θ and T, we need similar but more detailed and exact values for such pressures as are used in gas-thermometers.

Equations (A), (B), and (C) show the relation of λ to μ and ρ; *i. e.*, how λ might be computed from the results of either form of the plug experiment. I have not made any such computations, but it seems not beyond the bounds of possibility that with our present experimental refinements the quantity λ might be measured directly.

§ 7. The two most interesting problems of thermodynamics at the present day are both concerned with the scale of temperature. The first is the performance of the plug experiment at very low temperatures. For it is a highly interesting question whether such temperatures as $1°\!.7$ absolute, as given in a recent paper by Olszewski [†], would really be anything like that if we could measure them on the thermodynamic scale. At present we cannot say whether they would or not.

The other problem is the purely theoretical one of inventing a radiation scale which shall be based on the two laws of thermodynamics in their accepted form, and on nothing else except experimental facts that we are sure of and familiar with. The Stefan-Boltzmann law comes nearest to this. Boltzmann's deduction depended ultimately on the assumption that Maxwell's distribution of stress in the electromagnetic field was the real one. As Maxwell's reasoning on this point does not pretend to be conclusive, Boltzmann's deduction of

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[*] *Jour. de Phys.* (3) iii. p. 307 (1894).
[†] *Annalen der Physik*, 1905, no. 10.

Stefan's law for the black body could also not be considered conclusive. This state of affairs has been changed since we know, from the brilliant experiments of Lebedew and of Nichols and Hull, that the light-pressure does exist and does have, very approximately at all events, the value deduced *a priori* from Maxwell's theory. It still remains, however, highly desirable that another absolute radiation scale should be devised, so that we may have another and independent check on our high temperature measurements. Planck's formula rests on too many assumptions to be satisfactory theoretically; and Wien's displacement law, while its theoretical foundation is nearly as good as that of the Stefan-Boltzmann law, does not, by itself, offer a good method from the point of view of experimental accuracy.

U.S. Department of Agriculture,
    Washington, Jan. 17, 1906.

LXII. *Some Measurements of Wave-Lengths with a Modified Apparatus.* By Lord RAYLEIGH, *O.M.*, *Pres. R.S.**

AS the result of discussions held during the last three or four years, it seems to be pretty generally agreed that the use of the diffraction-grating in fundamental work must be limited to interpolation between standard wave-lengths determined by other means. Even under the advantageous conditions rendered possible by Rowland's invention of the concave grating, allowing collimators and object-glasses to be dispensed with, the accuracy attained in comparisons of considerably differing wave-lengths is found to fall short of what had been hoped. I think that this disappointment is partly the result of exaggerated expectations, against which in 1888 † I gave what was intended to be a warning. Quite recently, Michelson ‡ has shown in detail how particular errors of ruling may interfere with results obtained by the method of coincidences ; but we must admit that the discrepancies found by Kayser § in experiments specially designed to test this question, are greater than would have been anticipated.

Under these circumstances, attention has naturally been directed to interference methods, and especially to that so

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* Communicated by the Author.
† Wave Theory, *Enc. Brit.*; Scientific Papers, iii. p. 111, footnote.
‡ Astro-physical Journal, xviii. p. 278 (1903).
§ *Zeitschrift für wiss. Photographie*, Bd. ii. p. 49 (1904).

skillfully worked out by Fabry and Perot. In using an accepted phrase it may be well to say definitely that these methods have no more claim to the title than has the method which employs the grating. The difference between the grating and the parallel plates of Fabry and Perot is not that the latter depends more upon interference than the former, but that in virtue of simplicity the parallel plates allow of a more accurate construction. In Fabry and Perot's work the wave-lengths are directly compared with the green and red of *cadmium*; and they have obtained numbers, apparently of great accuracy, for artificial lights from vacuum-tubes containing various substances, *e. g.* mercury, for numerous lines from an iron arc, and also for various rays of the solar spectrum. While, so far as I can judge, there has been every disposition to receive with favour work which not only bears the marks of care but is explained with great discrimination, it must still be felt that, in accordance with an almost universal rule, confirmation by other hands is necessary to complete satisfaction. It was with this feeling that about a year ago I commenced some observations of which I now present a preliminary account. I was not without hope that I might be able to introduce some variations which would turn out to be improvements, and which would, at any rate, promote the independence of my results.

In this method the interference rings utilized are of the kind first observed by Haidinger, dependent upon obliquity. Their theory is contained in the usual formulæ for the reflexion and transmission of parallel light by a "thin plate." Thus, if λ be the wave-length of monochromatic light, $\kappa = 2\pi/\lambda$, δ the retardation, e the reflecting power of the surface, we have, in the usual notation for the intensity of reflected light [*],

$$R = \frac{4\,e^2 \sin^2\left(\frac{1}{2}\kappa\delta\right)}{1 - 2e^2 \cos\kappa\delta + e^4}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

and

$$\delta = 2\mu t \cos\alpha', \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

where t denotes the thickness of the plate, μ the refractive index, and α' the obliquity of the rays within the plate.

Another form of (1) is

$$\frac{1}{R} = 1 + \frac{(1 - e^2)^2}{4e^2 \sin^2\left(\frac{1}{2}\kappa\delta\right)}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3)$$

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[*] See, for example, Wave Theory, *Enc. Brit.*; Scientific Papers, iii. pp. 64, 65.

and from this we see that if $e=1$ absolutely,
$$1/\mathrm{R} = \mathrm{R} = 1$$
for all values of δ. If $e=1$ very nearly, $\mathrm{R}=1$ nearly for all values of δ for which $\sin(\tfrac{1}{2}\kappa\delta)$ is not very small. In the light reflected from an extended source, the ground will be of the full brightness corresponding to the source, but it will be traversed by *narrow* dark lines. By transmitted light the ground, corresponding to general values of the obliquity, will be dark, but will be interrupted by narrow bright rings whose position is determined by $\sin(\tfrac{1}{2}\kappa\delta)=0$. In permitting for certain directions a complete transmission in spite of a high reflecting power (e) of the surfaces, the plate acts the part of a resonator.

There is no transparent material for which, unless at high obliquity, e approaches unity. In Fabry and Perot's apparatus the reflexions at nearly perpendicular incidence are enhanced by lightly silvering the surfaces. In this way the advantage of narrowing the bright rings is attained in great measure without too great a sacrifice of light. The plate in the optical sense is one of air, and is bounded by plates of glass whose inner silvered surfaces are accurately flat and parallel *. The outer surfaces need only ordinary flatness, and it is best that they be not quite parallel to the inner ones.

It will be seen that the optical parts are themselves of extreme simplicity; but they require accuracy of construction and adjustment, and the demand in these respects is the more severe the further the ideal is pursued of narrowing the rings by increase of reflecting power. Two forms of mounting are employed. In one instrument, called the interferometer, the distance between the surfaces—the thickness of the plate—is adjustable over a wide range. In its complete development this instrument is elaborate and costly. The actual measurements of wave-lengths by Fabry and Perot were for the most part effected by another form of instrument called an *étalon* or interference-gauge. The thickness of the optical plate is here fixed ; the glasses are held up to metal knobs, acting as distance-pieces, by adjustable springs, and the final adjustment to parallelism is effected by regulating the pressure exerted by these springs.

The theory of the comparison of wave-lengths by means of this apparatus is very simple, and it may be well to give it, following closely the statement of Fabry and Perot †.

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* The most important requirement is the equidistance of the surfaces, and would not be inconsistent with equal and opposite finite curvatures.

† *Ann. de Chimie*, xxv. p. 110 (1902). A good account is given in Baly's ' Spectroscopy.'

Consider first the cadmium radiation λ. It gives a system of rings. Let P be the ordinal number of one of these rings, for example the first counting from the centre. This integer is supposed known. The order of interference at the centre will be $p = P + \epsilon$. We have to determine this number ϵ, lying ordinarily between 0 and 1. The diameter of the ring under consideration increases with ϵ; so that a measure of the diameter allows us to determine the latter. Let e * be the thickness of the plate of air. The order of interference at the centre is $p = 2e/\lambda$. This corresponds to normal passage. At an obliquity i the order of interference is $p \cos i$. Thus if x be the angular diameter of the ring P, $p \cos \frac{1}{2} x = P$; or since x is small,

$$p = P\left(1 + \frac{x^2}{8}\right),$$

In like manner, from observations upon another radiation λ' to be compared with λ, we have

$$p' = P'\left(1 + \frac{x'^2}{8}\right);$$

whence if e be treated as an absolute constant,

$$\frac{\lambda'}{\lambda} = \frac{P}{P'}\left(1 + \frac{x^2}{8} - \frac{x'^2}{8}\right). \quad \cdots \quad (4)$$

The ratio λ/λ' is thus determined as a function of the angular diameters x, x' and of the integers P, P'.

One of the principal variations in my procedure relates to the manner in which P is determined. MM. Fabry and Perot † say :—" L'étalon, une fois réglé, est mesuré en fonction des longueurs d'onde du cadmium, par les méthodes que nous avons précédemment décrites ; l'emploi de interféromètre est nécessaire pour cela." I wished to dispense with the sliding interferometer, and there is no real difficulty in determining P without it. For this purpose we use a modified form of (4), viz. :—

$$\frac{P'}{P} = \frac{\lambda}{\lambda'}\left(1 + \frac{x^2}{8} - \frac{x'^2}{8}\right), \quad \cdots \quad (5)$$

expressing P'/P as a function of λ/λ', regarded as known, and of the diameters. To test a proposed (integral) value of P, we calculate P' from (5). If the result deviates from an integer by more than a small amount (depending upon the accuracy of the observations), the proposed value of P is to be rejected.

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* Now with an altered meaning.
† *Loc. cit.* p. 112.

In this way, by a process of exclusion the true value is ultimately arrived at.

The details of the best course will depend somewhat upon circumstances. It will usually be convenient to take first a ratio of wave-lengths not differing much from unity. Thus in my actual operations the mechanical measure of the distance between the plates was 4·766 mm., and the first optical observations calculated related to the two yellow lines of mercury. The ratio of wave-lengths, according to the measurements of Fabry and Perot, is 1·003650; giving after correction for the measured diameters 1·003641 as the ratio P'/P. From the mechanical measure we find as a rough value of P, P = 16460. Calculating from this, we get P' = 16519·92, not sufficiently close to an integer. Adding 22 to P we find as corresponding values

$$P = 16482, \qquad P' = 16542·00,$$

giving P as closely as it can be found from these observations. This makes the value of P for the cadmium-red ring observed at the same time about 14824, and this should not be in error by more than ± 30.

Having obtained an approximate value of P for the cadmium red, we may now conveniently form a table, of which the first column contains all the so far admissible (say 60) integral values of P. The other columns contain the results by calculation from (5) of comparisons between other radiations and the cadmium red. The second and third columns, for example, may relate to cadmium green and cadmium blue. These almost suffice to fix the value of P, but any lingering doubt will be removed by additional columns relating to mercury green and mercury yellow (more refrangible). An extract from the table (p. 690) may make the matter clearer.

Inasmuch as the ratio of cadmium red to cadmium green is 1·2659650, very nearly 5 : 4, only every fourth number for red is admissible on this ground alone. If we consider a number such as 14803 not excluded by the comparison with cadmium green, we see that while it would pass the mercury green test, it is condemned by the cadmium blue and still more by the mercury yellow test. The only possible value of P is found to be 14814.

The criticism may probably suggest itself that, although other values of P may be excluded, the agreement of the row containing 14814 with integers is none too good. It is to be remembered that these observations were of a preliminary character, and were taken without the full precautions with

regard to temperature afterwards found to be necessary. The formula at the basis of the calculation assumes that e, the thickness of the plate, is constant, but in fact it changes

Cd red.	Cd green.	Cd blue.	Hg green.	Hg yellow.
14788	18721·03	19836·04	17435·24	
9				
14790				
1		19840·07		
2	18726·09			
3				
4		19844·09		
5	18729·89			
6	18731·15			
7		19848·12		
8				
9	18734·95	19850·80		
14800		19852·14		
1				
2				
3	18740·01	19856·16	17459·04	16518·68
4				
5				
6		19860·19		
7	18745·08			
8		19862·87		
9				
14810	18748·88			
1	18750·14	19866·90	17462·36	
2				
3				
4	18753·94	19870·92	17465·90	16530·96
5				

with temperature. On this account alone erroneous results will be obtained unless the observations are well alternated, so as to eliminate such effects. The numbers finally arrived at, in substitution for the row in the table, are

$$14814, \ 18753 \cdot 95, \ 19870 \cdot 95, \ 17465 \cdot 97, \ 16531 \cdot 00.$$

The deviations from integers still outstanding have their origin in a complication which must be admitted to be a drawback to the method and might at first sight be estimated even more seriously. The optical thickness e of the plate, on which everything depends, is not really constant, as has been assumed, when we pass from one part of the spectrum to another somewhat distant from the first. The question is discussed by Fabry and Perot. If, to take account of this

factor, we denote the thicknesses for the two wave-lengths by e_λ, $e_{\lambda'}$, we have

$$\frac{p'}{p} = \frac{e_{\lambda'}}{e_\lambda} \frac{\lambda}{\lambda'},$$

and accordingly in place of (5)

$$\frac{P'}{P} = \frac{\lambda e_{\lambda'}}{\lambda' e_\lambda}\left(1 + \frac{x^2}{8} - \frac{x'^2}{8}\right). \quad \ldots \quad (6)$$

But although I was prepared to find the calculated values of P' differing somewhat from integers, I was disturbed by the amount and at first by the direction of the difference. For in their paper of 1899 * Fabry and Perot remark :—" Le surface optique du métal pour la radiation rouge est, par suite, située un peu plus profondément dans le métal que celle de la lumière verte, et à une distance de 4 $\mu\mu$." At this rate e_λ (red) would exceed $e_{\lambda'}$ (green), and the introduction of the new factor in (6) would increase, and not remove, the discrepancy. It would seem, however, that the passage above quoted is in error and inconsistent with the discussion given in the later paper †, itself indeed embarrassed by several misprints ‡.

The amount of the correction required to bring the number for cadmium green up to an integer—about $2\frac{1}{2}$ parts in a million—is $2\frac{1}{2}$ times as great as one would expect from Fabry and Perot's indications §. As to this, it may be observed that the wave-lengths employed in the calculation of the cadmium radiations are those of Michelson, and were obtained by a method free of the complication now under discussion. If these are correct, as there is no reason to doubt, and if there is no mistake in the identification of the ring—and there can be none here—it follows that the change of optical thickness in passing from red to green is determined by the numbers given and may be used to correct ratios of wave-lengths not previously known with precision.

If we wish to make the results of the present method entirely independent, we must obtain material from observation sufficient to allow the variation of thickness with wave-length to be eliminated, that is, we must use the same silvered

* *Ann. de Chimie*, xvi. p. 311.
† *L. c.* pp. 120–124.
‡ Of these it may be worth while to note that the sign of 6·6 $\mu\mu$ on p. 123, line 5 should apparently be − instead of +.
§ It is known that the effect depends upon the thickness of the silver films; perhaps also upon the process used in silvering and upon the condition of the surfaces in other respects. Surfaces that have stood some time in air are almost certain to be contaminated with layers of volatile greasy matter.

plates at two different distances.   In Fabry and Perot's work the sliding interferometer was employed ; the silvered surfaces were brought to very small distances, and the coincidences of two band systems, *e. g.* cadmium red and cadmium green, were observed, the telescope being focussed upon the plate, and not as before for infinity.   It appears that excellent results were obtained in this way, affording material for eliminating the complication due to change of optical thickness.

It is rather simpler, in principle, and has the incidental advantage of allowing the sliding interferometer to be dispensed with, if we follow the same method for the small as for the greater distance.   If the calculation be conducted on the same lines as before by means of (5), we ought to obtain the same *fractional* part again in the value of P′, *e. g.* ·95 for cadmium green referred to cadmium red.   For, as we see from (6), the proportional error in P′/P as calculated from (5) is $(e_{\lambda'} - e_\lambda)/e_\lambda$.   In the second set of operations, writing η for e, we find as the proportional error $(\eta_{\lambda'} - \eta_\lambda)/\eta_\lambda$, in which $\eta_{\lambda'} - \eta_\lambda = e_{\lambda'} - e_\lambda$; so that the proportional errors are as $\eta_\lambda : e_\lambda$, or inversely as P or P′.   Thus the absolute error in P′, as calculated from (5), is unaffected by the change of e to η. If the fractional part is not recovered, within the limits of error, it is a proof that the assumed ratio of wave-lengths calls for correction, and the discrepancy gives the means for effecting such correction.

The above procedure is the natural one, when it is a question of identifying a ring or of confirming ratios of wave-lengths already presumably determined with full accuracy ; but when the object is to find more accurately wave-lengths only roughly known, it has an air of indirectness.   Otherwise, we have as before,

$$2e_\lambda = p\lambda, \qquad 2e_{\lambda'} = p'\lambda' \; ;$$

and again for a smaller interval between the surfaces,

$$2\eta_\lambda = \pi\lambda, \qquad 2\eta_\lambda = \pi'\lambda'.$$

Hence

$$2(e_\lambda - \eta_\lambda) = (p - \pi)\lambda, \qquad 2(e_{\lambda'} - \eta_{\lambda'}) = (p' - \pi')\lambda' \; ;$$

and $e_\lambda - \eta_\lambda = e_{\lambda'} - \eta_{\lambda'}$,   so that

$$\frac{\lambda}{\lambda'} = \frac{p' - \pi'}{p - \pi}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence p, π, p', π' are the ordinal numbers at the centre. They are to be deduced, as before, from the integral numbers proper to the rings actually observed and from the measured angular diameters of these rings.

It is obvious that p and π must not be nearly equal. If p be the larger number corresponding to the greater interval, π should not exceed $\frac{1}{4} p$. On the other hand, too great a reduction of π would lead to difficulties on account of the increased angular diameter of the rings. Perhaps it was for this reason that Fabry and Perot adopted an altered course. In my experiments the longer interval was, as already mentioned, about 5 mm., and the shorter interval was about 1 mm., so that the angular diameter of the rings was rather more than doubled in the latter case.

The facility with which angular diameters larger than usual could be observed is due, in part at any rate, to the special construction of my apparatus. MM. Fabry and Perot employed a fixed interference-gauge and a fixed telescope, measuring the diameters of the rings by an eyepiece micrometer. There are, I think, some advantages in a modified arrangement, whereby it becomes possible to refer the rings to a wire fixed in the optic axis of the telescope. To this end the wire is made vertical, and the rings are brought to coincidence with it by a rotation of the gauge, which is mounted upon a turntable giving movement round a vertical axis. The middle plane of the gauge is vertical and adjusted so as to include the axis of rotation. In this way of working the reference wire is backed always by the same light, whether opposite sides of one ring or of different rings are under observation. It is perhaps a more important advantage that the same part of the object-glass is always in use, and to a better approximation the same parts of the plates of the gauge. The diaphragm which limits the latter should be as close to the plates as possible (or to their image near the eye), but when the multiple reflexions are taken into account it is impossible to secure that exactly the same part should always be in action.

The revolving turntable carried with it a thick strip of plate-glass upon which was scratched a radial line. The point observed described a circle of 10 inches radius, and the rotation was measured by means of a travelling microscope reading to ·001 inch. The angles involved are sufficiently small to allow the diameter of a ring to be taken as proportional to the difference of readings at the microscope.

As regards the gauge itself, the plates are by Brashear. For the mounting of the 5 mm. gauge, which is of brass, I am indebted to my son Mr. R. J. Strutt. The 1 mm. gauge is of iron and was made by my assistant Mr. Enock. They are much after the design of Fabry and Perot. For the final adjustment to parallelism the eye is moved in various directions

across the line of vision so as to bring different parts of the plates into action, and for this purpose it may be desirable to increase the aperture. A dilatation of the rings means that the corresponding parts of the plates need approximation by additional pressure. The aperture employed in the actual measurements was of about 9 mm. diameter.

The (achromatic) object-glass of the telescope is of 15 inches focus. In rigid connexion with it is the vertical reference wire accurately adjusted to focus, and close to the wire a small frame suitable for carrying the horizontal slits (cut out of thin sheet zinc) necessary for the isolation of the various colours *. The eyepiece is a single lens of 5 inches focus, *mounted independently*, so that it can be re-adjusted without fear of disturbing the object-glass and reference wire. The change of position required for the best seeing in passing from red to blue or even from red to green is so great as to occasion surprise that good results can be attained in the absence of such a provision †.

The separation of the colours was usually effected by direct-vision prisms held between the eyepiece and the eye. Of these two were available. The larger containing (in all) three prisms was usually the more convenient, but sometimes a smaller and more dispersive combination containing five prisms was preferred. It is better to use more dispersion than unduly to narrow the slit. The refracting edges of the prisms are, of course, horizontal. In order to secure that the proper parts of the ring systems should be visible, the axis of the telescope was adjusted in the vertical plane with substitution for the slit of a horizontal wire coincident with the middle line of the former.

The advantage of this arrangement is that the ring systems (or at least so much of them as is necessary) of the various radiations emitted by one source of light are all in view at the same time.

In some cases, direct-vision prisms held between the 5-inch eye-lens and the eye do not suffice. The soda lines, for example, require a high dispersion. Even the yellow lines of mercury, which are about three times as far apart as the soda lines, could not be fully separated by the prisms already spoken of. Here a good deal depends upon chance. If the rings of one mercury system happen to bisect approximately those of the other system, both can be measured in the inter-ferometer-gauge, and the only question which remains open

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* Fabry and Perot, *C. R.* March 27, 1904.

† Especially in using the method of coincidences. I ought perhaps to mention that my eyes have now very little power of accommodation.

is the distinction of the two systems. For this purpose a prism of moderate power, by which one system is lifted a little relatively to the other, suffices. If, however, the two ring systems chance to be nearly in coincidence, a much more powerful dispersion is required in order to measure them separately.

In such cases recourse was had to a special direct-vision prism of glass and bisulphide of carbon through which a selected ray of the spectrum passes without refraction at all at any of the surfaces *. In this instrument the upper edge of the beam traverses 20 inches of glass and the lower edge 20 inches of bisulphide of carbon. This prism cannot be inserted between the eyepiece already described and the eye, which latter must be placed at the image of the object-glass. Additional lenses are therefore required. These are merely ordinary spectacle-lenses and constitute a telescope of unit magnifying power. A more precise description is postponed, as I am not sure that I have as yet hit upon the best arrangement. It may suffice to say that with this instrument rings formed of spectral rays even closer than the soda lines could be readily separated, and that without too great a contraction of the slit limiting the visible portion of the rings.

The source of light, sometimes very small, was focussed upon the diaphragm at the gauge, and it is necessary that the aperture be completely filled with light. This gives the ratio between the distances of the lens from the source (u) and from the gauge (v). Again, the angular diameter of the field of light, which must not be too small, fixes the ratio of the aperture of the lens to v; so that only the absolute scale of the three quantities is left open. It is desirable that the lens be achromatic. I have used a one-inch lens from a small opera-glass, and this worked well with $u = 2\frac{1}{2}$ inches and $v = 4$ feet.

As sources of light in experiments involving high interference, vacuum-tubes are by far the most convenient, and their introduction is one of the many services which Optics owes to Prof. Michelson. At the head of the list stands the helium tube, both on account of its not requiring to be heated and also of the brilliancy of the yellow radiation. Hitherto, however, the wave-lengths have not been measured with the highest accuracy. The tube that I have employed was made some years ago by my son and had already seen a good deal of service in experiments designed to answer the question : "Is Rotatory Polarization influenced by the Earth's

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* Nature, lx. p. 64 (1899) ; Scientific Papers, iv. p. 394.

Motion ? * " From the overpowering brilliancy of the yellow line, it may be inferred that the pressure is not very low. Mercury too, for which the principal wave-lengths have been determined with great accuracy by Fabry and Perot, is convenient as requiring only a very moderate heating ; and cadmium, in spite of the higher temperature demanded, is indispensable. Not only is the cadmium red by general consent the ultimate standard, but a comparison of the red and green ring systems, even without a prism, gives rapid information as to the condition of the gauge, slightly variable from time to time on account of temperature and of necessary readjustments. Thus, in most of my observations, the red ring under measurement was in very approximate coincidence with a green ring. If, owing to rise of temperature, this ring had so far expanded as to make it advisable to substitute the next interior one, there could still be no uncertainty as to the order (one higher) of the ring actually under observation.

As cadmium tubes appear to have been found troublesome, it may be well to describe a simple construction specially adapted to private workers whose skill in glass-blowing is limited. It was thought that alloying and consequent expansion of platinum sealings was a likely source of difficulty, and these were accordingly dispensed with. The diagram exhibits half the complete tube. The working capillary A, the enlargement B D, and the lateral tube C for attachment
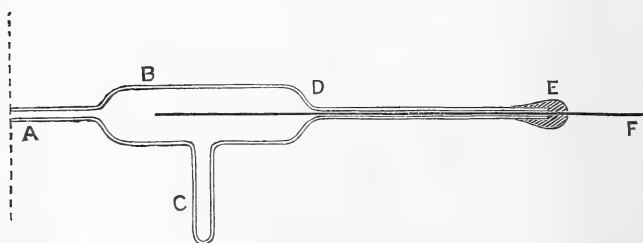
to the pump are much as usual. But the enlargement is continued by a second capillary D E, perhaps $1\frac{1}{2}$ mm. in diameter and 15 cm. long, through which passes with approximate fit a straight aluminium wire, serving as electrode. The air-tight joint at E between the wire and the glass is made with sealing-wax. The length D E must be sufficient

* Phil. Mag. vol. iv. p. 215 (1902).

to allow E to remain cool, although D, enclosed in a copper case, is hot enough to keep the cadmium vapour uncondensed. The lateral tube C projects from the case, and the cadmium condensed in it may need to be driven back from time to time by temporary application of the flame of a spirit-lamp or bunsen-burner.

This construction, used with cadmium, mercury, and thallium, has so far answered my expectations   Cadmium tubes, apart from failures by cracking, are said often to deteriorate rapidly.   My experience did not contradict this; for after four or five evenings' work the *red* radiation, which at first had been very brilliant, was no longer serviceable, although the green did not seem to have suffered much. At this stage the tube was re-exhausted and then appeared to behave differently, the red radiation being much better maintained.   One must suppose that something deleterious had been emitted and been pumped away.   There is much in the behaviour of vacuum-tubes which at present defies explanation.

To excite the electric discharge a large Ruhmkorff, actuated by five small storage-cells, was usually employed.   Sometimes, especially in the comparison of the cadmium radiations, an alternate current was substituted; but there was no perceptible difference in the measurements.   In this case a transformer of home construction was fed from a De Meritens magneto machine.

The radiations from zinc (and occasionally from cadmium) were obtained by an arrangement similar to Fabry and Perot's "trembler"[*].   The behaviour was very capricious. Sometimes, even when actuated by five secondary cells only, the zinc rings were magnificent; but the deterioration was usually rapid as the zinc points lost their metallic surfaces. This change appears to be independent of oxidation.   When the current was from a dynamo giving about 80 volts, the apparatus was less troublesome, but even then required careful management.   The fineness of the points needs to be accommodated to the current employed.

As an example of the observations and calculations therefrom, I will take a series of Dec. 20, 1905, relative to the three radiations from the cadmium vacuum-tube.   In this series the temperature conditions were more favourable than usual.

[*] *C. R.* 130. p. 406 (1900).

Cadmium (5 mm. gauge).

RED.		GREEN.		BLUE.	
Right.	Left.	Right.	Left.	Right.	Left.
·398	·221	·401	·217	·406	·213
·399	·221	·402	·217	·404	·211
·399	·220	·402	·216	·406	·210
·3987	·2207	·4017	·2167	·4053	·2113
Diff. = ·1780		Diff. = ·1850		Diff. = ·1940	

The numbers entered are the actual readings of the microscope in inches for settings on the right and left sides of the rings. Each horizontal row constitutes really a complete set. In order to eliminate temperature effects as far as possible, the readings are taken in a certain sequence. Thus in the first row the sequence was Red (R), Green (R), Blue (R), Blue (L), Green (L), Red (L). The differences, representing the diameters of the rings, are thus appropriate to the *middle* of the time occupied. If, as happened here and usually, the temperature was rising, so that the rings dilated, the first reading (·398) on the red is too small, but the error is compensated in the last reading (·221), which is equally too small. As a matter of convenience the next row would be taken in the reverse order, beginning with a repetition of Red (L), and so on.

Since the radius of the circle described by the point of observation is 10 inches, the angular diameters (x) of the rings are as follows:—

	RED.	GREEN.	BLUE.
x	·01780	·01850	·01940
x^2	$10^{-4} \times 3·168$	$10^{-4} \times 3·422$	$10^{-4} \times 3·764$
$\frac{1}{8} x^2$	$10^{-4} \times ·3960$	$10^{-4} \times ·4277$	$10^{-4} \times ·4705$
Diff.	....	$10^{-4} \times ·0317$	$10^{-4} \times ·0745$

The calculation now proceeds by means of (5). If P refer to cadmium red and P′ to green, we have with Michelson's

values of the wave-lengths:

$$\frac{P'}{P} = 1 \cdot 2659650 \ (1 - \cdot 00000317) = 1 \cdot 2659610,$$

which with $P = 14814$ gives

$$P' = 14814 + 3939 \cdot 945 = 18753 \cdot 945.$$

In like manner for the blue referred to the red,

$$\frac{P'}{P} = 1 \cdot 3413733 \ (1 - \cdot 00000745) = 1 \cdot 3413633,$$

whence

$$P' = 14814 + 5056 \cdot 955 = 19870 \cdot 955.$$

The wave-lengths of the various radiations from a single source can thus be compared with great ease, and but little fear of temperature error. A set of observations from which this error is practically eliminated can be made in a short time and a few repetitions give all the security necessary. But the situation is not so favourable when we compare radiations from different sources. More time is occupied and there is corresponding opportunity for temperature change. It is necessary to alternate the observations, taking the first source twice and the second once, or preferably the first three times and the second twice. Even with this precaution I believe that temperature change was the principal source of error in the results of a single evening's work.

In the observations with an interval of one millimetre between the silvered surfaces, the influence of temperature is of course much less perceptible. For a similar reason the identification of the rings is a much easier matter. I will give as a specimen a series of operations (Feb. 9) in which helium was compared with cadmium. The first and third sets, each containing a repetition, related to cadmium ; the second set (twice repeated) related to helium. Only the mean diameter for each set is here recorded :—

Cadmium.

	Red.	Green.	Blue.
I.........	·749	·636	·641
III.........	·752	·643	·643
Mean ...	·750	·639	·642

Helium.

	Red 7065	Red 6678	Yellow 5876	Green 5016	Green 4922	Blue 4713	Violet 4472
II.	·704	·673	·721	·630	·684	·665	·637

The first question is as to the ratio P'/P derived by (5) from these numbers for cadmium. The integral value of P for the cadmium red ring was 3328. From this we find

$$P' \text{ cad. green} = 4213\cdot946,$$
$$P' \text{ cad. blue } = 4464\cdot935,$$

on the basis of Michelson's wave-lengths. For the green the fractional part is practically identical with that deduced above from one set of observations with the 5 mm. gauge. In the case of the blue the fractional part is now distinctly lower.

The above are the results of work on single evenings. On the mean of all the comparisons with the two intervals there resulted :—

Cadmium.

	5 mm.	1 mm.
Red	14814	3328
Green	18753·95	4213·95
Blue...............	19870·95	4464·94

As already explained the agreement of the fractional parts constitutes a complete verification of Michelson's ratios of wave-lengths, accurate to one part in 2 millions in the case of red and blue and to a still closer accuracy in the case of red and green. And it appears further that the phase-changes, upon which depend the deviations from integers, are decidedly greater than in the examples recorded by Fabry and Perot.

The above results for cadmium suffice to indicate what deviations from integral values are to be expected when any radiation is compared with cadmium red assumed integral. In so far as the expected fractional parts appear in the results, so far are the ratios of wave-lengths assumed in the calculation verified. The following are the wave-lengths, reckoned in air at 15° C. and 760 mm. pressure, whose ratios to

cadmium red have been verified by my observations to about one part in a million :—

$$
\text{Cadmium} \left\{ \begin{array}{l} 6438\cdot4722 \\ 5085\cdot8240 \\ 4799\cdot911 \end{array} \right\} \text{Michelson.}
$$

$$
\text{Mercury} \left\{ \begin{array}{l} 5790\cdot659 \\ 5769\cdot598 \\ 5460\cdot742 \\ 4358\cdot343 \end{array} \right\} \text{Fabry \& Perot.}
$$

$$
\text{Zinc} \left\{ \begin{array}{l} 6362\cdot345 \\ 4810\cdot535 \\ 4722\cdot164 \\ 4680\cdot138 \end{array} \right\} \text{Fabry \& Perot.}
$$

$$
\text{Soda} \left\{ \begin{array}{l} 5895\cdot932 \\ 5889\cdot965 \end{array} \right\} \text{Fabry \& Perot.}
$$

I have spoken of an agreement to about 1 part in a million. In several cases the confirmation was decidedly closer. In one only, that of zinc red, did there appear an indication of a disagreement rather outside the limits of error. My observations would point to a wave-length about 1 millionth part greater than that of Fabry and Perot ; but in view of the difficulty of observations with the trembler, I am not disposed to insist upon it. The soda observations were on light from a cadmium vacuum-tube in which soda accidentally presented itself. The numbers quoted from Fabry and Perot relate to a soda-flame.

As an example in which the ratios of wave-lengths were less accurately known beforehand, I will give some details relative to helium, beginning with observations of Feb. 9 by the 1 mm. gauge, already referred to. The Table I. annexed gives, in the second column the wave-lengths of the various helium lines recorded by Runge[*], in the fifth the same reduced to Michelson's scale as employed by Fabry and Perot. The third column gives the corrections for obliquity as calculated from the observations with the 1 mm. apparatus already recorded, the fourth the differences from the corresponding quantity for cadmium red. Taking, for example, the helium ray of longest wave-length in comparison with cadmium red, we get by (5)

$$
P' = 3328 \frac{6438\cdot472}{7065\cdot22} (1 + \cdot000084) = 3033\cdot03.
$$

These numbers should be integers, were the wave-lengths

TABLE I. (1 mm.)

	λ (Runge).	$10^4 \times \frac{1}{8}x^2$.	Diff.	λ (reduced).	P.		Correction in millionths.	λ (corrected).
Cd red	......	7·04	......	......	3328		......	......
He	7065·48	6·20	·84	7065·22	3033·03	·02	+3	7065·24
,, ...	6678·37	5·67	1·37	6678·12	3201·01	·01	+0	6678·12
,, ...	5875·870	6·49	·54	5875·653	3646·98	·98	0	5875·653
,, ...	5015·732	4·96	2·08	5015·547	4273·06	·95	+26	5015·677
,, ...	4922·096	5·84	1·20	4921·912	4353·96	·95	+2	4921·922
,, ...	4713·252	5·53	1·51	4713·079	4547·03	·94	+20	4713·173
,, ...	4471·646	5·08	1·98	4471·480	4792·93	·93	0	4471·480

TABLE II. (5 mm.)

	λ.	P.		Correction in millionths.	λ (corrected).
Cd green ...	5036	18753·95	·95	......	......
He	7065	13498·97	·92	-4	7065·192
,, ...	6678	14282·07	·01	+4	6678·147
,, ...	5876	16232·89	·98	-6	5875·618
,, ...	5016	19016·46	·95	+27	5015·682
,, ...	4922	19378·00	·95	+3	4921·927
,, ...	4713	......	......	......	(4713·173)
,, ...	4471	......	......	......	(4471·480)

accurate, and were there no phase change. On account of the phase change as determined from the cadmium observations, the fractional parts should be those entered in the 7th column. The differences are trifling, except in the cases of 5016 and 4713. The proportional corrections by which the λ's of column 5 are to be increased are set out in column 8 expressed in millionths; but of course an accuracy of 4 or 5 millionths is hardly to be expected in results from a single set of observations with the 1 mm. gauge.

In the observations (Table II.) with the 5 mm. gauge the comparisons were with the cadmium *green*, for which P is assumed to be 18753·95, corresponding to 14814·00 for cadmium red. The numbers given embody the results of three days' observations, but they do not include the wavelengths 4713, 4472. The procedure is the same as for Table I. If the observations with the 5 mm. gauge stood alone, we should be in doubt whether P for 5016 should be 19016·95, or 19015·95. The results with the 1 mm. gauge show that the latter alternative must be chosen. Except in this respect, the 5 mm. results are independent; and they are of course to be preferred as presumably more accurate. The final numbers for helium are therefore those given in column 6 of Table II.

The only further remark that I will make is that the observations on the helium yellow (5876) are not improbably somewhat embarrassed by a companion of feeble luminosity which could not be separated. In the 5 mm. apparatus the two components would be nearly but not quite in coincidence.

LXIII. *The Symmetrical Vibrations of Conducting Surfaces of Revolution.* By J. W. Nicholson, *B.A., M.Sc., Scholar of Trinity College, and Isaac Newton Student in the University of Cambridge* [*].

IN a previous paper [†], the author investigated the possible electrical periods of the space between two perfectly conducting elliptic cylinders belonging to a confocal system. The vibrations of symmetrical type were alone considered. The object of the present paper is to obtain a list of the simpler surfaces of revolution, including cylinders as a particular case, whose vibrations admit readily of analytical treatment, and to examine certain important cases in detail.

The entire investigation is confined to symmetrical vibrations, which are the most interesting from a physical point of view.

The electric and magnetic vectors are first expressed in a convenient manner in terms of two potential functions satisfying a certain differential equation.

Let any point of space be defined by (1) cylindrical co-ordinates $(\rho\omega z)$, where z is the distance along an axis, ρ the perpendicular upon that axis, and ω the longitude of the meridian of the point ; and (2) coordinates $(\alpha\beta\omega)$, where

$$\rho + \iota z = f(\alpha + \iota\beta) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then the surfaces $\alpha = $ const., $\beta = $ const., $\omega = $ const., form three mutually orthogonal systems. The last is a system of planes, and the others are surfaces of revolution about z.

The space elements normal to the three surfaces, one of each system, through any point, are

$$dn_1 = \frac{d\alpha}{p_1}, \quad dn_2 = \frac{d\beta}{p_2}, \quad dn_3 = \frac{d\omega}{p_3}. \quad . \quad . \quad . \quad (2)$$

As usual (xyz), (abc), (uvw) will denote components of electric force, magnetic induction, and current, but measured along the directions whose cosines are

$$(dn_1, dn_2, dn_3)/(dn_1^2 + dn_2^2 + dn_3^2)^{\frac{1}{2}}.$$

The magnetic force is $(a, b, c)/\mu$, if μ be the permeability. For all investigations here contemplated, $\mu = 1$.

By the circuital relation expressing Faraday's law, since $\dfrac{\partial}{\partial\omega} = 0$ for symmetrical vibrations,

$$\frac{-\dot{a}}{p_2 p_3} = \frac{\partial}{\partial\beta} \cdot \frac{z}{p_3}, \qquad \frac{\dot{b}}{p_1 p_3} = \frac{\partial}{\partial\alpha} \cdot \frac{z}{p_3},$$

$$\frac{-\dot{c}}{p_1 p_2} = \frac{\partial}{\partial\alpha} \cdot \frac{\gamma}{p_2} - \frac{\partial}{\partial\beta} \cdot \frac{x}{p_1}, \quad . \quad . \quad . \quad (3)$$

where dots denote differentiations with respect to t.

The appropriate expression of Ampère's law leads to

$$\frac{\dot{x}}{V^2 p_2 p_3} = \frac{\partial}{\partial\beta} \cdot \frac{c}{p_3}, \qquad \frac{\dot{y}}{V^2 p_1 p_3} = -\frac{\partial}{\partial\alpha} \cdot \frac{c}{p_3},$$

$$\frac{-\dot{z}}{V^2 p_1 p_2} = \frac{\partial}{\partial\alpha} \cdot \frac{b}{p_2} - \frac{\partial}{\partial\beta} \cdot \frac{a}{p_1}, \quad . \quad . \quad . \quad (4)$$

writing $4\pi V^2 (u, v, w) = (x, y, z)$, where V is the velocity of propagation of electromagnetic disturbances in the medium between the conductors.

In order that these equations should be mutually compatible, the solenoidal conditions

$$\frac{\partial}{\partial \alpha} \cdot \frac{x}{p_2 p_3} + \frac{\partial}{\partial \beta} \cdot \frac{y}{p_1 p_3} = 0,$$

$$\frac{\partial}{\partial \alpha} \cdot \frac{a}{p_2 p_3} + \frac{\partial}{\partial \beta} \cdot \frac{b}{p_1 p_3} = 0,$$

must be satisfied, again writing $\dfrac{\partial}{\partial \omega} = 0$.

Thus if ϕ and ψ are two functions of α and β, we may write

$$x = p_2 p_3 \frac{\partial \phi}{\partial \beta}, \qquad y = -p_1 p_3 \frac{\partial \phi}{\partial \alpha}, \quad \cdot \quad \cdot \quad \cdot \quad (5)$$

$$a = p_2 p_3 \frac{\partial \psi}{\partial \beta}, \qquad b = -p_1 p_3 \frac{\partial \psi}{\partial \alpha}, \quad \cdot \quad \cdot \quad \cdot \quad (6)$$

whence by (3),

$$\frac{\dot{c}}{p_1 p_2} = \frac{\partial}{\partial \alpha} \cdot \left(\frac{p_1 p_3}{p_2} \frac{\partial \phi}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{p_2 p_3}{p_1} \frac{\partial \phi}{\partial \beta} \right).$$

Thus with (4), writing $\dfrac{1}{V^2} \dfrac{\partial^2}{\partial t^2} = -k^2$, where $\dfrac{2\pi}{k}$ is the wave-length,

$$\frac{\partial}{\partial \beta} \frac{p_1 p_2}{p_3} \left\{ \frac{\partial}{\partial \alpha} \left(\frac{p_1 p_3}{p_2} \frac{\partial \phi}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{p_2 p_3}{p_1} \frac{\partial \phi}{\partial \beta} \right) \right\} + k^2 \frac{\partial \phi}{\partial \beta} = 0.$$

Similarly from (3) and (4)

$$\frac{\partial}{\partial \alpha} \frac{p_1 p_2}{p_3} \left\{ \frac{\partial}{\partial \alpha} \left(\frac{p_1 p_3}{p_2} \frac{\partial \phi}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{p_2 p_3}{p_1} \frac{\partial \phi}{\partial \beta} \right) \right\} + k^2 \frac{\partial \phi}{\partial \alpha} = 0.$$

Thus

$$\frac{\partial}{\partial \alpha} \left(\frac{p_1 p_3}{p_2} \frac{\partial \phi}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{p_2 p_3}{p_1} \frac{\partial \phi}{\partial \beta} \right) + \frac{k^2 p_3}{p_1 p_2} \phi = 0. \quad \cdot \quad (7)$$

By the symmetry of the relations, ψ must satisfy the same characteristic equation.

The independence of the functions ϕ and ψ, and of the corresponding components of force (x, y, c), (a, b, z) derived therefrom, allows of the extension to surfaces of revolution of a result noticed by Macdonald [*] for the case of spheres. The result so extended may be stated :—

The most general symmetrical oscillation possible in the space between two surfaces of revolution belonging to one of three orthogonal systems of surfaces, is an additive combination of two types : (1) a type in which the components

[*] Electric Waves, Chap. 3.

of electric force at any point normal to two of the surfaces through that point are zero; and (2) a type in which the same property is true of the magnetic force. As a particular case, one of the surfaces of revolution may be absent.

The components of force are, in the most general oscillation,

$$x = p_2 p_3 \frac{\partial \phi}{\partial \beta}, \qquad y = -p_1 p_3 \frac{\partial \phi}{\partial \alpha},$$

$$a = p_2 p_3 \frac{\partial \psi}{\partial \beta}, \qquad b = -p_1 p_3 \frac{\partial \psi}{\partial \alpha},$$

$$z = \frac{-p_3}{V^2} \frac{\partial \psi}{\partial t}, \qquad c = \frac{-p_3}{V^2} \frac{\partial \phi}{\partial t} \text{ by (3) to (6). (8)}$$

Now the element of arc in any direction λ is

$$ds = \left\{ \left(\frac{\partial x}{\partial \lambda} \right)^2 + \left(\frac{\partial y}{\partial \lambda} \right)^2 + \left(\frac{\partial z}{\partial \lambda} \right)^2 \right\}^{\frac{1}{2}} d\lambda.$$

Taking $\lambda = \alpha, \beta, \omega$ successively,

$$dn_1 = \left\{ \left(\frac{\partial \rho}{\partial \alpha} \right)^2 + \left(\frac{\partial z}{\partial \alpha} \right)^2 \right\}^{\frac{1}{2}} d\alpha = \sqrt{\frac{\partial(\rho, z)}{\partial(\alpha, \beta)}} d\alpha,$$

by a property of conjugate functions,

$$dn_2 = \sqrt{\frac{\partial(\rho, z)}{\partial(\alpha, \beta)}} d\beta,$$

similarly

$$dn_3 = \rho d\omega.$$

Thus

$$\frac{1}{p_1^2} = \frac{1}{p_2^2} = \frac{\partial(\rho, z)}{\partial(\alpha, \beta)} \text{ and } p_3 = \frac{1}{\rho}, \quad \cdots \quad (9)$$

and (7) becomes

$$\frac{\partial^2 \phi}{\partial \alpha^2} + \frac{\partial^2 \phi}{\partial \beta^2} + \frac{1}{\rho} \left(\frac{\partial \phi}{\partial \alpha} \frac{\partial \rho}{\partial \alpha} + \frac{\partial \phi}{\partial \beta} \frac{\partial \rho}{\partial \beta} \right) + k^2 \phi \frac{\partial(\rho, z)}{\partial(\alpha, \beta)} = 0. \quad (10)$$

In order to obtain the electrical periods between two perfectly conducting surfaces of the family, it is necessary to find solutions of the form $A(\alpha) B(\beta)$, which is possible, provided that

$$\frac{\partial(\rho, z)}{\partial(\alpha, \beta)} = f_1(\alpha) + f_2(\beta), \quad \cdots \quad \cdots \quad (11)$$

$$\frac{1}{\rho} \frac{\partial \rho}{\partial \alpha} = \text{function of } \alpha \text{ only},$$

$$\frac{1}{\rho} \frac{\partial \rho}{\partial \beta} = \text{function of } \beta \text{ only}.$$

The two latter conditions may be combined to

$$\rho = f_3(\alpha) \cdot f_4(\beta) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The general solution of (11) has been given by Prof. Michell *. For our present purpose, it is necessary to select those solutions which satisfy the further conditions (12).

Excluding the cases in which the conjugate transformation involves elliptic functions, it appears that the surfaces for which our problem may be solved are the sphere, spheroid, cone, paraboloid and hyperboloid of revolution. Since a cylinder is a particular form of surface of revolution, in which the bounding curve describes a circle of infinite radius, it is evident that the cases of the circular, elliptic, parabolic, and hyperbolic cylinders, as well as that of two infinite planes meeting at any angle, may also be solved. It is found that the anchor-ring, a surface of considerable physical interest, does not admit of such a solution.

We proceed to discuss the vibrations between two surfaces of revolution of spherical and spheroidal shape. When they are perfectly conducting, the surface conditions are that the resultant magnetic induction and electric force at the surface are tangential and normal respectively. There are thus two distinct classes of vibrations, corresponding to $\phi = 0$, and $\dfrac{\partial \phi}{\partial \alpha} = 0$, if $\alpha = $ const. is one of the surfaces.

Vibrations of Concentric Spheres.

The periods of this simple system have been obtained by Macdonald †, as the roots of a complicated transcendental equation. If $(r\,\theta\,\omega)$ are spherical polar coordinates referred to the centre of the spheres,

$$z = r \cos \theta, \qquad \rho = r \sin \theta,$$

$$\frac{\partial(\rho, z)}{\partial(\alpha, \beta)} = r^2, \qquad \rho \frac{\partial}{\partial \alpha}\left(\frac{1}{\rho}\right) = -1, \qquad \rho \frac{\partial}{\partial \beta}\left(\frac{1}{\rho}\right) = -\cot \theta,$$

whence (10) becomes

$$r^2 \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial \theta^2} - \cot \theta \frac{\partial \phi}{\partial \theta} + k^2 r^2 \phi = 0, \quad . \quad . \quad (13)$$

whose solution, finite for all inclinations θ, is, if $\mu = \cos \theta$,

$$\phi = \sum_{n=0}^{\infty} r^{\frac{1}{2}} \sin^2 \theta \frac{d\mathrm{P}_n(\mu)}{d\mu} \left\{ \mathrm{A}\mathrm{J}_{n+\frac{3}{2}}(kr) + \mathrm{B}\mathrm{J}_{-n-\frac{1}{2}}(kr) \right\}. \quad (14)$$

* Messenger of Math. vol. xix.
† *Loc. cit.*

In the mode in which $\psi = 0$, the only non-zero electrical components are (x, y, c). Since $a = 0$, the magnetic force at the surfaces is tangential, and the surface condition is merely $y = 0$, or $\dfrac{\partial \phi}{\partial r} = 0$. This leads to Macdonald's equation for the periods. The mode defined by ψ leads to the other set of periods. The case $n = 0$ makes all the components zero, and leads to no vibration.

The most interesting case is that of a single sphere. The most general components in a symmetrical oscillation inside the conductor are, if c is the radius,

$$x = p_2 p_3 \frac{\partial \phi}{\partial \theta} = \frac{\mathrm{A}n(n+1)}{r^{\frac{3}{2}}} \mathrm{J}_{n+\frac{1}{2}}(kr) \cdot \mathrm{P}_n(\mu) \cos (k\mathrm{V}t + \epsilon),$$

$$y = -p_1 p_3 \frac{\partial \phi}{\partial (\log r)} = -\frac{\mathrm{A}}{r} \cdot \frac{d}{dr}\left(r^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(kr) \right) \frac{d\mathrm{P}_n(\mu)}{d\mu} \cos(k\mathrm{V}t + \epsilon),$$

$$c = -\frac{\mathrm{A}k}{\mathrm{V}r^{\frac{1}{2}}} \mathrm{J}_{n+\frac{1}{2}}(kr) \cdot \sin^2 \theta \cdot \frac{d\mathrm{P}_n(\mu)}{d\mu} \sin (k\mathrm{V}t + \epsilon),$$

taken for all roots k of

$$\frac{d}{dc}\left\{ c^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(kc) \right\} = 0, \quad \cdots \quad (15)$$

together with

$$a = \frac{\mathrm{B}n(n+1)}{r^{\frac{3}{2}}} \mathrm{J}_{n+\frac{1}{2}}(kr) \mathrm{P}_n(\mu) \cos (k\mathrm{V}t + \epsilon),$$

$$b = \frac{\mathrm{B}}{r} \frac{d}{dr}\left(r^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(kr) \right) \frac{d\mathrm{P}_n(\mu)}{d\mu} \cdot \cos (k\mathrm{V}t + \epsilon),$$

$$z = \frac{\mathrm{B}k}{\mathrm{V}r^{\frac{1}{2}}} \mathrm{J}_{n+\frac{1}{2}}(kr) \cdot \sin^2 \theta \cdot \frac{d\mathrm{P}_n(\mu)}{d\mu} \cdot \sin (k\mathrm{V}t + \epsilon),$$

taken for all roots k of

$$\mathrm{J}_{n+\frac{1}{2}}(kc) = 0. \quad \cdots \quad (16)$$

The gravest modes of the first and second systems (corresponding to $n = 1$ in each case) are given by the least roots of

$$\left(kc - \frac{1}{kc} \right) \tan kc + 1 = 0, \quad \cdots \quad (17)$$

and

$$\tan kc = kc, \quad \cdots \quad (18)$$

respectively.

Vibrations of Confocal Spheroids.

The proper transformation in the case of spheroids of *ovary* form is

$$z + \iota\rho = c \cosh(\alpha + \iota\beta), \quad \cdots \quad (19)$$

which gives

$$\rho = c \sinh\alpha \sin\beta = \frac{1}{p_3},$$

$$\frac{\partial(\rho, z)}{\partial(\alpha, \beta)} = c^2(\cosh^2\alpha - \cos^2\beta).$$

Thus the characteristic equation (10) becomes

$$\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial\beta^2} - \coth\alpha\frac{\partial\phi}{\partial\beta} - \cot\beta\frac{\partial\phi}{\partial\beta} + k^2c^2(\cosh^2\alpha - \cos^2\beta)\phi = 0.$$

To obtain a solution of the proper form, we may write, if λ is some constant,

$$\frac{\partial^2\phi}{\partial\alpha^2} - \coth\alpha\frac{\partial\phi}{\partial\alpha} + (k^2c^2\cosh^2\alpha - \lambda)\phi = 0, \quad \cdot \quad (20)$$

$$\frac{\partial^2\phi}{\partial\beta^2} - \cot\beta\frac{\partial\phi}{\partial\beta} + (\lambda - k^2c^2\cos^2\beta)\phi = 0. \quad \cdot \quad \cdot \quad (21)$$

In the sphere, $\lambda = n.(n+1)$. Its proper values for the spheroid are to be determined so as to make ϕ recur when β increases by 2π, and are of the form

$$\lambda = n(n+1) + \epsilon_n(kc)^2 + \delta_n(kc)^4 + \ldots$$

where ϵ_n, δ_n, $\ldots$ are functions of n only.

Write $\phi = \sinh\alpha . \sin\beta . A . B$, where A is a function of α only, and B of β only.

We thus obtain

$$\frac{d^2A}{d\alpha^2} + \coth\alpha\frac{dA}{d\alpha} + A(k^2c^2\cosh^2\alpha - \operatorname{cosech}^2\alpha - \lambda) = 0 \cdot (22)$$

$$\frac{d^2B}{d\beta^2} + \cot\beta\frac{dB}{d\beta} + B(\lambda - k^2c^2\cos^2\beta - \operatorname{cosec}^2\beta) = 0 \quad \cdot \quad (23)$$

These equations are included in a form considered by Niven * when discussing the conduction of heat in ellipsoids of revolution. His solution is unsuited to the present problem, but we may quote his value of λ. In the present case, it is found that

$$\lambda = n(n+1) + (kc)^2\frac{2n^2 + 2n - 3}{(2n-1)(2n+3)} - \tfrac{1}{2}(kc)^4\left\{\frac{n(n+1)(n+2)(n+3)}{(2n+1)(2n+3)^3(2n+5)}\right.$$

$$\left. - \frac{(n-2)(n-1)(n)(n+1)}{(2n-3)(2n-1)^3(2n+1)}\right\} + \cdots \quad \cdot \quad (24)$$

* Phil. Trans. 1880, p. 138.

Thus when $n = 0$, $\lambda = (kc)^2$ up to and including the seventh order. This is found to correspond to no vibration.

For the first three types of vibration, given by $n = 1, 2, 3$, the values of λ are

$$\lambda_1 = 2 + \frac{1}{5}(kc)^2 - \frac{4}{7.5^3}(kc)^4 + \frac{8}{21.5^5}(kc)^6 - \ldots \quad (25)$$

$$\lambda_2 = 6 + \frac{3}{7}(kc)^2 - \frac{4}{3.7^3}(kc)^4 + \frac{8}{33.7^5}(kc)^6 - \ldots \quad (26)$$

$$\lambda_3 = 12 + \frac{7}{15}(kc)^2 + \frac{1064}{77.5^3 \cdot 9^2}(kc)^4 - \ldots \quad (27)$$

After the first term, these series converge very rapidly for values of kc which are not very great, and more rapidly as n increases. Now for the fundamental, and for the vibrations near to it in a sphere of radius a, ka is not much greater than 1 or 2, and unless the spheroids differ greatly from the spherical shape, these series may be conveniently used in calculating the gravest vibrations of the doubly infinite system. For $c = ae$, where a is the semi-axis major and e the eccentricity.

Let kc be denoted by ϵ. We may pass from the spheroidal coordinates to spherical polars (r, θ, ω) by making α great and c small, so that $r = \frac{c}{2}e^{\alpha}$, $\beta = \theta$.

It is known that the equation corresponding to (20) when α is infinite and c zero, is satisfied by the form

$$\phi = e^{\iota kr} f_n(\iota kr),$$

where f_n denotes a terminating polynomial in descending powers of ιkr, which is useful for calculation for only moderate values of kr.

Write $\cosh \alpha = t$ in (20), and it becomes

$$(t^2 - 1)\frac{d^2\phi}{dt^2} + (\epsilon^2 t^2 - \lambda)\phi = 0. \quad \ldots \quad (28)$$

If in this transformed equation we write, following the method suggested by the case of the sphere,

$$\phi = e^{\iota \epsilon t}\left(\frac{c_r}{(\iota \epsilon t)^r} + \frac{c_{r+1}}{(\iota \epsilon t)^{r+1}} + \ldots\right)$$

we find that $r = 0$.

Thus
$$\phi = e^{\iota \epsilon t}\left(1 + \frac{c_1}{\iota \epsilon t} + \frac{c^2}{(\iota \epsilon t)^2} + \ldots\right), \quad \ldots \quad (29)$$

where the series will not, in general, terminate. Denoting the series in brackets by S, and substituting in the equation, we obtain, if $z = \iota \epsilon t$,

$$(z^2 + \epsilon^2)\left(\frac{d^2 S}{dz^2} + 2\,\frac{dS}{dz}\right) + (\epsilon^2 - \lambda)\,S = 0.$$

From this equation, the relations among successive coefficients become

$$2(m+1)c_{m+1} = (\epsilon^2 - \lambda + m\,.\,m+1)c_m - 2(m-1)\epsilon^2 c_{m-1}$$
$$+ (m-1)(m-2)\epsilon^2 c_{m-2}\quad .\quad.\quad (30)$$
$$2c_1 = (\epsilon^2 - \lambda),$$
$$4c_2 = (2 + \epsilon^2 - \lambda)c_1,$$
$$6c_3 = (6 + \epsilon^2 - \lambda)c_2 - 2\epsilon^2 c_1.$$

In the case of $n = 0$, $\epsilon^2 = \lambda$ to the seventh order. Thus $c_1 = 0$, and therefore c_2 and successive coefficients are all zero. In fact, the solution in this case is, to this order,

$$\phi = A\cos(kc\cosh\alpha) + B\sin(kc\cosh\alpha)\quad.\quad.\quad (31)$$

so far as α is concerned.

The conjugate equation in β is

$$\frac{d^2\phi}{d\beta^2} - \cot\beta\,\frac{d\phi}{d\beta} + k^2 c^2 \sin^2\beta = 0,$$

whose solution is accurately

$$\phi = C\cos(kc\cos\beta) + D\sin(kc\cos\beta).\quad.\quad.\quad (32)$$

Now this is a function, which, even for great values of kc, is perfectly periodic in β, and is therefore a possible solution for $n = 0$ for *any* value of kc. But there is only one such solution. Thus we infer without further calculation that for $n = 0$, all the terms of Niven's series will vanish except $(kc)^2$, which is the accurate value of λ_0. Although this result is interesting from the point of view of several problems, it is not to be expected, from the analogy of the sphere, that it corresponds to any vibration in this case, and in fact, on writing down the values of the electric and magnetic forces by previous equations, it is found impossible to make them finite on the axis ($\beta = 0$) unless they are everywhere zero. We therefore pass to the case $n = 1$, which leads to the

true fundamental series of periods. When $n=1$, $\lambda=2+\dfrac{\epsilon^2}{5}$ to the second order, by (25),

$$c_1 = \left(\frac{2}{5}\epsilon^2 - 1\right),$$

$$c^2 = \frac{\epsilon^2}{5}\left(\frac{2}{5}\epsilon^2 - 1\right),$$

$$c_3 = \frac{\epsilon^2}{5} \quad \text{to the same order,}$$

and c_4 and all higher coefficients commence with ϵ^4.

Thus a solution for $n=1$, with small eccentricity, is

$$\phi_1 = e^{\iota k c \cosh a}\left\{1 - \frac{5-2\epsilon^2}{5\iota\epsilon\cosh a} - \frac{\epsilon^2}{5(\iota\epsilon\cosh a)^2} + \frac{\epsilon^2}{5(\iota\epsilon\cosh a)^3}\right\} \quad (33)$$

Another solution ϕ_1' is obtained by changing the sign of ι.

When $n=2$, $\quad \lambda=6+\dfrac{3}{7}\epsilon^2$, by (26),

$$c_1 = \frac{2}{7}\epsilon^2 - 3, \quad c_2 = 3 - \frac{5}{7}\epsilon^2, \quad c_3 = \frac{9}{7}\epsilon^2, \quad c_4 = -\frac{9}{7}\epsilon^2,$$

and the higher coefficients commence with ϵ^4.

Thus

$$\phi_2 = e^{\iota k c \cosh a}\left\{1 - \frac{21-2\epsilon^2}{7\iota\epsilon\cosh a} + \frac{21-5\epsilon^2}{7(\iota\epsilon\cosh a)^2}\right.$$
$$\left. + \frac{9}{7}\cdot\frac{\epsilon^2}{(\iota\epsilon\cosh a)^3} - \frac{9}{7}\cdot\frac{\epsilon^2}{(\iota\epsilon\cosh a)^4}\right\} \quad (34)$$

and a solution ϕ_2' is obtained by changing the sign of ι.

Similarly,

$$\phi_3 = e^{\iota k c \cosh a}\left\{1 - \frac{90-4\epsilon^2}{15\iota\epsilon\cosh a} + \frac{225-22\epsilon^2}{15(\iota\epsilon\cosh a)^2} - \frac{75-24\epsilon^2}{5(\iota\epsilon\cosh a)^3}\right.$$
$$\left. - \frac{10\epsilon^2}{(\iota\epsilon\cosh a)^4} + \frac{10\epsilon^2}{(\iota\epsilon\cosh a)^5}\right\} \quad (35)$$

Each succeeding function necessitates the addition of a new term. For vibrations between two confocal spheroids, a linear combination of any pair (ϕ, ϕ') must be taken. Inside a single spheroid, we require a function finite at the centre. By comparison with the Bessel functions, it appears that if $n=1$, 3, 5,... the real part of ϕ must be taken, and if $n=2$, 4, 6, ..., the imaginary part. Thus the first three

appropriate functions for internal vibrations are

$$\phi_1 = \left(1 + \frac{1}{5}\operatorname{sech}^2\alpha\right)\cos(kc\cosh\alpha)$$

$$- \left\{1 - \frac{2}{5}k^2c^2 + \frac{1}{5}\operatorname{sech}^2\alpha\right\}\frac{\sin(kc\cosh\alpha)}{kc\cosh\alpha}, \quad (36)$$

$$\phi_2 = \left(3 - \frac{2}{7}k^2c^2 + \frac{9}{7}\operatorname{sech}^2\alpha\right)\frac{\cos(kc\cosh\alpha)}{kc\cosh\alpha}$$

$$+ \left\{1 - \frac{21 - 5k^2c^2}{7k^2c^2\cosh^2\alpha} - \frac{9}{7}\frac{\operatorname{sech}^4\alpha}{k^2c^2}\right\}\sin(kc\cosh\alpha), \quad (37)$$

$$\phi_3 = \left\{1 - \frac{75 - 22k^2c^2}{15(kc\cosh\alpha)^2} - \frac{10}{k^2c^2}\operatorname{sech}^4\alpha\right\}\cos(kc\cosh\alpha)$$

$$- \left\{6 - \frac{4}{15}(kc)^2 - \frac{75 - 24k^2c^2}{(5kc\cosh\alpha)^2} - \frac{10\operatorname{sech}^3\alpha}{k^2c^2}\right\}\frac{\sin(kc\cosh\alpha)}{kc\cosh\alpha}. \quad (38)$$

In finding periods corresponding to any value of n, the equation in β need not be solved, as it is known from the case of the sphere that the forces will be finite when n is not zero. We shall only examine the periods corresponding to $n = 1$.

If a is the major semi-axis, and e the eccentricity, $c = ae$, $e = \operatorname{sech}\alpha$, where α relates to the boundary. The periods belonging to case 2 correspond to the surface condition $\phi_1 = 0$, when $n = 0$.

Thus if $\sigma = ka$, the period equation becomes

$$\frac{\tan\sigma}{\sigma} = 1 + \frac{2}{5}\sigma^2e^2 : \quad \cdots \quad (39)$$

the corresponding equation for the sphere being $\tan\sigma = \sigma$.

If δ be the positive correction for eccentricity of any root θ of $\tan\sigma = \sigma$,

$$\delta = \frac{2}{5}e^2\theta^2 \Big/ \frac{d}{d\theta}\left(\frac{\tan\theta}{\theta}\right)$$

$$= \frac{2}{5}e^2\theta, \quad \cdots \quad \cdots \quad (40)$$

writing $\tan\theta = \theta$, $\sec^2\theta = 1 + \theta^2$.

The effect of a small eccentricity is therefore to increase ka in the ratio $1 + \frac{2}{5}e^2$, if a sphere become a spheroid with the same major axis. The wave-lengths are less in the spheroid in the ratio $1 - \frac{2}{5}e^2$, and the correction is therefore

very small. An eccentricity of $\frac{1}{2}$ reduces the wave-lengths in the ratio $\frac{9}{10}$ approximately.

In case 1, $n=1$, the period equation is $\dfrac{\partial \phi}{\partial \alpha} = 0$.

This equation supplies the true fundamental of the system. With the previous notation, after some reduction, it becomes,

$$\left(\frac{1}{\sigma} - \sigma\right) \tan \sigma = 1 + \frac{2e^2}{5}\left(\frac{\sigma^4 - 3}{\sigma^2 - 1}\right). \quad . \quad . \quad (41)$$

Let θ be a root of $\tan \sigma = \dfrac{\sigma}{1 - \sigma^2}$, the period equation for the sphere, and δ be the correction for eccentricity. Then it may be at once shown that

$$\delta = \frac{2}{5}\, e^2 \cdot \frac{\theta^4 - 3}{\theta^2 - 1} \bigg/ \frac{d}{d\theta}\left\{\frac{\tan \theta}{\theta} - \theta \tan \theta\right\}$$

$$= -\frac{2}{5}\, e^2 \cdot \frac{\theta^4 - 3}{\theta(\theta^2 - 2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

For the higher roots, this becomes $\delta = -\dfrac{2}{5}\, e^2 \theta$. The correction is therefore equal and opposite to that in case 2, if the vibrations are not too near the fundamental. The fundamental of the sphere corresponds to a value of θ approximately equal to $2 \cdot 744$ or $7\pi/8$.

The Oblate or Planetary Spheroid.

The proper substitution in this case is

$$\rho + \iota z = c \cosh (\alpha + \iota \beta), \quad . \quad . \quad . \quad (43)$$

giving $\qquad \rho = c \cosh \alpha \cos \beta,$

$$\frac{\partial (\rho, z)}{\partial (\alpha, \beta)} = c^2 (\cosh^2 \alpha - \cos^2 \beta);$$

and the equation for ϕ becomes

$$\frac{\partial^2 \phi}{\partial \alpha^2} + \frac{\partial^2 \phi}{\partial \beta^2} - \tanh \alpha \frac{\partial \phi}{\partial \alpha} + \tan \beta \frac{\partial \phi}{\partial \beta} + \epsilon^2 (\sinh^2 \alpha + \sin^2 \beta)\phi = 0. \quad . \quad (44)$$

In this, write $\left(-\epsilon^2,\ \alpha - \dfrac{\iota \pi}{2},\ \beta - \dfrac{\pi}{2}\right)$ for $(\epsilon^2,\ \alpha,\ \beta)$ respectively, and it becomes identical with (20). Thus to apply the previous results to an oblate spheroid, it is only necessary to make these three substitutions in them. In previous investigations, $\cosh \alpha$ becomes $\iota \sinh \alpha$; $\sinh \alpha$, $\iota \cosh \alpha$; $\cos \beta$, $-\sin \beta$; $\sin \beta$, $\cos \beta$; and $kc \cosh \alpha$, $kc \sinh \alpha$. The

major axis of the previous spheroid is to be replaced by the minor axis of the new one. The sign of e^2 is to be changed wherever it occurs. Thus the corrections of the fundamental series of periods for eccentricity are equal and opposite to those of the other type of spheroids. The detailed theory for the oblate spheroids will not be given, as the previous formulæ may be at once transformed properly.

The Higher Periods of Spheres and Spheroids.

In cases 1 and 2 of the sphere, when n is small, and the higher periods are being treated, the period equations become, adopting the ordinary asymptotic expansion of Bessel functions,

$$\left. \begin{matrix} \cos \\ \sin \end{matrix} \right\} \left(ka - (n+1) \frac{\pi}{2} \right) = 0. \quad \ldots \quad (45)$$

Thus if m be a large integer, a possible wave-length for a sphere of radius a is $\dfrac{4a}{m}$ approximately.

When n is appreciable, the ordinary expansion of Bessel functions ceases to apply. But it was shown by the author * that the proper expansion in this case is

$$J_{n+\frac{1}{2}}(kr) = \sqrt{\frac{2}{\pi}} \frac{\cos\left\{ \sqrt{k^2 r^2 - n \cdot n + 1} - \overline{n+\tfrac{1}{2}}\cos^{-1}\dfrac{(n \cdot n + 1)^{\frac{1}{2}}}{kr} - \dfrac{\pi}{4} \right\}}{(k^2 r^2 - n \cdot n + 1)^{\frac{1}{4}}}, \quad (46)$$

and the two period equations become

$$\sqrt{k^2 a^2 - n \cdot n + 1} - (n+\tfrac{1}{2}) \cos^{-1}\sqrt{\dfrac{n \cdot n + 1}{k^2 c^2}} = (4s+3) \frac{\pi}{4} \quad (s \text{ large}) \quad . \quad (47)$$

and

$$\tan\left\{ \sqrt{k^2 a^2 - n \cdot n + 1} - (n+\tfrac{1}{2}) \cos^{-1}\sqrt{\dfrac{n \cdot n + 1}{k^2 a^2}} - \frac{\pi}{4} \right\}$$

$$= -\frac{n \cdot n + 1}{2 (ka)^{\frac{1}{2}}(k^2 a^2 - n \cdot n + 1)^{\frac{7}{4}}}. \quad . \quad (48)$$

These two formulæ will supply the correction necessary to the periods deduced from the ordinary expansion, when n becomes appreciable, if expanded in powers of $\dfrac{n}{ka}$.

However great n may be, these equations will give the higher modes of the series defined by n, and will do so more accurately as n increases, if the higher roots are taken. But

* Phil. Mag. Feb. 1906, p. 195.

these high frequencies are less interesting physically. The fundamentals of series defined by large values of n are themselves high modes, in comparison with *the* fundamental of the system.

We now proceed to treat the higher vibrations of spheroids corresponding to any values of n. The equation in β is

$$\frac{d^2\phi}{d\beta^2} - \cot\beta\frac{d\phi}{d\beta} + (\lambda - k^2c^2\cos^2\beta)\phi = 0.$$

Adopting the substitution *

$$\phi = e^{\iota kc\omega} \cdot \psi, \quad \ldots \quad \ldots \quad (49)$$

we obtain

$$(\psi'' - \cot\beta\psi') - \iota kc(2\omega'\psi' + \psi\omega'' - \omega'\psi\cot\beta)$$
$$+ \psi k^2c^2\left(\frac{\lambda}{\epsilon^2} - \cos^2\beta - \omega'^2\right) = 0.$$

Neglecting the first bracket, and equating the brackets of different orders in kc to zero,

$$\omega' = \sqrt{\frac{\lambda}{\epsilon^2} - \cos^2\beta},$$

$$\frac{2\psi'}{\psi} + \frac{\omega''}{\omega'} - \cot\beta = 0.$$

Thus if $\omega = \displaystyle\int^\beta \sqrt{\frac{\lambda}{\epsilon^2} - \cos^2\beta}\, d\beta$, the solution becomes

$$\phi = \frac{A\cos kc\omega + B\sin kc\omega}{\left(\dfrac{\lambda}{\epsilon^2} - \cos^2\beta\right)^{\frac{1}{4}}} \cdot \sqrt{\sin\beta}, \quad . \quad . \quad (50)$$

except close to the axis of the spheroids.

From this we may obtain the limiting value of λ when n, although possibly large, is small in comparison with ϵ or kc. The denominator of ϕ as written has a period 2π, and therefore ϕ has, if ω has. This occurs if $\lambda = \epsilon^2$, if the solution remains finite and uniform at $\beta = 0$. Now writing

$$\lambda = \epsilon^2 + n \cdot n + 1,$$

and putting $\cos\beta = \mu$, we obtain

$$(1 - \mu^2)\frac{d^2\phi}{d\mu^2} + (n \cdot n + 1 + \epsilon^2 - \epsilon^2\mu^2)\phi = 0, . \quad . \quad (51)$$

This is satisfied by a convergent series in rising powers of μ, even in μ if n is odd, and odd if n is even. By putting $\mu = \pm 1 - \eta$, the solution is finite at $\mu = \pm 1$, and therefore on the axis of the spheroids. Thus a solution of the required

* Webb, Proc. Roy. Soc. 1904, p. 315.

form exists. The asymptotic expansion therefore represents (except on the axis) a function finite at all points, if it satisfies the odd and even conditions.

Thus B$=0$ if n is odd, and A$=0$ if n is even. The continuation of the function on the axis is not given by the expansion.

When kc is great in comparison with n, λ therefore becomes $(kc)^2$, and the equation in α for ϕ then has the simple solution given by (31). Thus

$$\phi = \mathrm{C} \cos{(kc \cosh \alpha)} + \mathrm{D} \sin{(kc \cosh \alpha)} \quad . \quad . \quad (52)$$

This is, of course, another asymptotic expansion, suitable for the space between two spheroids, and representing the proper function except at the origin, where its continuation must be differently expressed. D$=0$ if n is odd, and C$=0$ if n is even, if we are treating the interior of a spheroid. When the eccentricity is very small, the β factor becomes incorrect, and must be replaced by $(1-\mu^2)\dfrac{d\mathrm{P}_n(\mu)}{d\mu}$ where $\mu = \cos \beta$. But the α factor remains correct, for $c \cosh \alpha$ becomes r when the eccentricity is small, and the α factor becomes the ordinary expansion of

$$r^{\frac{1}{2}}\{\mathrm{J}_{n+\frac{1}{2}}(kr) + \nu \mathrm{J}_{-n-\frac{1}{2}}(kr)\},$$

which is the true factor for a sphere (*cf. supra*).

If n is retained, the more accurate expression for ϕ becomes

$$\phi = \frac{\mathrm{A}\sqrt{\sinh \alpha \sin \beta} \cos{(\epsilon\omega' - \epsilon_1)} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (\epsilon\omega - \epsilon_2)}{(\epsilon^2 \sinh^2 \alpha - n \cdot n + 1)^{\frac{1}{4}}(n \cdot n + 1 + \epsilon^2 \sin^2 \beta)^{\frac{1}{4}}} \cdot \cos k\mathrm{V}t, \quad (53)$$

where ϵ_1, ϵ_2 are arbitrary constants, and

$$\epsilon\omega' = \int^{\alpha}(\epsilon^2 \sinh^2 \alpha - n \cdot n + 1)^{\frac{1}{2}} d\dot{\alpha}, \quad . \quad . \quad . \quad (54)$$

$$\epsilon\omega = \int^{\beta}(n \cdot n + 1 + \epsilon^2 \sin^2 \beta)^{\frac{1}{2}} d\beta, \quad . \quad . \quad . \quad (55)$$

This is proved for the α factor just as in (50).

If n be neglected, the high periods between two spheroids (α_1, α_2) in Case 2 are given by

$$\mathrm{A} \cos{(kc \cosh \alpha_{1,\,2})} + \beta \sin{(kc \cosh \alpha_{1,\,2})} = 0,$$

whence

$$k(a_1 - a_2) = s\pi.$$

In Case 1,

$$k(a_1 - a_2) = (2s+1)\frac{\pi}{2}, \quad . \quad . \quad . \quad . \quad (56)$$

where a_1, a_2 are the major axes. These are replaced by the minor axes in the case of oblate spheroids.

The high periods of a single spheroid are given by

$$(1) \quad \left.\begin{matrix}\cos\\\sin\end{matrix}\right\} \int^{\alpha} (k^2 c^2 \cosh^2 \alpha - n \cdot n+1)^{\frac{1}{2}} d\alpha = 0, \quad . \quad (57)$$

$$(2) \quad \frac{d}{d\alpha} \left\{ \frac{\sqrt{\sinh \alpha}}{(k^2 c^2 \cosh^2 \alpha - n \cdot n+1)^{\frac{1}{4}}} \right.$$

$$\times \left.\begin{matrix}\cos\\\sin\end{matrix}\right\} \int^{\alpha} (k^2 c^2 \cosh^2 \alpha - n \cdot n+1)^{\frac{1}{2}} d\alpha \left. \right\} = 0, \quad . \quad (58)$$

where the upper or lower functions are taken according as n is odd or even. These give the high vibrations of the singly infinite system defined by n.

The investigation given in a previous paper * for the elliptic cylinders becomes, if stated in full detail, exactly similar to the present. The β factor is restricted in a similar way, but its adjustment for uniformity on the axis is incorrectly given. The true value, in the notation there employed, oscillates between $\mu^{-\frac{1}{4}} \cosh k \sqrt{b^2 + \mu}$ and $\mu^{-\frac{1}{4}} \sinh k \sqrt{b^2 + \mu}$ according to the value of n. The equation there numbered (51) ought to be written

$$\frac{d}{d\lambda} \left\{ \frac{\left.\begin{matrix}\cos\\\sin\end{matrix}\right\} \left(k\sqrt{b^2 + \lambda} - \frac{\pi}{4} \right)}{\lambda^{\frac{1}{4}}} \right\} = 0, \quad . \quad . \quad (59)$$

according to the value of n. The $\frac{\pi}{4}$ is determined most readily from the corresponding Bessel function.

Series of Periods corresponding to High Values of n.

The solution may be expressed approximately in the general case when n is large, even though kc may not be so large that the value of λ is $(kc)^2$. Writing $\lambda = n \cdot n+1 + \lambda'$, or, when n is large, $\lambda = n^2 + \lambda'$, the asymptotic expansion of the equations for ϕ when n is large, found just as in the case when kc is large, lead to

$$\phi = \frac{A \cos(\omega_1 + \epsilon_1)\cos(\omega_2 + \epsilon_2)\cos kVt}{(\omega_1' \omega_2')^{\frac{1}{2}}} \sqrt{\sinh \alpha \sin \beta} \quad . \quad (60)$$

where

$$\omega_1' = \frac{d\omega_1}{d\alpha} = (\epsilon^2 \cosh^2 \alpha - \lambda' - n^2)^{\frac{1}{2}}, \quad . \quad . \quad (61)$$

$$\omega_2' = \frac{d\omega_2}{d\alpha} = (n^2 + \lambda' - \epsilon^2 \cos^2 \beta)^{\frac{1}{2}}, \quad . \quad . \quad (62)$$

* *Loc. cit.*

and the value of ϕ may be expressed in terms of elliptic functions, as was previously done for the elliptic cylinder (*loc. cit.*). It may be noted that ω_1 and ω_2 are both real for all roots ϵ. This formula holds for all values of β if n is not very small compared with kc, and does not fail on the axis.

It is convenient to note here that in the case of the elliptic cylinder corresponding to the above, n^2 should be $-n^2$, and the quantity θ there employed must be retained in full as a series in powers of $(kc)^2$. The series in two dimensions, given by Mathieu in connexion with the vibrations of an elliptic membrane, is,

$$\theta = n^2 + \frac{(kb)^4}{2(n^2-1)} + \frac{(kb)^8 \cdot 5n^2 + 7}{32(n^2-1)^3(n^2-4)} + \ldots, \quad (63)$$

corresponding to

$$\frac{d^2M}{d\eta^2} + (\theta - 2k^2b^2\cos 2\eta)M = 0. \quad \ldots \quad (64)$$

The period equations for the vibrations between two spheroids, corresponding to large values of n, may be written down at once from (60).

Selecting the function corresponding to the internal Bessel function of the sphere, by writing $c = \lambda' = 0$, it appears that ϵ_1 in (60) is $\frac{\pi}{4}$ for the single spheroid. Thus for the higher series, the period equations for a single spheroid are

$$\int_0^a (\epsilon^2 \cosh^2 \alpha - \lambda' - n^2)^{\frac{1}{2}} = (4s+1)\frac{\pi}{4}, \quad \ldots \quad (65)$$

and

$$\frac{d}{d\alpha}\left\{ \left(\frac{\sinh\alpha}{\omega_1'}\right)^{\frac{1}{2}} \cos\left(\omega_1 - \frac{\pi}{4}\right) \right\} = 0. \quad \ldots \quad (66)$$

In the case of the elliptic cylinder, there is only one such equation, namely

$$\frac{d}{d\alpha} \cdot \frac{\cos\left(\omega_1 - \frac{\pi}{4}\right)}{(\omega_1')^{\frac{1}{2}}} = 0, \quad \ldots \quad \ldots \quad (67)$$

the value of λ' being given by (63) above.

Vibrations in the External Space.

The vibrations in the space external to a single conductor have, in general, a large modulus of decay, and cannot be maintained for a great space of time. The form of ϕ suitable for the space, employing imaginary quantities, must

contain an exponential factor of negative imaginary argument. By this consideration, all previous formulæ for periods may be at once transformed into those proper to the external space.

In the case of the sphere, when $n=1$, the proper form of ϕ is, so far as concerns r,

$$\phi = A e^{-\iota k r}\left(1 + \frac{1}{\iota k r}\right). \quad . \quad . \quad . \quad . \quad (68)$$

In the second type of vibration, $\phi = 0$ when $r = a$.

Thus
$$1 + \frac{1}{\iota k a} = 0,$$

and
$$e^{\iota k V t} = e^{-\frac{V t}{a}};$$

so that there is really no vibration at all, but a simple decay at a very rapid rate, on account of the large value of V.

In the first type, $\dfrac{\partial \phi}{\partial r} = 0$, when $r = a$, leading to

$$1 + \iota k - k^2 a^2 = 0,$$

or
$$k a = \tfrac{1}{2}(\iota + \sqrt{3}) \quad . \quad . \quad . \quad . \quad . \quad (69)$$

The time factor becomes

$$e^{-\frac{V t}{2a}} \cdot e^{\iota \sqrt{3}\frac{V t}{2a}},$$

and therefore the real value of ka is $\dfrac{\sqrt{3}}{2}$, but there is, in addition, a rapid decay of the vectors at half the rate of the previous type. In the case of the prolate spheroid, when $n=1$, by changing the sign of ι in (33),

$$\phi = e^{-\iota \epsilon \cosh \alpha}\left\{1 + \frac{1 - \frac{2}{5}\epsilon^2}{\iota \epsilon \cosh \alpha} - \frac{\epsilon^2}{5(\iota \epsilon \cosh \alpha)^2} - \frac{\epsilon^2}{5(\iota \epsilon \cosh \alpha)^3}\right\} (70)$$

Taking the second type of vibration, corresponding to $\phi = 0$ at $c \cosh \alpha = a$, where a is the major semiaxis, and putting $\cosh \alpha = \dfrac{1}{e}, \quad ka = \sigma$,

$$1 + \frac{1 - \frac{2}{5}\sigma^2 e^2}{\iota \sigma} + \frac{\sigma^2 e^2}{5\sigma^2} + \frac{\sigma^2 e^2}{5\iota \sigma^3} = 0,$$

or neglecting e^4,

$$1 + \iota \sigma - \frac{2}{5}\sigma^2 e^2 = 0.$$

If $\sigma = \iota + \delta$, we find

$$\delta = \frac{2}{5} e^2 \iota \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

The vectors therefore decay more rapidly than in the case of the sphere, the rate being increased in the ratio $\left(1 + \frac{2}{5} e^2\right)$. For the other type, the period equation reduces to

$$\iota \left(\frac{1 - \sigma^2}{\sigma}\right) = 1 + \frac{2}{5} e^2 \left(\frac{\sigma^4 - 3}{\sigma^2 - 1}\right).$$

If

$$\sigma = \frac{1}{2} (\iota + \sqrt{3}) + \delta,$$

$$\delta = (\iota \sqrt{3} - 7) \frac{e^2}{5 \sqrt{3}} \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Thus the new rate of decay is $\dfrac{V}{a} \left(\dfrac{1}{2} + \dfrac{e^2}{5}\right)$, and the decay is again more rapid than in the case of the sphere. The real value of ka becomes $\dfrac{\sqrt{3}}{2} - \dfrac{7 e^2}{5 \sqrt{3}}$, and is decreased. For the oblate spheroid, the sign of e^2 must be altered. A particular case of the oblate spheroid is the circular disk, which is worthy of separate treatment.

All formulæ above for the high periods may be transformed as shown, so as to be suitable for the external space.

Vibrations of other Surfaces of Revolution.

The cone is best treated by spherical polar coordinates. The transformation appropriate to the paraboloid is

$$\rho + \iota z = (\alpha + \iota \beta)^2. \quad . \quad . \quad . \quad . \quad (73)$$

In two dimensions, this will solve the problem of the parabolic cylinder. The appropriate functions for this case have been treated by Whittaker *.

Hyperboloids of revolution form the conjugate case to spheroids, but are of little physical interest. The hyperbolic cylinder may be solved in the same manner.

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* Proc. Lond. Math. Soc. 1902.

LXIV. *On some Properties of the α Rays emitted by Radium and by Bodies rendered Active by the Radium Emanation.* *By* H. Becquerel [*].

SOME months ago [†], in consequence of an investigation of Rutherford's relating to the retardation suffered by the α rays in passing through thin aluminium sheets, I had resumed some old experiments on the rays of radium. I had arranged, in particular, a differential experiment which enabled me to receive, on the same photographic plate, the two portions of a pencil of α rays proceeding from a linear source, passing through a slit parallel to the source and plate, and deviated by a magnetic field—a pencil one half of which had traversed nothing but air, while the other had, in addition, passed through one or more thin sheets of aluminium. Under these conditions, the two parallel bands which are the traces of the deflected pencil corresponding to the two directions of the magnetic field should be more widely separated in the portion of the pencil which has traversed the aluminium than in the other. The first experiments made with the α rays of radium did not exhibit the deflexion expected.

Immediately on my return to Paris last October, I resumed these experiments, using as sources either salts of radium or wires excited by the radium emanation, which M. Curie was kind enough to render active in his laboratory. The results obtained were in accordance with Rutherford's statement (fig. 1).

Fig. 1.

–Aluminium Sheet.

–Air only.

D. 19/10/1905.

The following are, for instance, the means of the results of several experiments :—

			Double deflexion of pencil		Ratio.
No. of experiment.	Nature of source.	Field Intensity.	in air.	after passing through a sheet of aluminium $e = 0.015$ mm.	
D. 19/10/1905 ...	Excited wire.	9659	2·360 mm.	2·658 mm.	1·122
A. 21/10/1905 ...	Radium salt.	9659	2·412 ,,	2·671 ,,	1·107
A. 24/10/1905 ...	,, ,,	9384	2·222 ,,	2·423 ,,	1·090

The above tests were made with the same apparatus (distance *a* from source to slit, 2·145 cm.; distance *b* from

* Communicated by the Author; from the *Comptes Rendus* of February 12, 1906.

† *Comptes Rendus,* t. cxli. 11th September, 1905, p. 485.

slit to plate, 1·94 cm.; total distance $a + b = 4·085$ cm.). The magnetic field did not vary by more than 0·3 to 0·5 per cent. in the course of an experiment.

Since the time when I carried out these experiments, Rutherford has published similar ones [*], and has besides proved the important fact that the separation of the bands is greater in air than *in vacuo*.

These results might be explained by supposing that the greater deflectibility of the pencil corresponds to an increased curvature of the trajectory—such increase being due to a decrease in the velocity of the particles carrying positive electric charges, which constitute the α rays. This decrease of velocity should, moreover, take place progressively along the air path.

I proceeded to study with greater precision than heretofore the air trajectory of the α rays when deflected by a magnetic field, employing a photographic method which I had already used for a number of years.

The method consists in receiving a pencil, defined by a linear source and a slit parallel to it and at a distance a from it, on an inclined photographic plate resting against the screen containing the slit and normal to the plane of the undeflected pencil. The trace of this pencil will be a straight line ; but if it is deviated in a direction parallel to the plate, first one way and then the other, the trace consists of two divergent curves which intersect each other at the point of contact of the slit with the plate. The distance of a point in the plate from its lower edge is proportional to the height y of this point above the horizontal plane against which the plate rests, and if the inclination of the plate be known, this height may be determined. Further, by measuring the distance apart $2x$ of two points in the two curves having the same y, the required trajectory may be constructed point by point.

The greater part of the new tests were carried out by using as source a platinum wire 0·1 to 0·2 mm. in diameter, uniformly excited by the radium emanation. The method of procedure generally adopted was as follows :—Owing to the rapid decrease in the intensity of the source, the magnetic field was reversed every five minutes in order to equalize the impressions of the two deflected pencils ; the source was first screened by an aluminium sheet 0·015 mm. thick, and then, at the end of about half an hour, the aluminium sheet was removed, and the photographic plate was displaced laterally so as to obtain, side by side with the first curves and

[*] Phil. Mag. January 1906, p. 166.

for the same magnetic field, the trajectories of the rays which had merely traversed an air space (fig. 2).

Fig. 2.
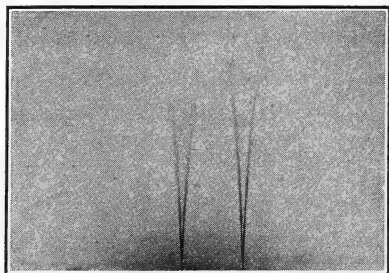
Aluminium Sheet.   Air only.
A. 17/11/1905.

It was directly found that the rays which have traversed aluminium give a shorter image than the others.

Numerous points, whose positions were determined with a micrometric apparatus reading to $\frac{1}{600}$ mm., have given the mean results shown below.   Values of the distance apart $2x$ of the divergent curves on the plates were obtained for distances from the lower edge corresponding to integral numbers n of turns of the screw used in measuring the vertical coordinates, the pitch of the screw being $0·94684$ mm.   The corresponding heights y were calculated from the inclination of the plate in each experiment.

If the results of the observations are plotted on a large scale, a sinuous curve is obtained whose mean position gives the true trajectory.   Theoretically, in a uniform magnetic field and *in vacuo,* this trajectory should be circular.   I had previously realized that, for increasing values of y, the circles normal to the field which pass through the source, the upper portion of the slit and a point of the trajectory, have progressively increasing radii, and the actual experiments confirm this result.   But, as will be seen, this conclusion does not apply to the radius of curvature of the trajectory, and is due to an inexact interpretation founded on imperfect knowledge of the actual distance of the slit from different points of the photographic plate.

If the origin of coordinates is taken at the intersection of the edge of the plate with the bisector of the two pencils, it is observed that the curves do not pass through this origin, but cut the y-axis below the edge of the plate.   Different trials have shown that it is possible to draw through the mean position of the points representing the observations a

parabolic arc whose radius of curvature varies sufficiently slowly within the given limits to render it indistinguishable from a circle.

Let $y = kn$ be the height of a point above the plane against which rests the photographic plate, $k\epsilon$ the distance of the slit below this plane, $kn_0 = a$ the distance of the source below the same plane. Then the equation of a parabola whose axis is parallel to the x-axis, and which passes through the source, the slit, and the point (x, y), may be written

$$2\mathrm{R}x = (y + a)(y + k\epsilon) \;*$$

or

$$2\mathrm{R}x = k^2(n + n_0)(n + \epsilon).$$

The values of $z = \dfrac{2x}{n + n_0}$, which are nearly coincident with the straight line $z = (n + \epsilon)\delta$, were calculated in $\frac{1}{600}$ths of a mm., so that

$$\mathrm{R} = \frac{k^2}{\delta} \times 600,$$

a value which is practically identical with the radius of curvature of the parabola

$$\rho = \mathrm{R}\left[1 + \frac{1}{\mathrm{R}^2}\left(y + \frac{a + k\epsilon}{2}\right)^2\right]^{\frac{3}{2}}.$$

The following tables show the extent of the agreement between the observations and the mode of interpretation explained above.

Experiment C. 19/10/1905.—$a = 25 \cdot 15$ mm. ; $k = 0 \cdot 920$; H = 10,809. Trajectory through air only.

$n=$	1	2	3	4	5	6	7	8	9	10	11
$\delta = 1\cdot5546$; $2x$ (obs.)=	57·5	105·2	157·4	210·2	272·5	333·0	391·4	458·5	529·8	602·7	681·3
$\epsilon\delta = 0\cdot43$ { observed=	2·01	3·55	5·14	6·65	8·35	9·91	11·31	12·87	14·47	16·02	17·64
$(n+\epsilon)\delta$ { calculated=	1·98	3·54	5·09	6·65	8·20	9·76	11·31	12·87	14·42	15·97	17·53

$n=$	12	13	14	15	16	17	18	19	20
$2x$ (obs.) =	755·0	829·6	925·0	1018·7	1096·2	1193·0	1297·0	1397	1493·7
$(n+\epsilon)\delta$ { observed=	19·06	20·42	22·23	23·90	25·11	26·74	28·43	29·97	31·39
{ calculated=	19·08	20·64	22·19	23·75	25·30	26·86	28·41	29·97	31·52

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* In this form will be recognized the principal term of the expression which gives the radius of a circle passing through the three points,

$$\mathrm{R}^2 = \left[\frac{(y+a)(y+k\epsilon)}{2x} + \frac{x}{2}\right]^2 + \frac{(a - k\epsilon)^2}{4}.$$

Experiment A. 17/11/1905.—$a = 11\cdot65$ mm. ; $k = 0\cdot937$;
$$H = 12{,}148.$$

		In Air only.			Through Aluminium.	
	$\delta = 1\cdot7248$	$\epsilon\delta = 1\cdot11$ $(n+\epsilon)\delta$		$\delta = 1\cdot9334$	$\epsilon\delta = 1\cdot39$ $(n+\epsilon)\delta$	
$n.$	$2x$ (observed).	observed.	calculated.	$2x$ (observed).	observed.	calculated.
3......	92·4	6·26	6·28	143·9	9·13	9·13
4......	134·1	8·50	8·01	188·1	11·22	11·06
5......	166·3	9·92	9·73	233·2	13·13	12·99
6......	205·9	11·59	11·46	285·1	15·19	14·93
7......	252·1	13·44	13·18	333·2	16·86	16·86
8......	294·6	14·91	14·91	392·5	18·90	18·79
9......	343·8	16·55	16·63	451·4	20·74	20·73
10......	395·2	18·16	18·35	515·7	22·65	22·66
11......	456·7	20·06	20·08	584·1	24·56	24·59
12......	519·2	21 85	21·81	660·4	26·66	26·53
13......	576·4	23·27	23·53	724·1	28·10	28·46
14......	644·3	25·01	25·25	790·0	29·52	30·39
15......	722·2	26·98	26·98	888·7	32·01	32·32
16......	791·3	28·50	28·70	985·2	34·26	34·26
17......	872·6	30·34	30·43			
18......	950·8	31·94	32·15			
19......	1034·5	33·62	33·88			
20......	1131·3	35·61	35·60			
21......	1233·3	37·63	37·33			
22......	1318·7	39·05	39·05			
23......	1437·0	41·24	40·78			

Experiment A. 16/11/1905.—$a = 25\cdot15$ mm.; $k = 0\cdot920$;
$$H = 12{,}148.$$

		In Air only.			Through Aluminium.	
	$\delta = 1\cdot7318$	$\epsilon\delta = 0\cdot70$ $(n+\epsilon)\delta$		$\delta = 1\cdot9037$	$\epsilon\delta = 0\cdot84$ $(n+\epsilon)\delta$	
$n.$	$2x$ (observed.)	observed.	calculated.	$2x$ observed.	observed.	calculated.
1......	72·9	2·55	2·43	81·0	2·83	2·75
2......	118·3	3·99	4·17	137·8	4·65	4·65
3......	182·0	5·94	5·90	204·3	6·67	6·55
4......	233·9	7·40	7·63	268·9	8·50	8·46
5......	305·4	9·36	9·36	336·1	10·30	10·36
6......	371·8	11·06	11·09	412·4	12·26	12·26
7......	440·3	12·75	12·83	486·5	14 05	14·17
8......	516·6	14·50	14·56	551·7	15·49	16·07
9......	598·4	16·34	16 29			
10......	678·0	18·02	18·02			
11......	769·8	19·93	19·76			
12......	842·7	21·27	21·49			

Experiment A. 18/6/1903.—$a = 20\cdot4$ mm.; $k = 0\cdot983$;
H = 9955. Air only.

$n =$	2	3	4	5	6	7	8	9	10	11
$\delta = 1\cdot6808$; $2x$ (obs.)=	101·0	141·2	193·1	245·7	291·9	357·0	417·6	483·2	550·2	624·5
$\epsilon\delta = 1\cdot07$ { observed=	4·44	5·94	7·80	9·54	10·91	12·86	14·52	16·24	17·88	19·67
$(n+\epsilon)\delta$ { calculated=	4·44	6·12	7·80	9·48	11·16	12·84	14·52	16·20	17·88	19·57

It will be noticed that within the limits of each table, *i. e.* within a length of about 2 cm., the radius of curvature varies but little, or at least that the variation, if it exists, is within the limits of experimental error. The results may be summarized thus :—

No. of experiment.	Field intensity H.	a (cms.).	Value of product $RH = \dfrac{k^2 H \times 600}{\delta}$ in air.
1. A. 17/11/1905.........	12,148	1·165	$3·71 \times 10^5$
2. A. 18/ 6/1903.........	9,955	2·040	$3·43 \times 10^5$
3. C. 19/10/1905.........:	10,809	2·515	$3·53 \times 10^5$
4. A. 16/11/1905.........	12,143	2·515	$3·56 \times 10^5$

The two experiments carried out with the same apparatus but with different field intensities have given concordant results. Such, however, is not the case if we compare two experiments corresponding to different arrangements of apparatus. It would appear that the principal cause of the divergence is the imperfect knowledge of the coefficient k which enters in the second power, and which in these experiments could not be determined as accurately as the other data. With this restriction, a comparison of experiments 1 and 4 of the preceding table would indicate a decrease of curvature along the trajectory.

I would more particularly call attention to the table relating to the old plate (A. 18/6/1903), in connexion with which radium rays were used and the results obtained with which have already been published. It is seen that, taking into account the distance ϵ, the numbers correspond very closely to a circular trajectory. It becomes therefore necessary to reject definitely the explanation which I had previously advanced and the hypothesis of an increase in the radius of curvature along the trajectory.

A comparison of the trajectories of rays which have traversed a sheet of aluminium 0·015 mm. thick and of rays which have only traversed air leads to conclusions similar to those which have been explained at the beginning of this paper, as is shown by the following summary :—

No. of experiment.	Values of δ.		Ratio.
	In Air.	Through Aluminium.	
A. 16/11/1905	1·7318	1·9037	1·099
A. 17/11/1905	1·7248	1·9334	1·121

I may finally add that measurements made with respect to the interior and with respect to the exterior edges of the deflected images, with a view to detecting dispersion, have not yielded differences exceeding errors of observation.

To sum up, the measurements described confirm the existence of a retardation of the α rays when they traverse a sheet of aluminium such as has been observed by Rutherford. The α rays of radium have behaved in these experiments like the α rays of bodies rendered active by the emanation.

I have been obligingly assisted in the above experiments by M. Matout, preparation-room assistant at the Museum.

LXV. *Notices respecting New Books.*

Ueber den Gegenwärtigen Stand der Frage nach einer Mechanischen Erklärung der Elektrischen Erscheinungen. Von HANS WITTE, *Dr. phil.* Mit 14 Figuren und einer Tafel. Berlin: E. Ebering. 1906. Pp. xii+232.

FEW aspects of physical science have claimed the attention of so many distinguished investigators as the dynamical interpretation of electromagnetic phenomena. The number of different dynamical theories of the electromagnetic field is somewhat perplexing, and the student who first approaches the subject cannot help feeling bewildered by the enormous mass of material to be dealt with. The author of the monograph under review, which is the first of a series to be issued under the general title of *Naturwissenschaftliche Studien*, has performed a signal service by his remarkably lucid and thoroughly systematic examination of the various possible dynamical theories of electromagnetic phenomena. In it he has included all past and present theories, and has clearly exhibited their relationships. The work is divided into 4 sections. Section I. deals with general notions, fundamental laws and classification of theories. The very brief Sections II. and III. are devoted to action-at-a-distance and emission theories respectively—now only of historical interest. Section IV., which forms the bulk of the book, contains an examination of wave theories. A useful bibliography is given at the end of the book. We can heartily recommend this unique monograph to all advanced students of electromagnetic theory.

LXVI. *Intelligence and Miscellaneous Articles.*
To the Editors of the Philosophical Magazine.

GENTLEMEN,—   24th April, 1906.

SINCE the publication of my note on a mercury-sealed tap in your March number I have received from Dr. Thiele, of Dresden, a copy of a paper published in the *Annalen der Physik* in 1901, Bd. 6, in which he describes a tap of design identical with mine, as well as certain improvements upon it.

I much regret that I was not previously acquainted with Dr. Thiele's work, and I shall esteem it a favour if you will insert this letter in your Magazine.

I am, Gentlemen,
Yours obediently,
A. P. CHATTOCK.

THE

LONDON, EDINBURGH, and DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JUNE 1906.

LXVII. *The Field of Force in a Discharge between Parallel Plates.* By J. S. TOWNSEND, *M.A., F.R.S.*[*]

1. FROM the theory which has been given by the author[†] to explain the large difference between the sparking-potential and the potential required to maintain a discharge, it is to be expected that large variations in the electric field accompany changes in the current between two parallel plates in a gas. A number of experiments have been made with a view to obtaining evidence as to the nature of these changes. The results are in good accordance with the general indications given by the theory, but, owing to imperfections in the experimental methods which are employed, it is impossible to obtain accurate determinations of the electric field.

The only practical method that has been employed to determine the potentials at various points of the discharge is by means of a wire placed in the path of the discharge. It has generally been assumed that the wire does not disturb the potential in the discharge, and when the gas is conducting the wire is supposed to acquire the potential at any point which would exist there if the wire were removed. It is difficult to see why this should be universally the case; and on consideration it would appear that when the wire is near one of the electrodes, the potential-fall between the electrode and the wire is probably much greater than the potential-fall

[*] Communicated by the Author.
[†] Philosophical Magazine, March 1905.

between the electrode and the same point in the gas when the wire is removed.

In the following investigation it will be assumed that the conduction in the gas is produced in the manner which has been explained in the previous paper. According to the theory, practically all the ionization is produced by the collisions of positive and negative ions with the molecules of the gas. On this supposition the following result has been obtained :

$$e \times n_1 \times u = i \times Z(x) \times \int_0^x [Z(x)]^{-1} \times \beta \times dx,$$

where $Z(x) = \epsilon^{\int_0^x (a-\beta)dx}$, i being the current per unit area of the electrodes, x the distance of any point in the gas from the negative electrode, e the charge on an ion, u and v the velocities of the negative and positive ions respectively, n_1 the number of negative ions per c.c. at any time at the point x, n_2 the number of positive ions, a the number of molecules of the gas ionized by a negative ion in moving through a distance of one centimetre under the force X, β the number ionized by a positive ion under similar conditions.

It will suffice for present purposes to consider very small currents, so that the potential will not be appreciably affected by the charges on the ions, and the electric force will be practically uniform between the plates. As the quantities a, β, u, and v depend only on the electric force and the pressure of the gas, these quantities will not vary with the distance x, so that the above equation reduces to the form

$$e \times n_1 \times u = \frac{i\beta}{a-\beta}(\epsilon^{(a-\beta)x} - 1).$$

The equation

$$e \times n_2 \times v = \frac{ia}{\beta-a}(\epsilon^{(\beta-a)(a-x)} - 1)$$

may be similarly found.

The ratio

$$\frac{n_1 u}{n_2 v} = \frac{\beta}{a} \times \frac{\epsilon^{(a-\beta)x} - 1}{1 - \epsilon^{(\beta-a)(a-x)}}$$

represents the proportion of the current carried by the negative ions to that carried by the positive ions. The values of a and β have been found over considerable ranges of forces and pressures, so that with the aid of the above formula it is possible to find the ratio of the currents carried by the ions of opposite sign at various points in the gas.

It is interesting to consider the cases for two different pressures above the critical pressure.

Thus let the plates be 8 millimetres apart, and let the gas be hydrogen at 2 millimetres pressure. In this case the sparking-potential is 282 volts, which is also the potential required to maintain a small current. The value of $\dfrac{X}{p}$ will therefore be 176, and the corresponding values of $\dfrac{\alpha}{p}$ and $\dfrac{\beta}{p}$ are 2·41 and ·055 respectively *; so that $\alpha = 4·83$ and $\beta = ·110$. The following are the values of the ratio $\dfrac{n_1 u}{n_2 v}$ corresponding to different distances from the negative electrode:—

x ...	0.	1.	2.	3.	4.	5.	6.	7.	8.
$\dfrac{n_1 u}{n_2 v}$.	0	·0143	·038	·078	·150	·287	·59	1·58	∞

It may thus be seen that for the greater part of the distance between the plates the current consists principally of a stream of positive ions, since the ratio $\dfrac{n_1 u}{n_2 v}$ is less than unity for more than 6 millimetres of the distance. At a point between 6 and 7 millimetres from the negative electrode, the numbers of positive and negative ions passing unit section are equal, and from that point up to the positive electrode the current carried by the negative ions exceeds the current carried by the positive ions.

It is interesting to see how the numbers change as the pressure and sparking-potential rise. As before, let the plates be 8 millimetres apart and the pressure 9·27 millimetres instead of 2 millimetres. In this case the sparking-potential is 487 volts, and the values of α and β are 6·03 and ·05 respectively. The corresponding ratios for various values of the distance x are :—

x ...	0.	1.	2.	3.	4.	5.	6.	7.	8.
$\dfrac{n_1 u}{n_2 v}$.	0	·007	·020	·044	·090	·190	·502	1·20	∞

If the negative electrode be taken as at zero potential and the positive electrode at potential V, the potential at a point

* J. S. Townsend, Phil. Mag. Nov. 1903.

3 B 2

in the gas 2 millimetres, for instance, from the negative electrode would be $\dfrac{V}{4}$ when the current is small.

The question then arises, whether a fine wire placed at a distance of 2 millimetres from the negative electrode would assume a potential of $\dfrac{V}{4}$. It is easy to see that if the wire is brought to a potential $\dfrac{V}{4}$ and then insulated, it would rapidly acquire a higher potential. Considering the case in which the hydrogen is at a pressure of two millimetres, the calculations which have been made show that when the field is undisturbed, the wire being at potential $\dfrac{V}{4}$, the number of negative ions passing the wire per second is less than 4 per cent. of the number of positive ions. The number of positive ions discharged on the wire would be largely in excess of the number of negative ions; it would therefore acquire a positive charge, and its potential would rise above $\dfrac{V}{4}$. The positive charge on the wire would tend to repel positive ions and attract negative ions until the numbers of each kind acquired per second become equal. It would be almost impossible, owing to the mathematical difficulties, to calculate the amount to which the potential would rise; but it would be undoubtedly very considerable. A comparison between the tables given for the two pressures shows that the discrepancy between the potential of the wire and the undisturbed potential of the gas increases as the pressure rises.

It will be noticed that between 6 and 7 millimetres from the negative electrode the positive and negative currents are equal, so that in that neighbourhood the wire would most probably take up the potential of the gas.

It may also be seen that the wire would assume a potential lower than that of the gas when placed near the positive electrode.

2. It is interesting to examine on these principles what might be expected with pressures above the critical pressure when a large current is flowing, and the number of ions in the gas produce an appreciable alteration in the distribution of electric force between the electrodes. According to the theory, the charge in the gas would alter the field of force so as to give rise to a fall of potential, across a layer of gas near the negative electrode, of the same order as the minimum sparking-potential. Nearly all the ions which constitute the

current are generated in this layer where the force is large, and the negative ions proceed from it under a comparatively small force along the positive column to the positive electrode. The layer near the cathode where the ions are generated contains approximately the same amount of gas as is required between parallel plate electrodes to obtain the minimum sparking-potential. The currents of positive and negative ions will be equal at a point near the end of this layer, so that a wire placed in that position would assume the true potential of the gas. For points very near the negative electrode, the current of positive ions exceeds the current of negative ions, and the wire would assume a potential higher than the potential of the gas.

Along the positive column the electric force is comparatively small, so that in this portion of the discharge comparatively few molecules are ionized. The stream of negative ions must therefore be in excess of the positive stream, so that in the positive column the wire would assume a potential lower than the gas. As the positive electrode is approached the number of positive ions in the current diminishes and the number of negative ions increases, and at the electrode the current consists altogether of negative ions. When the wire is near the electrode, it would therefore assume a potential much lower than that of the gas. This is a well-known experimental result; but the above investigation shows that it does not follow that there is any discontinuity in the electric force near the electrode when the wire is removed. There is therefore no reliable evidence to show that the so-called anode fall of potential really exists.

3. In order to investigate experimentally the field of force and a current between parallel plates the apparatus which is shown in fig. 1 (p. 734) was used. The current passed between the two zinc plates A and B. The upper plate B was fixed to a micrometer-screw S which passed through the brass plate C, and in order that the screw should not shake in its bearing the upper portion was reduced so as to fit exactly into a hole in the cock-piece F. The plate C was fixed firmly to the lower plate A by means of four ebonite pillars E. The axle S carried a divided circle D, so that the distance between the plates A and B could be determined accurately. By means of the arrangement shown in the figure, it was possible to move the fine wire W to any position between the electrodes. The wire was stretched between the ends of two ebonite rods R which passed through holes in the brass plate C. The split brass tubes T were fixed to the brass plate, and fitted over the ebonite rods, so that the latter moved up and down

easily. The upper ends of the rods were screwed to a piece
of brass X having a hole in the centre through which the
upper portion of the axle S passed freely without touching.

Fig. 1.
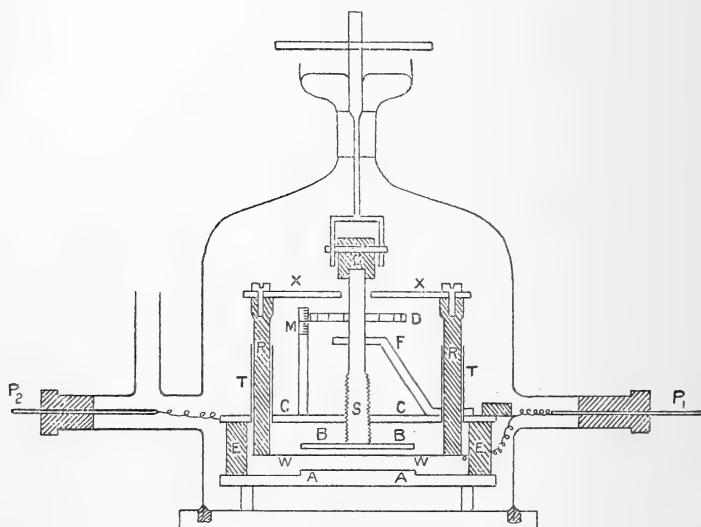
The wire W, the brass piece X, and the two rods R thus
formed a rigid rectangular frame which could be moved up
or down, and the brass tubes T exerted sufficient pressure on
the ebonite rods to keep the frame in any position in which
it might be placed. In order to raise the frame the screw
S was rotated until the disk D moved the brass piece X to
the required position. The frame was lowered by screwing
down the axle so as to make the short piece of ebonite G on
the top of the axle to press on the upper side of X. It was
arranged that the wire W should be one millimetre below the
plate B when the ebonite G was pressing on X. When the
distance between the plates A and B did not exceed
8 millimetres it was possible to place the wire accurately at
any distance from the lower plate, except that it could not be
brought nearer than one millimetre to the upper plate.

Electric connexion was made with the upper plate and
the wire through ebonite plugs in the side tubes of the glass
cover, and the connexion with the lower plate was made by
the base.

A sensitive electrostatic voltmeter which gave readings to

$\frac{1}{2}$ a volt from 80 to 265 volts, and an aluminium-leaf electroscope graduated to read potentials above 150 volts were used to determine the potential-differences between the wire and the electrodes. When either the voltmeter or the electroscope were connected to the wire and charged with a small replenisher, they maintained their charges without any appreciable diminution for several minutes, showing that the insulation was satisfactory. The cases of the voltmeter and electroscope were also insulated.

4. An unexpected difficulty arose when preliminary experiments were being made, which was subsequently traced to an effect arising from the capacity of the insulated system of the voltmeter or the electroscope in connexion with the wire in the gas. The following experiment illustrates the general nature of the effects which are produced. The plates were set at 8 millimetres apart and the wire was placed midway between the plates, the gas being hydrogen at 6·7 mms. pressure. The terminals of a battery of 452 volts were connected through a galvanometer and resistances amounting to 371,000 ohms to the electrodes, which gave a constant current of $2·1 \times 10^{-4}$ ampere through the gas. The potential-fall along the external resistances being 78 volts, the difference of potential between the electrodes in the gas was 374 volts. This was verified by another electrostatic voltmeter which gave a constant deflexion of 374 volts when it was connected to the electrodes.

When the sensitive electrostatic voltmeter was used to find the potential-differences between the wire and the electrodes, the following results were obtained :—

volts

Potential-fall between wire and negative electrode . . 178

    ,,   ,,   ,,   positive   ,,   less than 80

The sum of the potentials should be 374 volts instead of being less than 258 volts, which shows that the potential of the wire must be disturbed by connecting it to the voltmeter. It was found that the discrepancy could not be due to faults in the insulation or to errors in the voltmeter, as the readings of the latter were found to be very accurate over the whole range of the scale when tested with voltages that had been measured with a standard milliampere balance.

In the above experiments the connexions to the voltmeter were made with ordinary copper wires having small resistance.

A number of experiments were made to find the cause of the disturbance of the potential of the exploring wire and to eliminate the effect if possible.

It was found that perfectly consistent results were obtained when the connexions from the electrode and the exploring-wire in the gas to the terminals of the voltmeter were made through high resistances of about 5 megohms.   A dilute salt solution in a capillary tube 3 or 4 centimetres in length is a convenient form of resistance for this purpose as it has a small capacity.   It is necessary to have a high resistance in each line leading to the voltmeter, and it is of advantage to have the resistance tubes as near as possible to the electrodes and the exploring-wire.   With the high resistances thus placed in the lines conducting to the electrostatic voltmeter, the following readings were observed :—

Potential-difference between exploring-wire and negative electrode 270 volts

,,   ,,   ,,   positive   ,,   102 volts

The sum of the potentials only differs by 2 volts from the ascertained difference of potential between the electrodes, so that the disturbing effects produced by the capacity of the voltmeter are eliminated when high resistances are used in the connexions.

The following experiments are of interest in this connexion as they show the effect of capacity in a simple manner.

A high resistance was joined to the base, another to the rod P_1, and a third to the rod P_2, and the connexions to the voltmeter were made through these resistances, so that the potential-differences between the wire and each of the electrodes were not disturbed by using the electrostatic voltmeter to make the necessary observations.   The effects produced by connecting one set of plates of a condenser to the wire W and the other set of plates to either A or B may then be easily examined.   For this purpose, a small air-condenser was used which consisted of an inner plate P 12 centimetres square and two outer plates Q in metallic connexion and slightly larger than the plate P.   The air-space between the inner plate P and the plates Q was approximately 1 centimetre, so that the capacity of P was about 23 electrostatic units.   The plate P was fixed in a vertical position to the rod P_1 making contact with the wire W through the ebonite plug in the side tube of the cover-glass, and the plates Q were set up on a paraffin block so as to be equidistant from it on either side.   There was thus complete air insulation between the plates P and Q ; nevertheless the potential of the wire when the current was flowing depended upon whether the outer plates Q were joined to the positive or negative electrode in the gas.

Using the same current as in the previous experiments the following results were obtained :—

	No capacity attached to W.	Inner plate P joined to W, and outer plates Q to negative electrode.	Inner plate P joined to W, and outer plates Q to positive electrode.
Potential-difference between wire and positive electrode.	102	178	82
Potential-difference between wire and negative electrode.	270	194	285?
Sum	372	372	367?

In the last experiment the pointer of the voltmeter was considerably over the scale, so the potential was determined approximately with a graduated aluminium-leaf electroscope. The potential must of course in this case be 290 (372−82), since the difference of potential between the wire and the electrode is not affected by the voltmeter as the connexions to it are made through high resistances.

These and other experiments show that when one set of plates of a small condenser is joined to the wire W and the other set to one of the electrodes, the potential-difference between the wire and that electrode is diminished. The condenser has no effect when the connexions to it are made through high resistances.

The effect may also be observed by having the leaves of an electroscope connected to the wire and the case to the negative electrode. No change in the divergence of the leaves is produced when the electrostatic voltmeter is connected up through high resistances, but when the connectors are of small resistance an increase in the divergence of the leaves is observed when one terminal of the voltmeter is joined to the wire in the gas and the other to the positive electrode. The opposite effect is produced when the voltmeter terminal is joined to the negative electrode. Similar results were obtained when air was used instead of hydrogen for these experiments.

It is difficult to see how these effects arise, but it is most probable that they are due to the unsteadiness of the current. A telephone placed in the circuit leading from the battery to the electrodes gave a rumbling noise, which showed that there must have been slight variations in the current. The noise was generally diminished (without altering the intensity of the current as given by a galvanometer) by increasing the potential of the battery and the external resistance, but no reliable method was found of producing silence in the telephone.

It has been suggested that the unsteadiness is due to faults in the connexions of the battery, but although the connexions were carefully examined and renewed in places where there might possibly have been a fault, still there was no diminution in noise in the telephone.   It was also found that the telephone gave no sound when the wires leading to the electrodes in the gas were connected together, and the resistance of the circuit increased so as to bring the current to its original value.

It appears therefore that the slight variations in the current must be attributed to the nature of the conductivity of the gas.

Experiments were also made with a current (maintained by a battery of lead cells) through a gas at low pressure in a glass tube of diameter 2·4 centimetres.   The electrodes were flat aluminium disks 14 centimetres apart, and a wire projected into the tube in the usual manner near one of the electrodes so as to obtain the cathode fall of potential.   The telephone showed that in this case also the current was not perfectly uniform, but the effect of the capacity of the voltmeter could scarcely be detected, and the ordinary value of the cathode fall of potential was obtained.   The potential of the wire is therefore more stable in a long discharge which fills a glass tube, but of course this apparatus would have been useless for these investigations, as one of the principal objects was to obtain small currents in a uniform field.

5. A number of experiments were made to determine the potential of the wire at different distances from the large parallel plate-electrodes, using large resistances in the connexions to the voltmeter.   The following curves represent the results for hydrogen at different pressures.   The three curves shown in fig. 2 give the potentials of the wire at different distances from the electrodes for three different currents.   The currents expressed in amperes per square centimetre of the electrodes corresponding to the curves 1, 2 and 3 were $1·2 \times 10^{-6}$, 6×10^{-6}, and 3×10^{-5} respectively.

The pressure of the hydrogen was 1·37 millimetres, which corresponds to the minimum sparking-potential for the distance of 8 millimetres between the plates.   The small difference between the curves 1 and 2 shows that when the current is small, the potential of the wire is not affected by changing the intensity of the current.   It is most likely in these cases that the charge on the ions is too small to affect the field of force appreciably, and that if the wire were removed the field would be uniform.   The ratios of the currents carried by ions of different sign are independent of the absolute value of

the current when the force is uniform ; so that the potential assumed by the wire ought to be the same for all small currents, although it may differ from the potential of the gas.

Fig. 2.—Hydrogen-pressure 1·37 mm.

1. Current=1·210⁻⁶ amperes per sq. cm. of electrodes.
2.   „   6×10⁻⁶   „   „   „
3.   „   3×10⁻⁵   „   „   „
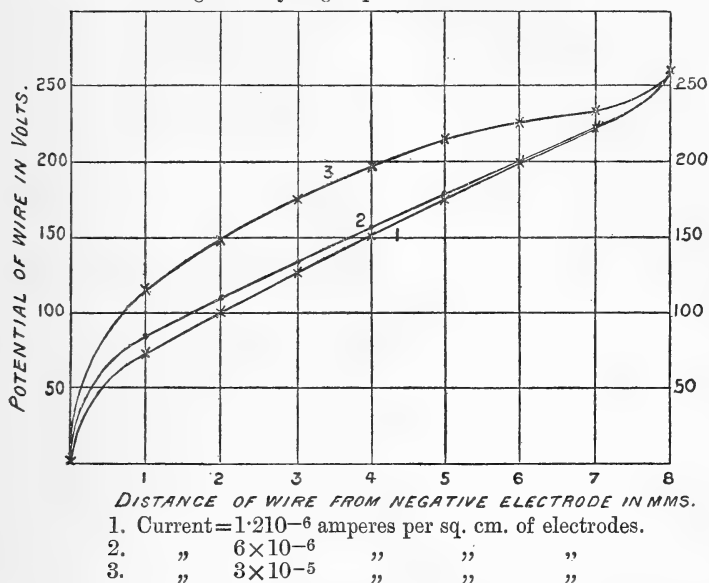
It would be easy to make a calculation of the charge in the gas in this case, and to estimate the extent to which the charge on the ions disturbs the uniformity of the field, if the velocity of the positive ion were known. A rough estimate may be made from the numbers found for the velocity at higher pressures and smaller forces by supposing that the velocity is proportional to the electric force and inversely proportional to the pressure. The value thus found for the larger forces and smaller pressures would be a lower limit to the velocity. It may be conjectured from the determinations at higher pressures both of the rate of diffusion and the velocity under an electric force, that the ions are accompanied in their motion by a group of molecules. If the group remained the same for the lower pressures and higher forces, then the velocity would be inversely as the pressure and proportional to the electric force. It is most probable that for large forces and small pressures the group would diminish ; and so the velocity under an electric force would increase more rapidly than the quotient $\dfrac{X}{p}$. It may be shown from

the experiments that have been made that, on the average, the velocity does not increase much more rapidly than $\dfrac{X}{p}$ for the forces and pressures used in the experiments.

For this purpose, it is necessary to examine to what extent a current of $1\cdot2 \times 10^{-6}$ ampere disturbs the uniformity of the field. The effect of the charge is greatest near the negative electrode, and the calculations of the ratio of the positive and negative currents which have been given above show that the stream of positive ions exceeds that of the negative ions for the greater part of the distance between the plates. The quantity $n_2 \times e \times v$ measured in electrostatic units is of the order $1\cdot2 \times 10^{-6} \times 3 \times 10^9$ for the space near the negative electrode, as nearly all the current is carried by positive ions; so that the charge $n_2 \times e$ becomes $\dfrac{3\cdot6 \times 10^3}{v}$, v being the velocity of the positive ion.

The velocity of positive ions in hydrogen at 760 mms. pressure under a force of 1 volt per centimetre is $6\cdot7$ cms. per second*; so that for the force $260 \div \cdot8$ volts per centimetre and a pressure of $1\cdot37$ millimetres the velocity would be $1\cdot2 \times 10^6$ cms. per second. Near the negative electrode the positive charge would be 3×10^{-3} electrostatic units per cubic centimetre, if all the current were carried by the positive ions.

A simple calculation shows that if a charge of 3×10^{-3} electrostatic units per cubic centimetre be placed in a layer of the gas 6 millimetres wide near the cathode, the potentials in volts at different distances from the cathode are:—

Distance from Cathode.	0.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.
Potential $= \dfrac{V'x}{\cdot8} +$	0	·36	·62	·77	·80	·70	·51	·25	0

V' being the potential-difference between the plates, which in this case was 256 volts.

As these changes are so small the current might have been considerably larger without having much influence on the potentials. The actual potential of the wire at 2 millimetres from the negative electrode was 102 volts; so that this large increase over the potential $64\cdot62$ is most probably due to the fact that the stream of positive ions is so much greater than that of negative ions. There is a marked difference in the

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* Zeleney, Phil. Trans. 1900, p. 193.

curves obtained experimentally when the current rises to 3×10^{-5} amperes.

It may be deduced from the calculations which are given above that in this case the charge on the positive ions must produce a large effect on the distribution of force in the field, and the experiments show that the potential of the wire at 2 millimetres from the negative electrode changes from 102 to 147 volts.

It will be noticed that the curve corresponding to a current of 6×10^{-6} ampere lies a little above the curve for $1 \cdot 2 \times 10^{-6}$ ampere, and the calculations show that the charge in the gas would begin to produce increases of 2 or 3 volts in that case.

The numbers which have been given for the increases of potential are not very accurate. They must be considered as indicating the order of effects which would be produced under the assumptions that have been made in calculating the velocity. The velocity $1 \cdot 2 \times 10^{6}$ which has been found is the final velocity that a positive ion attains under these conditions, and the estimate of the effect of the charge has been made on the supposition that a positive ion travels along its whole path in the gas with this mean velocity. As a matter of fact, it would be more correct to consider the ion as starting from rest from the place where it is generated, so that the ions take a longer time to traverse their paths, and consequently the charge in the gas has been underestimated. It is interesting to find how long it takes an ion to traverse 6 millimetres if it starts from rest when $\dfrac{X}{p}$ has the value 233, as in these experiments.

The equation of motion of a body of mass m and charge e in an electric field of intensity X is

$$m \frac{d^2x}{dt^2} = eX - k \frac{dx}{dt},$$

k being a constant depending on the retardation of the ion. The final velocity $\dfrac{eX}{k}$ has been estimated at $1 \cdot 2 \times 10^{6}$ cms. per second, X being $1 \cdot 1$ approximately in electrostatic units. Experiments on diffusion at the higher pressures show that m is roughly 30 times the mass of a molecule of oxygen; so that the value of $\dfrac{m}{e}$ in the above equation becomes $3 \cdot 6 \times 10^{-12}$. When these values of the constant are substituted, the solution of the equation becomes

$$x = 12 \times 10^5 \times t - 4 \cdot 8 [1 - \epsilon^{-2 \cdot 5 \times 10^5 \times t}].$$

The time required for the ion to travel 6 millimetres may be obtained by substituting for x the value ·6, and it is thus found to be approximately $2 \cdot 2 \times 10^{-6}$ seconds. This is more than four times the length of time 5×10^{-7} seconds required to travel the distance 6 millimetres at the uniform velocity $1 \cdot 2 \times 10^{6}$ cms. per second. It must, however, be borne in mind that it requires some time for a large group of molecules to collect round a positive ion, and during the first part of its path the ion is travelling with a mass practically equal to the mass of a molecule of hydrogen. It is quite conceivable that during the short time of the order 10^{-7} second the positive ion moves as if it had a comparatively small mass, and possibly the value 5×10^{-7} seconds is a more correct estimate of the time required to travel a distance of 6 millimetres.

These considerations show that it is very difficult to make an accurate calculation of the charge in the gas even for a field of force that is practically uniform; nevertheless it is of interest to see that the experimental results are in fair agreement with the rough indications given by the theory.

The theory shows that the potentials at 2, 4, 6, and 8 millimetres from the negative electrode ought to be the same as the potentials at 1, 2, 3, and 4 millimetres respectively from the negative electrode when the pressure of the gas is doubled and the distance between the plates reduced from 8 to 4 millimetres. It was found that the potentials were 107, 155, 205, and 260 at the distances mentioned when the pressure of the hydrogen was $1 \cdot 37$ millimetres and the distance between the plates 8 millimetres; and for the pressure $2 \cdot 74$ millimetres with the plates 4 millimetres apart, the potentials were 108, 154, 202, and 257 at the corresponding reduced distances.

6. Further experiments were made at a pressure of $2 \cdot 74$ millimetres with the plates 8 millimetres apart. The results are shown by the curves, fig. 3. The velocity of the positive ions is much *slower* in this case as the potential required to maintain a small current rises only to 316 volts as compared with 256 volts at the lower pressure. In order that the field of force should not be appreciably influenced by the charge, it is necessary to use smaller currents since the charge in the gas is inversely proportional to the velocity. It was found that the field, as indicated by the potential of the wire, did not vary for currents between 2×10^{-6} and 2×10^{-7} ampere, so that for these small currents the charge is too small to make an appreciable effect on the field of force. It will be seen that for a current of 4×10^{-6} ampere a large

change occurs in the field of force, and a cathode-fall of potential of about 240 volts is developed in the layer 4 millimetres thick near the cathode. The curve shows that the potential between the electrodes required to maintain the current diminishes when the cathode-fall of potential is developed.

Fig. 3.—Hydrogen-pressure 2·74 millimetres.
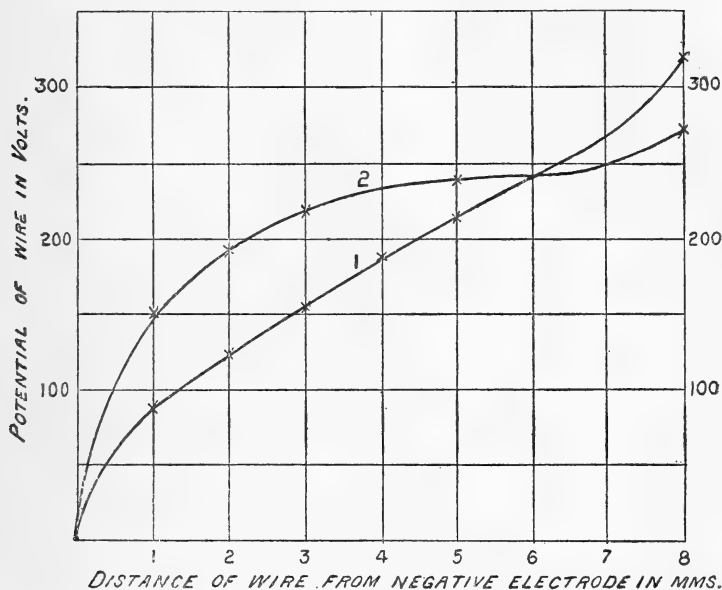
1. Currents less than 2×10^{-6} ampere per sq. cm.
2. Current $= 4 \times 10^{-6}$ ampere per sq. cm.

The calculations which have been given for the distribution of the current between positive and negative ions show that at a point between 6 and 7 millimetres from the negative electrode, the two streams are equal over a range of pressures between 2 and 9·27 millimetres, and the wire should take up a potential nearly equal to the potential of the gas at those points. The latter potential would be represented by a straight line joining the two ends of the curve for the smaller currents, and the figures show that these lines would intersect the potential curves determined by the wire at points about 6 millimetres from the negative electrode.

It is not easy to find other potential curves intermediate between those given in the figure as the current is unsteady for values between 2×10^{-6} and 4×10^{-6} ampere, where the charge in the gas exercises an appreciable effect on the field.

It would be possible, no doubt, to use a very high electro-motive force and a large external resistance, and most probably a steady current of any value could then be obtained, but for a preliminary investigation it was not considered necessary to make experiments in that direction.

Experiments were also made at higher pressures, but owing to continued use of the apparatus the electrodes underwent some change and the glow did not spread uniformly over the surface of the electrodes. It frequently happened that the current passed through a narrow column of gas, so that it was impossible to make observations of potential or to determine the density of the current per square centimetre of the electrodes. From a theoretical point of view it is better to make experiments at the lower pressures (2 or 3 times the critical pressure) in order to obtain experimental evidence of the transformation of the electric field, as the potential of the wire does not differ so much from that of the gas at these pressures for the smaller currents. It may be seen from the numbers that have been given for the ratio of the currents of positive and negative ions that the number of negative ions in the gas near the negative electrode diminishes as the pressure rises.

7. Another set of experiments of a different kind have been made to test the results which are indicated by theoretical considerations. It has been shown that the sparking potential is the same as the potential required to maintain a small current, but as the field of force near the negative electrode is increased by the positive charge in the gas, the potential required to maintain a large current diminishes. If this theory is correct, it should be possible to produce sparking or to maintain a small current between parallel plate-electrodes, if the force near the negative electrode is increased by having a positively charged body near it. This can easily be arranged by having a grating of fine wires near the negative electrode and charging it by means of a battery or a small replenisher.

A simple form of apparatus to illustrate this effect is shown in fig. 4. A pair of plane parallel electrodes N and P were fitted in a glass tube of about $2\frac{1}{2}$ centimetres in diameter. A grating of fine wires was mounted on a brass ring R which was fixed to the end of an ebonite tube T. The ebonite tube was shaped so that the ring fitted over the end and the wires of the gauze rested on the front of the ebonite that projected into the ring. The negative electrode fitted inside the ebonite tube and could be placed at any convenient distance from the grating. The wires of the grating were

about ·1 millimetre in thickness and 1 millimetre apart, and were parallel to the surfaces of the electrodes.

Fig. 4.
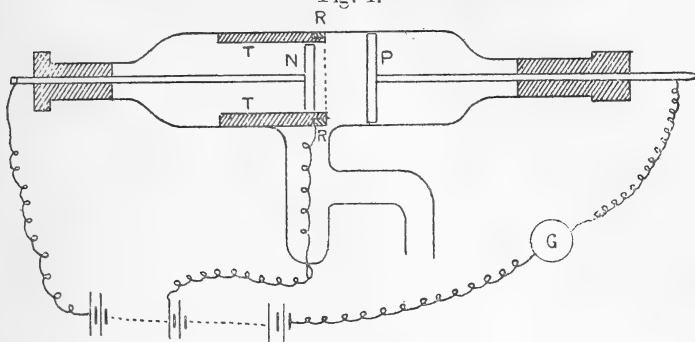
A battery of known electromotive force was connected through a galvanometer to the electrodes in the tube, and the potential of the wire was gradually raised until sparking took place.

The following table represents the results of the experiments, V denotes the potential-difference between the electrodes, and V′ the potential-difference between the negative electrode and the gauze when sparking occurred.

V	550	600	640	680	700
V′.........	240	210	170	150	100

It thus appears that the sparking potential between the two electrodes may be lowered by 150 volts by increasing the strength of the field near the negative electrode.

The same results were obtained by a different arrangement for charging the grating. A known electromotive force was applied to the electrodes, and the grating was insulated and connected to one terminal of an electrostatic voltmeter, the other terminal being connected to the negative electrode. A small quantity of radium was held outside the tube opposite the space between the grating and the positive electrode. The grating thus acquired a positive charge slowly, and when the potential as shown by the voltmeter attained a certain value, sparking took place between the electrodes. The greater the potential-difference between the electrodes, the less is the charge acquired by the grating before the spark passes. The results obtained in this way were practically the same as those given in the above table.

LXVIII. *The Black Spot in Thin Liquid Films.*
By E. S. Johonnott, Jun., *Ph.D.**

IN some previous work† on this subject it was found that, in general, the black film consisted of two portions. That first formed varied in thickness between 40 $\mu\mu$ and 12 $\mu\mu$. The first black was found to break naturally into the second when the atmosphere was incompletely saturated. With a thoroughly enclosed film it was necessary to heat the system in order to form the second black film ; which had, apparently, a constant thickness of 6 $\mu\mu$ in all cases.

Effect of Change in Pressure.—Recently it was discovered that great changes in the character of these films could be produced by changing the pressure on the atmosphere surrounding the films. A sudden increase in the pressure was accompanied by a rapid thinning of the film ; while, on the other hand, a sudden diminution of the pressure was accompanied by a correspondingly rapid increase in the thickness. On subjecting the first black film to sudden increase in pressure it was possible to convert it quickly into a second black ; if then the pressure was suddenly diminished, the second black became a first black. In many cases it was possible to continue this thickening until the film showed the yellow of the first order.

The evident explanation of these effects is that the heating due to adiabatic compression is accompanied by an increased vapour-pressure and, consequently, evaporation from the surface of the film. While, on the other hand, adiabatic expansion causes cooling and condensation on the film.

Fig. 1 represents a simple arrangement for producing these effects. The film is formed on a rectangular frame of glass-fibre, F, mounted on a rubber stopper. This stopper fits firmly into the mouth of an inverted beaker containing the solution ‡. Blowing or sucking in a rubber tube, attached at P, the pressure in the upper vessel is varied without contaminating the solution. If the upper vessel is carefully chosen the films may be observed quite advantageously with a low-power microscope. A simple gas-jet makes an

* Communicated by the Author.

† Phil. Mag. vol. xlvii. p. 501 (June 1899).

‡ The solution was made in the following manner :—7 grams of oleic acid (refined) and 2 grams of caustic soda are thoroughly boiled in about 100 c.c. of distilled water. This gives about 7·5 grams of oleate of soda which is diluted to about 50 of water to 1 of soda and again boiled. An excess of acid is not harmful, while an excess of soda is decidedly so.

excellent illuminant, and at the same time may be used to heat the film.

Fig. 1.
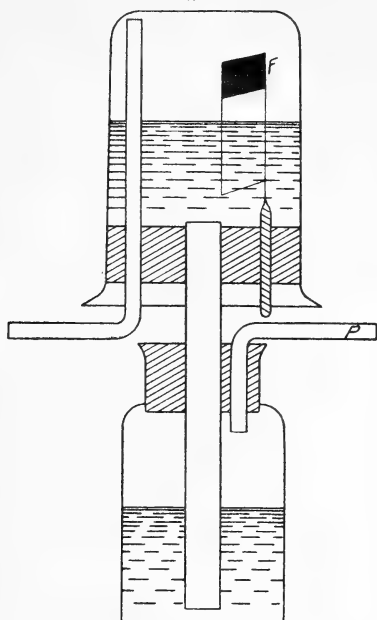
Let the temperature be adjusted with the flame and the pressure with the mouth until the first black film is about to break into the second. Then pinching the tube with the fingers will cause the second black to appear as small perfectly round spots in the first black film ; releasing the pressure, after the film has become the second black, will cause the first black to appear, in a similar manner, as spots in the second black.

Thickness of the Black Film.—In view of these circumstances, it was thought advisable to repeat the measurements on the thickness of the black film. The work was begun in the Ryerson Physical Laboratory of the University of Chicago, and through the courtesy of the officers of instruction, in kindly lending the necessary apparatus, was carried to completion elsewhere.

In order to vary the thickness, by changing the pressure, and at the same time to measure the thickness, a different form of the interferometer was required. In fig. 2 is shown a familiar form of the Michelson interferometer that was used. The object that was to be gained was simply that both

optical paths should lie alike in the atmosphere of the films. The long brass box in which the films were formed is shown at B. The films were 221 in number and formed on frames of glass-fibre as in the former work. The box was sealed,

Fig. 2.
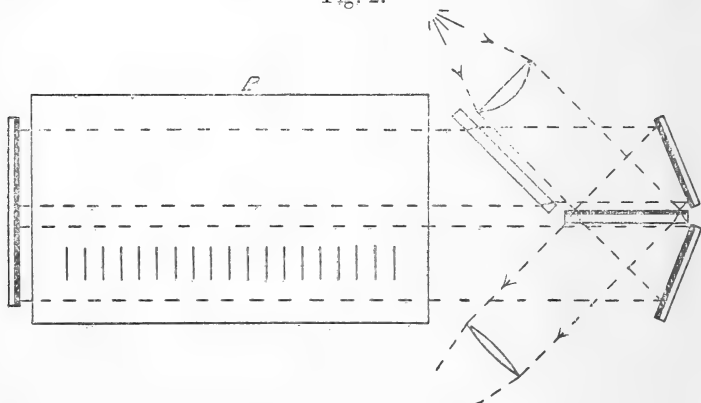
and connected to a manometer and suitable apparatus for varying the pressure. The fringes were viewed with a telescope. All other details were much the same as in the earlier work, and may be found described therein.

At the first trial it was apparent that the reading of the number of fringes displacement could not be made as promptly as the thickness could be altered by varying the pressure. This was due to the distortion produced in the fringes by quickly changing the pressure. It was very difficult to obtain trustworthy data when working with rapid changes in the pressure, and, on the whole, this part of the work was somewhat disappointing. Nevertheless, with less rapid alterations in the pressure, it was possible to pass the black film through all its phases with little if any distortion in the fringes.

A large portion of the time was spent in working with no variation in the pressure. The method usually employed was to form the films in the morning and take readings of the thickness throughout the day, with a more or less rapidly rising temperature. The box was surrounded with a large water-jacket to maintain a uniform temperature. It was found that this outer box filled with air served the purpose quite as well.

The following table was taken with no variation in pressure. The box containing the films was firmly sealed. The temperature recorded was the mean of the readings of two thermometers, placed within the box, B.

TABLE 1.

Age of Film.		Fringes Deflect.	No. Films.	Mean Thickness.	Mean. Temp.	Remarks.
hr.	min.			$\mu\mu$.	° C.	
1	18	4·0	221	16·0	25·5	All first black.
4	33	4·1	221	16·4	25·0	,, ,,
7	58	6·0	221	24·0	23·5	,, ,,
21	33	3·8	221	15·2	25·3	,, ,,
26	23	3·0	217	12·2	26·5	Two second black.
26	51	2·5	217	10·15	28·0	
27	3	1·5	215	6·16	28·7	All second black.

In the light of the results recorded in the former paper, this table requires little comment. The increase in the thickness with falling temperature ; the limiting thickness of the first black film at about $12 \mu\mu$; and the mean thickness of the second black at about $6 \mu\mu$, were therein illustrated a number of times.

In the above table, however, since the number of films was over four times as great, the number of fringes displacement was correspondingly greater.

In the former paper it was remarked that, " a number of observations indicated that the thickness of the second black film was not constant." With the number of films there employed the deflexion was only about ·35 fringes, when all of the frames held second black films. In the present case the corresponding deflexion was 1·5 fringes. With this greater sensibility it was at once apparent that the second black film was of variable thickness. While the limiting thickness of the second black was approximately $6 \mu\mu$, the upper limit, in many cases, was at least 50 per cent. greater.

In fact, in behaviour, the black films were found to be quite similar; *i. e.*, with a sealed atmosphere and a rising temperature, a rapid thinning immediately after birth was characteristic of both types. This rate of thinning became smaller and smaller as the limiting thickness was approached. The rate of thinning of the second black was so great, at first, that usually the limiting thickness was reached before observations could be made. The persistence of the black films at the limiting thickness, together with the tendency to reach this condition quickly, seems to indicate that the films were of quite uniform thickness, at the limit.

The data in Table II. were taken from a number of tables similar to Table I. The thickness is given in micro-millimetres ($\mu\mu$).

TABLE II.

A.	B.
Limiting Thickness of First Black Film.	Mean Thickness of the Second Black Film.
11·05, 12·20, 12·20, 13·15, 11·10, 12·30, 11·75, 12·60, 13·30.	6·03, 6·24, * 9·25, 5·52, 6·17, 5·05, 6·28, * 7·30, * 9·50, 5·40, 6·40, * 8·0.
Mean ...... 12·18.	Mean ...... 6·76.

It would be difficult, in section B, to distinguish between the values corresponding to the unstable condition and the condition of limiting thickness. The larger values, marked with the asterisk, were, however, in all cases followed by smaller values included in the table. If we take the mean of the values, not including these larger values, to determine the limiting thickness of the second black, we obtain $5·88 \, \mu\mu$. This would indicate that the limiting thickness of the second black is, approximately, half that of the first. No very great thickness was measured in the first black, since it was impossible to count the coloured fringes for such large deflexions.

Variable Atmospheric Pressure.—The following readings were obtained when changing the pressure on the atmosphere surrounding the films.

TABLE III.

Time.	Fringes deflect.	No. films.	Mean thick.	Temp. ° C.	Pressure change.	Remarks.
A.M.			$\mu\mu$		cms. Hg	
9.08	films	formed.		17·8		
9.32	3·0	101	26·8	20·0	none.	All first black.
9.50	2·3	100	20·3	20·0	,,	,, ,,
10.20	2·0	98	18·0	21·0	,,	,, ,,
					−4·6	
	3·4	98	30·6	...	...	,, ,,
					+9·2	All second black.
10.23	·9	98	8·1	21·6	...	,, ,,
	System allowed to come to normal pressure.					
10.40	2·0	97	18·2	...	none.	All first black.
					−6·4	
	4·2	96	38·5	...	...	,, ,,
					+12·8	
10.44	·9	95	8·4	21·2	...	All second black.

A large bottle, containing some of the solution, was connected to the film-box by a long rubber tube. The pressure was altered by changing the elevation of this bottle.

As shown in the table, the films were allowed to thin to the first black, then the pressure was diminished by 4·6 cms. of mercury. This caused the films to thicken from 18 $\mu\mu$ to 30·6 $\mu\mu$. The pressure was then increased by 9·2 cms., and the films were quickly thinned to the second black at a mean thickness of 8·1 $\mu\mu$.

Surface-tension in the Black Film.—In this connexion it may be of some interest to describe some work that was undertaken to determine the surface-tension in the black film. The method of Sondhaus [*] was employed.

A ring, 15 centimetres in diameter, was made of glass-fibre. This was attached to one arm of a sensitive Oertling balance and lifted from the surface of the solution. The film formed, although a catenoid, was quite approximately a cylinder.

The following table is a typical record of the observations made.

TABLE IV.

Age of film.	Supporting force.	Surface-tension.	Temp. ° C.	Remarks.
min.	mgs.	mgs. per cm.		
0	2495	26·5	24·0	No colour.
2	2475	26·35	...	Coloured quickly.
8	2465	26·20	...	5 per cent. first black.
18	2455	26·07	...	All ,, ,,
25	2450	26·00	...	10 per cent. second black.
37	2445	25·95	25	All ,, ,,
44	2440	25·90	...	,, ,, ,,
56	2435	25·83	26	,, ,, ,,

With these results plotted in a curve, it is quite evident that to the discontinuity in the thickness, there, apparently, corresponds no discontinuity in the tension. Otherwise, however, the varying rate with which the tension falls is quite similar to the manner in which the film thins. And it should be observed that these phenomena point to a variable condition in the second black film, both with respect to tension and thickness.

The Five Black Films.—Viewing the films with a micro-

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[*] Sondhaus, Pogg. *Ann.* Ergbd. viii. p. 266.

scope, in the arrangement shown in fig. 1, it is oftentimes possible to distinguish as many as five black films. These break, one into another, in the same manner in which the black film is ordinarily formed, each succeeding film being, apparently, the thinner. The first three formed are, however, quite evanescent and seldom extend over an elevation greater than two or three millimetres. The fourth and fifth are identical with those which have been termed the first and second black films, respectively. Oftentimes all five films are in the field of view at once.

Speculations.—The apparent continuity of the surface-tension in passing from one black film to the other has been observed by Professors Reinold and Rücker. Based on what may be termed Lord Kelvin's[*] minimum surface-tension principle, they suppose the tension to remain constant as the films thin to the limiting thickness of the coloured film. Then the tension is supposed to fall as the film continues to thin until a minimum tension is reached, after which the tension rises. Now it is evident that, for these thinner films, equilibrium could not be reached unless the thinning continues so far that the tension again rises to that of the coloured film.

On this theory it should be possible to have the black film in equilibrium with the coloured film of various thicknesses. Their experimental observations substantiate this theory in a very interesting manner. If a discontinuity occurs in the thickness between the successive black films, it is reasonable to suppose, as Poynting and Thomson [†] have remarked, that a variation in the tension, similar to that which has been described, takes place.

Laplace's theory of capillarity shows that the inference to be drawn is that we have alternate molecular forces of attraction and repulsion as we pass from one black film to the next. Or according to the minimum surface-tension principle, there should be as many undulations in the curve representing thickness and tension as there are black films.

Another theory of these discontinuities might be proposed, which may be little more than an analogy.

Long ago, Professor James Thomson [‡], in discussing the unstable condition supposed to exist between the super-

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[*] Phil. Trans. 1886, Part ii. p. 679.
[†] 'Properties of Matter,' p. 166.
[‡] Maxwell, 'Theory of Heat,' p. 127.

saturated vapour and superattenuated liquid states, says :—
" We cannot expect experimental evidence on this part of the
curve unless this state of things may exist in some part of
the thin superficial stratum of transition from the liquid to
its own gas, in which the phenomena of capillarity take
place."

Suppose that we imagine a liquid film in equilibrium with
its surrounding saturated atmosphere. Let heat be applied
to the system. Then may it not be that the film will become
coated with a thin stratum of superattenuated liquid which
will continue to thicken until, possibly, its outer portion
passes over into the unstable condition, causing the attenuated
layer quickly to evaporate ? If, on the other hand, heat taken
from the system should produce a corresponding layer
of supersaturated vapour next to the film, it should be
expected that a thickening of the film would result from
condensation.

Observations made with the apparatus shown in fig. 1
accord with this theory. For example, suppose that we
obtain a first black film near its limiting thickness, so that a
slight increase in the pressure will cause the second black
film to appear. This second black at first appears as small,
perfectly round, black spots in the first black film, which
expand to replace the first black as the pressure is increased.
If then the pressure be slightly lessened, perfectly round
spots of the first black will appear in the field of the second
black film. This latter effect is more difficult to obtain than
the former, indicating that the supersaturated vapour con-
dition is more difficult to realize than the superattenuated
liquid. In fact, the thickening process ordinarily is so rapid
that a second black film will appear to pass, as has already
been remarked, continuously, to even the yellow of the
first order.

Of course this is merely speculation. Possibly the best
excuse for making such is contained in the following words
of Maxwell :—" The surface which forms the boundary be-
tween a liquid and its vapour is the seat of phenomena, on
the careful study of which depends much of our future pro-
gress in the knowledge of the constitution of matter."

Rose Polytechnic Institute,
Terre Haute, Indiana, Feb. 14, 1906.

LXIX. *The α Particles of Uranium and Thorium. By* W. H. Bragg, *M.A., Elder Professor of Mathematics and Physics in the University of Adelaide* *.

THIS paper is divided into two parts. The first contains a discussion of the magnitude of the ionization current due to a layer of radioactive material scattered on the floor of an ionization-chamber, and covered by a uniform sheet of metal foil. The result is expressed in a formula which is somewhat complicated in its general form, but is capable of simplification under suitable conditions. Account is taken of the variation of ionization with velocity. The second contains an account of experiments which show:—

(*a*) That the values of the current in various cases, calculated from the formula, agree very well with the results of observation.

(*b*) That the ranges of the α particles of uranium and thorium are very nearly, perhaps exactly, equal to the range of the α particle of radium.

(*c*) That the rate at which thorium atoms break down is nearly ·19 of the similar rate for uranium.

Part I.

The method which was used by Mr. Kleeman and myself in the determination of the ranges of the α particles emitted by radium and its products does not lend itself to the corresponding determinations in the cases of uranium and thorium. It is a necessary feature of the method that all α particles except those moving normally to the horizontal layer of radioactive material should be prevented from reaching the ionization-chamber, below which the radium is placed. This is done by the use of a bundle of vertical tubes which stop all α particles other than those moving in the desired direction. But this limitation diminishes very greatly the number of effective α particles, and in the cases of uranium and thorium the effect is reduced below the limits of convenient measurement. This is the case even when a large surface of radioactive material is employed. In order to determine the ranges of uranium and thorium another method must be devised.

I have therefore calculated the ionization due to a radioactive layer over which a screen has been placed. The result is a function of the relation of the stopping power of the

screen to the range of the particle : so that if experiment is made the one can be found in terms of the other. The stopping power of the screen may be made the subject of a direct measurement, and so the range of the α particle can be determined. I find it better, however, to compare the range of the uranium or thorium α particle with that of radium, working the experiment by a substitution method : for the range in the case of radium is known with some accuracy, and the method itself is accurate enough when employed in comparing ranges, but a little uncertain in its application to direct determinations, as will be explained later.

Experiments of this kind have already been made by several observers, notably by Professor Rutherford and Miss Brooks (Phil. Mag. July 1900). But at the time when they were made it was believed that the "α rays" were absorbed according to an exponential law: it was not known that each α particle possessed a definite range or penetrating power. Consequently the results were not in all cases expressed in such a way as to render them available for the calculation of the range. I have therefore found it convenient to repeat them.

In the following theoretical treatment of the question three cases are considered :—

(a) When the layer of radioactive material is so thick that the α rays from the bottom of it are unable to reach the air above : such a thickness is of the order ·002 cm.

(b) When the layer is extremely thin.

(c) When the layer is thicker than in (b) but not so thick as in (a).

The first and second are really special cases of the third. Uranium and thorium are conveniently treated under case (a); radium under (c) ; and the induced activities under (b).

CASE (a) : *Thick layer of radioactive material over which a sheet of absorbing material is laid.*—Let the surface of the radioactive material be of unit area : the full range of the α particle in air be R, and the range lost by passing normally through the absorbing sheet be D.

Let the stopping power of the radioactive material per radioactive atom be s. This means that if an α particle passes, parallel to the axis, along a cylinder containing only as much matter as goes with one radioactive atom of the radioactive material, the loss of range is, on the average of a great number of such passages, s times the loss when an average air molecule is substituted for the other matter. The length of the cylinder is, of course, immaterial.

The α particles, when they emerge into the air, will penetrate to distances depending on the quantity of matter traversed before emerging. Consider in the first place all those whose ranges in air after emergence lie between r and $r + dr$. These move at various inclinations to the surface : the number emerging at any inclination depends on that inclination, and may be reckoned as follows.

Consider only those whose inclinations to the normal lie between θ and $\theta + d\theta$. All these come from a layer of a certain thickness at a certain depth below the surface. The depth does not concern us, but the thickness does, for we need to know the number of radioactive atoms in the layer.

Let n be the number of radioactive atoms in a cub. cent. of the material, n_0 the number of molecules in a cub. cent. of air.

An α particle loses the same range in traversing a distance δr in air, as in traversing a distance $n_0 \delta r / ns$ in the radioactive
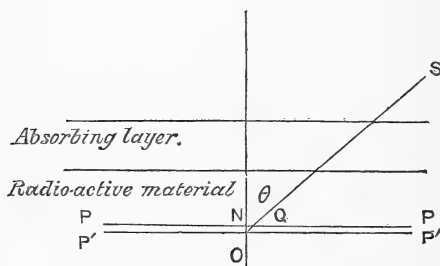
material. Hence if PP′ is the layer in question, O the radioactive atom, OS the course of the α particle, we have $OQ = n_0 \delta r / ns$ and $ON = n_0 \delta r \cos \theta / ns$. This last expression is also the volume of the layer from which the α particles come, since we are considering unit area of the material ; and therefore the number of radioactive atoms in it is $n_0 \delta r \cos \theta / s$.

Let each atom emit N α particles per second. N is a very small fraction. Then the number emitted by each particle between the inclinations θ and $\theta + d\theta$ is

$$\text{N} . 2\pi \sin \theta . \delta\theta / 4\pi = \text{N} \sin \theta . \delta\theta / 2.$$

Hence finally the number of α particles whose ranges in air after emergence lie between r and $r + \delta r$, and which have inclinations to the normal varying from θ to $\theta + \delta\theta$, is

$$\text{N} \sin \theta . n_0 \cos \theta \delta r \delta\theta / 2s.$$

The limits of θ are zero and such a value of θ that the α particles which come from the very surface of the radioactive

material and move at this inclination to the normal have a range r in the air after penetrating the absorbing sheet.

This value of θ is given by $D \sec \theta + r = R$.

Integrating between these limits we find that the total number of α particles whose ranges lie between r and $r + \delta r$

$$= \frac{N n_0}{4s} \left\{ 1 - \frac{D^2}{(R-r)^2} \right\} \delta r.$$

Each of these moves a distance r through the air of the ionization chamber before it ceases to ionize. Now I have shown (Phil. Mag. Nov. 1905) that the α particle spends energy on ionization at a rate which is inversely proportional to the energy which it possesses; so that we may say that $\delta e = k \delta r / e$, where e is the energy of the particle and δe the energy spent in traversing a distance δr. Hence we find that $e \propto \sqrt{(r + c)}$, where c is a constant. The value of the latter I have shown (*loc. cit.*) to be 1·33. Hence the ionization produced by the α particle in the last r cm. before it ceases to ionize may be written

$$l(\sqrt{r + 1 \cdot 33} - \sqrt{1 \cdot 33}).$$

Even if this expression should prove to be based on imperfect theory, it nevertheless expresses the actual fact very nearly.

Finally, therefore, the whole ionization ($= i$)

$$= \frac{N l n_0}{4s} \int_0^{R-D} \left\{ 1 - \frac{D^2}{(R-r)^2} \right\} (\sqrt{r+d} - \sqrt{d}) dr,$$

the proper limits being given to r in the integral.

After some reduction the value of this integral can be found to be

$$\frac{N l n_0}{4s} \left\{ \tfrac{2}{3} (R - D + d)^{\frac{3}{2}} - \tfrac{2}{3} d^{\frac{3}{2}} - R \sqrt{d} - D \sqrt{R + d - D} + 2D \sqrt{d} \right.$$
$$\left. + \frac{D^2}{\sqrt{R+d}} \log \frac{\sqrt{R}(\sqrt{R+d} + \sqrt{R+d-D})}{\sqrt{D}(\sqrt{R+d} + \sqrt{d})} \right\}.$$

If we put $D = 0$, we obtain the value of the current (I) when the radioactive material is uncovered, viz.:

$$I = \frac{N l n_0}{4s} \{ \tfrac{2}{3} (R + d)^{\frac{3}{2}} - \tfrac{2}{3} d^{\frac{3}{2}} - R \sqrt{d} \}.$$

Simpler formulæ may be found by neglecting the variation of ionization with velocity. If we put the ionization due to

traversing r cm. equal to $l'r$, we find that

$$i = \frac{N l' n_o}{8s} \left\{ (R-D)(R-3D) + 2D^2 \log \frac{R}{D} \right\},$$

and $I = \frac{N l' n_0}{8s} R^2.$

Curves may be plotted from these formulæ giving the relation between i/I and D/R. In the case of the latter

Fig. 1.
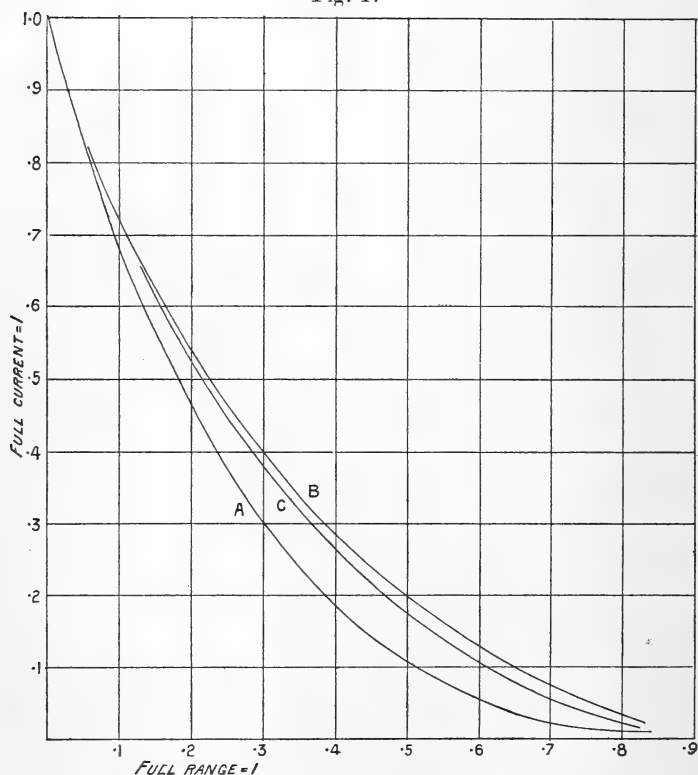
formula the form of the curve is independent of the value of R; but in the more accurate formula it is not. Actual trial shows, however, that the curve in this case shifts very little when the value of R is altered, and always lies close to that given by the simpler formula. In the interpretation of the experiments with uranium and thorium, I have used the curve A (see fig. 1) in which $R = 3$: it gives results of

sufficient accuracy, though the actual value of the range of the α particle from these substances is nearly 3·5.

The curve is plotted from the following figures:—

D/R...	·067	·110	·167	·250	·333	·443	·568	·667	·833
i/I......	·773	·657	·532	·378	·262	·148	·069	·030	·010

CASE (b). *Very thin layer of radioactive material.*—Let D be the air equivalent of the layer of radioactive material, *i. e.* the loss of range which an α particle would experience in crossing the layer normally.

The limits of θ are now $\theta_1 + \theta_2$, where

$$D \sec \theta_1 + r = R \quad \text{and} \quad (D + D') \sec \theta_2 + r = R.$$

Hence the total number of particles whose ranges lie between r and $r + \delta r$ is

$$\frac{Nn_0}{4s} \left\{ \frac{(D+D')^2}{(R-r)^2} - \frac{D^2}{(R-r)^2} \right\} \delta r$$

$$= \frac{Nn_0 DD'}{2s} \frac{\delta r}{(R-r)^2}, \text{ neglecting } D'^2.$$

Hence the ionization (i)

$$= \frac{Nln_0}{2s} \int_0^{R-D} DD' \frac{\sqrt{r+d} - \sqrt{d}}{(R-r)^2} dr$$

$$= \frac{Nln_0 D'}{2s} \left\{ \sqrt{R+d-D} - \sqrt{d} - \frac{D}{\sqrt{R+d}} \log \frac{\sqrt{R}(\sqrt{R+d} + \sqrt{R+d-D})}{\sqrt{D}(\sqrt{R+d} + \sqrt{d})} \right\};$$

and when $D = 0$, the ionization (I)

$$= \frac{Nln_0 D'}{2s} \{ \sqrt{R+d} - \sqrt{d} \}.$$

If we had here neglected the variation of ionization with velocity and supposed the ionization caused in traversing a distance r to be proportional to r, we should have found that

$$i \propto \frac{Nn_0 D'}{s} \left(R - D + \frac{D}{R} \log \frac{D}{R} \right).$$

More difference is made in this case by neglecting the variation of ionization with velocity. Curve B is plotted from the more accurate formula for the values $R = 7$, $d = 1·33$.

The curve is plotted from the following figures:—

D/R ...	·061	·124	·250	·357	·500	·690	·833
i/I......	·807	·672	·467	·335	·193	·077	·023

CASE (c). *Layer of moderate thickness.*—Let the air equivalent of the radioactive layer be D'.

This case must be considered in two parts :—

(a) When r is such that $D + D' + r < R$: the limits of θ being then $\cos^{-1}\dfrac{D+D'}{R-r}$ and $\cos^{-1}\dfrac{D}{R-r}$, and those of r, 0 and $R - D - D'$.

(b) When r is such that $D + D' + r$ R, the limits of θ being then 0 and $\cos^{-1}\dfrac{D}{R-r}$, and those of r, 0 and $\cos^{-1}\dfrac{D}{R-r}$.

Hence

$$i = \frac{Nln_0}{4s}\int_{R-D-D'}^{R-D}\left\{1 - \frac{D^2}{(R-r)^2}\right\}(\sqrt{r+d}-\sqrt{d})dr$$

$$+ \frac{Nln_0}{4s}\int_0^{R-D-D'}\left\{\frac{(D+D')^2 - D^2}{(R-r)^2}\right\}(\sqrt{r+d}-\sqrt{d})dr.$$

and

$$\frac{4si}{Nln_0} = \tfrac{2}{3}(R+d-D)^{\frac{3}{2}} - \tfrac{2}{3}(R+d-D-D')^{\frac{3}{2}} - 2D'\sqrt{d} - D\sqrt{R+d-D}$$

$$+ (D+D')\sqrt{(R+d-D-D'} + \frac{D^2}{\sqrt{R+d}}\log\frac{\sqrt{D+D'}(\sqrt{R+d}+\sqrt{R+d-D})}{\sqrt{D}(\sqrt{R+d}+\sqrt{R+d-D-D'})}$$

$$- \frac{D'^2 + 2DD'}{\sqrt{R+d}}\log\frac{\sqrt{R}(\sqrt{R+d}+\sqrt{R+d-D-D'})}{\sqrt{D+D'}(\sqrt{R+d}+\sqrt{d})}$$

Curve C shows the result of plotting this formula for the case when $R = 3\cdot5$, $D' = \cdot5$, and $d = 1\cdot33$.

The curve is plotted from the following figures :—

D/R...	·057	·143	·200	·257	·380	·500	·714
i/I......	·833	·642	·539	·449	·288	·174	·044

PART II.

The apparatus employed was of the usual form, and very similar to that described by Rutherford ('Radioactivity,' 1905, p. 98). As shown in fig. 2, the material was laid on the high-potential plate B, at such a distance from the upper plate A that no α particle could reach it. Thus every α particle ran to its extreme range; and, to make more certain of catching all the ions, the upper plate was extended downwards at the sides.

Uranium.

The uranium was used in the form of the green oxide, U_3O_8, and was freed for the time from UrX. This was not necessary, but convenient, as it diminished the β ray correction.

Fig. 2.
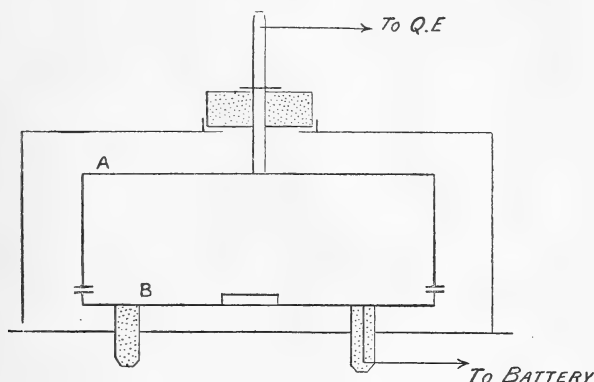
The uranium was ground to a fine powder and placed in a shallow depression turned in a metal plate, the diameter of the recess being 3·17 and the depth $\frac{1}{32}$ inch, which was far more than enough to make sure that the α rays from the lowest stratum could not get out. The surface of the material was carefully smoothed by the aid of a polished metal plate. A potential of 300 volts was used, which was nearly sufficient to saturate : more was not necessary, as only relative ionizations were in question. Aluminium-foil was used as the absorbing layer : the weight and area of each piece being measured so as to obtain the product of the density (ρ) and the thickness (d). In the following table (p. 762) the first column gives the value of ρd of the foil used, and the second the corresponding current, being the mean of five readings of the leak for ten seconds.

The last line shows that when two layers of tinfoil were added to the aluminium-foil already covering the uranium, the leak was reduced to 34. Each layer of foil was equivalent to about 17 mm. of air, and the aluminium to about 21; so that the whole cut off the α rays completely, for their range was known to be not more than 3·5 cm. This leak of 34 was therefore due to β rays and the normal leak of the apparatus. The third column shows the result of subtracting 34 from all the figures of the second column and reducing to a decimal of I. The numbers so obtained have then been

I. $\rho d \times 10^6$.	II. i.	III. $i/\text{I.}$	IV. From Curve A.	V. ρd of full range $\times 10^5$.
0	1044	1·000		
317	811	·768	·0705	448
633	635	·595	·139	462
949	494	·456	·205	463
1265	376	·339	·275	460
1620	275	·239	·353	458
1930	201	·165	·425	455
2610	97	·062	·580	449
3290	55	·021	·718	458
+2 layers of tinfoil.	34	0		

considered as so many ordinates of the thick-layer curve A; and the corresponding abscissæ found and placed in the fourth column. It was then possible to obtain from each reading a determination of the ρd of that aluminium sheet which the α particle of Ur could just penetrate. For example, the table shows that the sheet for which $\rho d = \cdot 000949$ is 20·5 per cent. of the necessary thickness, and so the ρd of the fully intercepting sheet is $\cdot 000949 / \cdot 205 = \cdot 00463$. All these separate determinations are shown in the fifth column. Their close agreement with each other is good evidence of the truth of the formula used in obtaining them.

Taking $\cdot 00456$ as the mean of the figures in the last column, the final result may be thus stated:—

The α particle of uranium can just penetrate a sheet of aluminium for which $\rho d = \cdot 00456$.

Although this may at once be interpreted so as to give the range of the α particle in air, yet it is better to measure the ranges of radium and thorium also in terms of aluminium-foil.

Radium.

The application of this method to thorium and radium is somewhat more difficult than in the case of uranium, since it is necessary for success that the radioactive material should issue α particles of one range only. It might no doubt be possible to allow for the existence of other α particles if their range and relative number were known. Some method of this kind must doubtless be used for Th X, and Th emanation. But it is obviously more direct, in the case of radium and thorium, to deal with the pure materials, if possible.

A very small quantity of radium bromide was evaporated on a platinum plate, which was then raised to a bright red heat for some minutes. This freed the radium from emanation, Ra A, and some Ra C. The ionization current due to this plate fell off quickly; and in three hours was down to half its first value. The remaining RaC had then disappeared. It was reheated, to get rid of any fresh emanation ; and it was then assumed that the great majority of the α particles emitted from the plate were due to radium itself (Bragg & Kleeman, Phil. Mag. Sept. 1905, p. 324).

The following table shows the result of experiment with the plate so prepared. The aluminium-foils were the same as those used in the uranium experiments. The curve employed was of course not the same, since the layer was thin. According to the experiments of Kleeman and myself such a layer may be considered as equivalent to 5 mm. of air. The results were therefore calculated from curve C.

I. $\rho d \times 10^6$.	II. i.	III. i/I.	IV. From Curve C.	V. ρd of full range $\times 10^5$.
0	1454	1·000		
317	1126	·772	·075	423
633	927	·634	·143	443
949	765	·522	·204	473
1265	645	·426	·265	477
1617	488	·331	·343	472
1933	388	·261	·407	476
2613	216	·150	·532	473
3289	104	·072	·675	488
Tinfoil	11	0		

These figures show an agreement between calculation and observation which is nearly as good as in the case of uranium. There is indeed a gradual increase in the figures of the last column; but such an effect should be expected, as there were present some α particles of longer range than those of radium, but none of shorter range. Neglecting this effect, the mean of the results of the last column is 466; and we may take this to express the product of the thickness and density of that aluminium sheet which the α particle of radium can just penetrate, with the reservation that it is probably too high by a small but uncertain amount, the error being caused by a slight want of purity in the material used. This statement

applies only in considering the relative ranges of uranium and thorium; other considerations enter when the method is used to determine ranges absolutely.

The falling of the first two results in column V. below the average of the column may in part be due to the fact that the first layer or two must cut off the easily absorbed radiation whose existence has been proved by J. J. Thomson and by Rutherford. I am not aware of any measurement of the amount of ionization due to this radiation. If in this experiment only 4 per cent. of the whole ionization current when the material is uncovered is supposed due to this cause, and if the foil whose $\rho d = \cdot 000317$ cuts off three quarters of it and the next addition of foil the remainder, then the figures in the last column become, in order, 488, 487, 487, 496, 486, 490, 502, 514. Thus the existence of a small quantity of radiation of this sort would explain the present discrepancies in the experiment. It will be seen later that a similar effect occurs with thorium. It is not so noticeable in the uranium experiment, as will be seen on turning back to the table of results. Still the first of these is rather smaller than the others; and a special measurement made with a very thin layer for which $\rho d = \cdot 000133$ gave a value for the full range equal to $\cdot 00426$, which is smaller still.

Ra C.

As a further test of the method, I have used it to determine the range of the a particle of Ra C. The special difficulty in this case was due to the rapid decline of the activity. It was avoided by taking readings of the current due to the uncovered active plate before and after each measurement with the covered plate. The geometric mean of these was taken and compared with the smaller measurement. The readings were properly spaced in time, so as to make this correct. The leak due to β rays and external causes was found by placing six sheets of tinfoil, each equivalent to about 15 mm. of air, over the active plate. Tinfoil was used as the absorbing sheet. The results were as follows:—

I. ρd of tinfoil $\times 10^6$	II. $i/\mathrm{I}.$	III. From Curve B.	IV. ρd of full range $\times 10^5.$
479	·518	34·2	224
958	·235	73·0	210
1437	·072	112·5	204

The results are in rather better agreement with theory than the last column seems at first sight to show ; for the first and third observations fell on such points of the curve that errors of measurement were magnified in calculating the result. The second measurement has the greatest chance of being accurate, and is very nearly the mean of the three. An independent experiment showed that tinfoil for which $\rho d =$ ·000536 was equivalent to 1·88 cm. of air. Hence the range of the a particle found by this method $= 210 \times 1\cdot88/535 = 7\cdot4$. The value found by Kleeman and myself, using the direct and more accurate method, was 7·1 nearly.

Thorium.

The material was used in the form of thorium oxide which had been freed as far as possible from other radioactive substances by means of the processes described by Rutherford and Soddy. The treatment employed, which included heating to a bright red heat as the final stage, was judged to have been successful for the following reasons. In the first place, the recovery of activity was not marked by an initial drop, so far as could be observed : in the second, it rose at a rate which showed that it would be halfway to the final value in four days, the final value being about four times the initial. In the third place, no emanation came off the material when first prepared ; even when no draught was employed the readings did not alter in 15 minutes : and in the fourth place, the observed results fitted closely to the calculated curves, showing only a slight variation as in the case of the radium.

The results of one experiment are shown in the following table :—

I. ρd of Al foil $\times 10^6$.	II. $i/\mathrm{I}.$	III. From Curve A.	IV. ρd of full range $\times 10^5$.
244	·813	·055	444
474	·670	·108	439
780	·544	·162	480
1061	·412	·227	468
1573	·271	·328	480
2073	·173	·417	499
2607	·106	·504	517

The mean of the figures in the last column is ·00477.

In another experiment the thorium was precipitated twice at intervals of two days and then five times at intervals of 12 hours.　The results were as follows :—

I. ρd of Al foil $\times 10^6$.	II. i/I.	III. From Curve A.	IV. ρd of full range $\times 10^5$.
534	·655	·114	470
1046	·425	·221	473
1633	·248	·347	471
2133	·154	·438	486

In this case the mean of the figures in the last column is ·00475.　As in the case of radium, this result is probably a little too high as it is impossible to get rid of all the radioactive products of thorium, and all these have ranges higher than thorium itself.　For Rutherford has shown that the *a* particle of the induced activity of thorium has the same penetrating power as the α particle of the induced activity of radium ; and some rough experiments which I have made with ThX go to show that, as in the case of Ra, the second and third active products have ranges intermediate between the first and fourth [*].

It may also be calculated from an experiment of Rutherford's ('Radioactivity' 2nd ed. p. 263) that the range of the emanation α particle is about 6 cm. ; but there is some uncertainty as to the stopping power of the mica sheet which he used.

The general conclusion is therefore that uranium, thorium, and radium eject α particles of nearly, if not exactly, the same speed.　Considering the many parallelisms already known to exist between the processes of disintegration of these substances and their products, this new fact is certainly suggestive.　It would be very interesting to know the ranges of Th X and Th emanation.

An expression is found in Part I. of this paper for the total ionization over an uncovered deep layer of active

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[*] An experiment by Schmidt (*Phys. Zeit.* No. 25, p. 897) has shown that Ra A has two-thirds of the penetrating power of Ra C.　Hence its range must be the longer of the two intermediate ranges, determined by Kleeman and myself, viz. 4·83 ; and the range of the emanation must be 4·23.　Thus in the radioactive sequence each explosion is more violent than the last.

material. By its aid we may find the relative numbers of α particles emitted by uranium and thorium, when the ionization currents due to known areas of the layers have been measured. Since the ranges are so nearly alike, it is sufficient to use the simpler formula

$$I = \frac{N l' n_0}{8s} \cdot R^2.$$

If now the suffixes U and T refer to uranium and thorium, we have

$$\frac{I_U}{I_T} = \frac{N_U R_U^2 s_T}{N_T R_T^2 s_U};$$

and therefore

$$\frac{N_T}{N_U} = \frac{T_T R_U^2 s_T}{T_U R_T^2 s_U}.$$

Each time that a thorium experiment was completed, a comparison was made of the currents I_T and I_U. In the first case I_T/I_U was found to be ·234, in the second ·234. $(R_U/R_T)^2$, as may be seen from the results given above, can be taken as equal to $(456/476)^2 = ·916$.

Also
$$\frac{s_T}{s_U} = \frac{\sqrt{232} + 2\sqrt{16}}{\sqrt{239} + \frac{8}{3}\sqrt{16}} = \frac{23·2}{26·2},$$

assuming the square-root law (Bragg and Kleeman, Phil. Mag. Sept. 1905) to hold for uranium and thorium.

Hence finally

$$\frac{N_T}{N_U} = ·234 \times ·916 \times \frac{23·2}{26·2}$$
$$= ·190.$$

This result may be a little too small, since the range of the α particle of thorium may be slightly overestimated. The square of the range enters into the formula of comparison; but on the other hand, any α rays of long range which have not been removed from the thorium would make I_T too large. On the whole, therefore, the actual value cannot be far from ·20; *i. e.* the uranium atoms break down very nearly five times as fast as the thorium.

I have preferred to make the method one of comparison of ranges rather than of absolute determination. For there are two or three difficulties in using it for the latter purpose. In

the first place, it is not easy to make the thin aluminium-leaf lie very close to the radiating surface ; and the layers of air close to the surface contribute a relatively large number of ions.   To make this error uniform I have used a net of very fine wires, with a mesh of $\frac{3}{8}$ of an inch to keep the foils down. Again, there is a disturbing effect due to the secondary ionization of the absorbing sheet.   Mme. Curie has called attention to effects of this kind (Rutherford, ' Radioactivity,' 1905, p. 189).   I find that there is slightly more ionization when, of two layers of foil, Al and Sn, the latter is on top. Using tinfoil, the range always comes out rather larger than when aluminium-foil is employed.   This may possibly be because the tinfoil lies closer to the surface.   All these effects are small, and disappear in a comparison method.

By direct measurement I found that aluminium-foil for which $\rho d = \cdot 00329$ was equivalent to 2·30 cm. of air.   Hence the range of the α particle of radium as measured by this method $= 2\cdot 30 \times 466/329 = 3\cdot 26$.   The actual value, as found directly, is 3·54.

One other difficulty lies in the way of a direct determination. As has already been mentioned by Kleeman and myself (Phil. Mag. Sept. 1905), the loss of range of the α particle of Ra C in going through a given sheet of material appears from the ionization curves to be slightly greater than the loss of an α particle of Ra A; and it is not quite clear whether this difference is real or apparent.

I owe my grateful thanks to Dr. W. T. Cooke who carried out for me all the necessary chemical operations.

Since the above was written I have received the February number of the Philosophical Magazine containing an article by Mr. N. R. Campbell on "The Radiation from Ordinary Materials."   In finding the formulæ necessary to his investigation, Mr. Campbell has anticipated part of the work in Part I. of this paper ; but as the fuller treatment which I have given is required for my own experiment, I have thought it better to allow it to stand without alteration.

In a footnote Mr. Campbell expresses his inability to understand why I inserted an obliquity factor $\cos \theta$ into the preliminary calculations in my first paper on the α rays (Phil. Mag. Dec. 1904).   The mistake is mine.   I did not discover it until I had occasion to reconsider the matter in connexion with the present investigation.

LXX. *On the Number of Corpuscles in an Atom.*
   By Prof. J. J. Thomson, *M.A., F.R.S.**

I CONSIDER in this paper three methods of determining the number of corpuscles in an atom of an elementary substance, all of which lead to the conclusion that this number is of the same order as the atomic weight of the substance. Two of these methods show in addition that the ratio of the number of corpuscles in the atom to the atomic weight of the element is the same for all elements. The data at present available indicate that the number of corpuscles in the atom is equal to the atomic weight. As, however, the evidence is rather indirect and the data are not very numerous, further investigation is necessary before we can be sure of this equality; the evidence at present available seems, however, sufficient to establish the conclusion that the number of corpuscles is not greatly different from the atomic weight.

It will be seen that the methods are very different and deal with widely separated physical phenomena; and although no one of the methods can, I think, be regarded as quite conclusive by itself, the evidence becomes very strong when we find that such different methods lead to practically identical results.

To enable the argument to be more easily followed, I shall begin with a general description of the methods and the results to which they lead, and postpone the details of the theory of each of the methods to the latter part of the paper.

The first method is founded on the dispersion of light by gases. If we regard an atom as consisting of a number of corpuscles dispersed through a sphere of uniform positive electrification, it is evident that the dispersive power of a medium consisting of these atoms will depend upon the mass of the positive electrification as well as upon the mass of the corpuscles, and will vanish if either of these masses is zero. For consider what takes place when the electric force in the light-wave strikes the atom. Since the wave-length is large compared with an atom, the latter may be regarded as being in a uniform electric field; under this field the corpuscles will be displaced in one direction, the positive sphere in the opposite; and if the force persists long enough, this displacement will go on until the force exerted on the corpuscles in their displaced position by the positive electricity is equal and

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* Communicated by the Author.

opposite to the force exerted on them by the electric force in the light-wave. The displacement of the corpuscles relatively to the sphere of positive electricity will polarize the atom, and the collection of polarized atoms will increase the specific inductive capacity, and therefore the refractive index of the medium. If the mass of either the positive electricity or of the corpuscles were zero, then, however short the time for which the electric force in the light-wave acted, the corpuscles and the positive electricity would adjust themselves in exactly the same way as if the electric force were continuous; so that the specific inductive capacity and the refractive index would be the same for short waves as for long, and there would be no dispersion. If, however, the masses of both the positive electricity and the corpuscles are finite, the relative displacement of the corpuscles and the positive electricity will depend upon the period of the electric force; and since the specific inductive capacity and refractive index depend upon this displacement, the refractive index of the medium will depend upon the period of the electric force, and there will be dispersion.

In the latter part of the paper the expression for the refractive index of a monatomic gas is investigated; and it is shown that if μ is the refractive index of such a gas for light of frequency p, then

$$\frac{\mu^2-1}{\mu^2+2} = \frac{\frac{4}{3}\pi N E(Me+mE)}{\frac{4}{3}\pi\rho(Me+mE)-Mmp^2}, \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

where N is the number of atoms in unit volume of the gas, m the mass of a corpuscle, M the mass of the sphere of positive electrification, e the charge on a corpuscle, E the whole charge on the sphere of positive electrification, ρ the density of the electrification in this sphere; e, E, and ρ are expressed in electrostatic measure.

If the term in p^2 is small, equation (1) may be written

$$\frac{\mu^2-1}{\mu^2+2} = \frac{NE}{\rho}\left(1 + \frac{M}{E}\frac{m}{e}\frac{p^2}{M+nm}\frac{3E}{4\pi\rho}\right),$$

since $E = ne$, where n is the number of corpuscles in the atom. If a is the radius of the sphere of positive electrification, $E = \frac{4}{3}\pi\rho a^3$, *i. e.* $\frac{NE}{\rho} = N\frac{4}{3}\pi a^3 =$ volume of the atoms per cubic centimetre of gas; this is the value of μ^2-1/μ^2+2 when $p=0$, *i. e.* for infinitely long waves. Writing P_0 for

this quantity, we see, if λ is the wave-length of the light,

$$\frac{\mu^2-1}{\mu^2+2}=\mathrm{P}_0+\mathrm{P}_0{}^2\frac{\mathrm{M}}{\mathrm{E}'}\frac{m}{e'}\frac{1}{\mathrm{N}(\mathrm{M}+nm)}\frac{3\pi}{\lambda^2}, \quad . \quad . \quad (2)$$

where E' and e' are the values of E and e in electromagnetic measure. This is the expression for the refractive index of a monatomic gas. I have not been able to find any determinations of the dispersion of such gases. Lord Rayleigh, however, found that the dispersion of helium was of the same *order* as that of diatomic gases. If the atoms in the molecules of a diatomic gas are not charged, the preceding expression will hold for the refractive index of such a gas ; if, however, the atoms carry electrical charges, the theory subsequently given shows that this expression has to be modified. We know that as a matter of fact the atomic refraction of some elements, oxygen for example, depends upon the kind of compound in which the oxygen is found. This variation may be ascribed to the charges carried by the atoms in the molecules. The atomic refraction of hydrogen seems, however, to be constant; and I shall, in the absence of data for monatomic gases, apply the preceding formula to this gas.

From Ketteler's measurements of the refractive index of hydrogen for light of different wave-lengths, we find that for hydrogen at atmospheric pressure

$$\frac{\mu^2-1}{\mu^2+2}=\frac{1}{3}\left\{2{\cdot}8014\times10^{-4}+\frac{2\times10^{-14}}{\lambda^2}\right\}.$$

Comparing this with the equation

$$\frac{\mu^2-1}{\mu^2+2}=\mathrm{P}_0+\mathrm{P}_0{}^2\frac{\mathrm{M}}{\mathrm{E}'}\frac{m}{e'}\frac{1}{\mathrm{N}(\mathrm{M}+nm)}\frac{3\pi}{\lambda^2},$$

we find

$$\frac{\mathrm{M}}{\mathrm{E}'}\frac{m}{e'}\frac{1}{\mathrm{N}(\mathrm{M}+nm)}=6\times10^{-8};$$

but $\quad m/e'=1/1{\cdot}7\times10^7 \quad$ and $\quad \mathrm{N}e'={\cdot}8$; hence

$$\frac{\mathrm{M}}{\mathrm{M}+nm}\frac{e'}{\mathrm{E}'}=1,\text{ approximately ;}$$

or, since $\mathrm{E}'=ne'$,

$$\frac{\mathrm{M}}{\mathrm{M}+nm}\frac{1}{n}=1,\text{ approximately.}$$

This result shows (1) that n cannot differ much from unity, and (2) that M, the mass of the carriers of positive electricity,

cannot be small compared with nm, the mass of the carriers of negative electricity. Hence we infer that n, the number of corpuscles in a hydrogen atom, is not much greater than unity. This result has been deduced from the consideration of the properties of a diatomic molecule; and if the atoms in the molecule were charged, the expression for $(\mu^2-1)/(\mu^2+2)$ would have to be modified; but since the dispersion of helium is by Lord Rayleigh's result comparable with that of hydrogen, we see, since the dispersion is proportional to $\dfrac{M}{M+nm}\dfrac{1}{n}$, that there cannot be a very large number of corpuscles in the helium atom; for if n were large, the dispersion of helium would be far too small.

2nd Method. Scattering of Röntgen Radiation by Gases.— It is shown in my 'Conduction of Electricity through Gases' that when Röntgen rays pass through a medium in which there are N corpuscles per cubic centimetre, the energy in the radiation scattered per cubic centimetre of the medium is $\dfrac{8\pi}{3}\dfrac{Ne^4}{m^2}$ E, where E is the energy of the primary radiation passing through the cubic centimetre, e the charge, and m the mass of the corpuscle. Barkla has shown that in the case of gases the energy in the scattered radiation always bears, for the same gas, a constant ratio to the energy in the primary whatever be the nature of the rays, *i. e.* whether they are hard or soft; and secondly, that the scattered energy is proportional to the mass of the gas. The first of these results is a confirmation of the theory, as the ratio of the energy scattered to that in the primary rays is $\dfrac{8\pi}{3}$ N(e^4/m^2), and is independent of the nature of the rays; the second result shows that the number of corpuscles per cub. centim. is proportional to the mass of the gas: from this it follows that the number of corpuscles in an atom is proportional to the mass of the atom, *i. e.* to the atomic weight. Barkla measured the ratio of the energy in the scattered radiation to that in the primary in the case of air, and found that it was equal to $2 \cdot 4 \times 10^{-4}$. Thus, for air

$$\frac{8\pi}{3}\frac{Ne^4}{m^2} = 2 \cdot 4 \times 10^{-4}.$$

Now $e/m = 1 \cdot 7 \times 10^7$ and $e = 1 \cdot 1 \times 10^{-20}$; hence

$$Ne = 10.$$

But if n is the number of molecules per c. c.,

$$ne = \cdot 4 ;$$

hence

$$N = 25n.$$

From this we deduce that there are 25 corpuscles in each molecule of air, and this indicates that the number of corpuscles in the atom is equal to the atomic weight ; for the scattering by air is very nearly the same as that by nitrogen, and 25, the number of corpuscles in the molecule deduced from Barkla's experiment, is near to 28, the number in each molecule if the number in the atom were equal to the atomic weight.

3rd Method. Absorption of β Rays.—We regard the absorption of the β rays as due to the effect of the collisions between these rays and the corpuscles which they meet with in their path through the absorbing substance. If λ is the coefficient of absorption, it is shown in the latter part of the paper that

$$\lambda = \frac{4\pi N e^4}{m^2} \frac{V_0^4}{V^4} \log\left\{ \frac{1}{2} \frac{aV^2}{V_0^2} \frac{m}{e^2} - 1 \right\},$$

where N is the number of corpuscles per cubic centimetre, V the velocity of the β particles, V_0 the velocity of light, e the charge on a corpuscle in electromagnetic measure, m the mass of a corpuscle, and a a length comparable with the distance between the corpuscles in an atom.

If δ is the density of the absorbing substance, M the mass of an atom, n the number of corpuscles in the atom, we have $\frac{N}{n} M = \delta$; so that

$$\lambda = \delta \cdot 4\pi \frac{e^2}{m^2} \frac{en}{M} \cdot e \frac{V_0^4}{V^4} \log\left(\frac{1}{2} \frac{aV^2}{V_0^2} \frac{m}{e^2} - 1 \right).$$

Now λ/δ is approximately constant whatever be the nature of the absorbing substance; hence since the logarithmic term only varies slowly, we conclude that n must be proportional to M, *i. e.* that the number of corpuscles in an atom is proportional to the atomic weight. To find the number of corpuscles in an atom, let us apply the formula to the case of the β particles from uranium, for which, as Becquerel has shown, $V = 1\cdot6 \times 10^{10}$; and Rutherford finds that for copper and silver $\lambda/\delta = 7$. Putting $e/m = 1\cdot7 \times 10^{10}$, $e = 10^{-20}$, $V_0 = 3 \times 10^{10}$, we get

$$\frac{ne}{M} = \frac{1\cdot4 \times 10^4}{\log\left(\frac{mV^2}{V_0^2} \frac{a}{e^2} - 1 \right)}.$$

The value of the logarithmic term is somewhat uncertain, involving as it does the indeterminate quantity a ; it cannot, however, be large enough to alter the order of the term on the right-hand side.   If M′ is the mass of the hydrogen atom,

$$\frac{e}{M'} = 10^4 ;$$

hence we have

$$n = \frac{M}{M'} \frac{1 \cdot 4}{\log \left(\frac{m V^2}{V_0^2} \frac{a}{e^2} - 1 \right)} ;$$

i. e., n is of the same order as M/M′ the atomic weight of the element.

Thus these three very different methods all lead to the result that the number of corpuscles in the atom of an element is of the same order as the atomic weight of the element ; and from the first method we conclude that the mass of the carrier of unit positive charge is large compared with that of the carrier of unit negative charge.   If we suppose the whole mass of an atom to be that of its charged parts, e/m for positive unit charge would be of the order 10^4.

An obvious argument against the number of corpuscles in the atom being as small as these results indicate, is that the number of lines showing the Zeeman effect, which must therefore be due to the vibrations of corpuscles, in the spectrum, say, of iron is very much greater than the atomic weight of iron.   This objection would be conclusive if it could be shown that all these lines are due to the vibrations of corpuscles inside the normal atom of iron; but I submit that there is no evidence that this is the case.   When an atom of an element is giving out its spectrum either in a flame or in an electric discharge, it is surrounded by a swarm of corpuscles; and combinations, not permanent indeed, but lasting sufficiently long for the emission of a large number of vibrations, might be expected to be formed.   These systems would give out characteristic spectrum-lines; but these lines would be due, not to the vibrations of corpuscles inside the atom, but of corpuscles vibrating in the field of force outside the atom.   Such lines would not be reversed by cold vapour, though they might be by very hot vapours, by the vapours in flames or in the neighbourhood of an electric discharge : the number of lines showing the Zeeman effect reversed by cold vapours is, however, very limited.

We shall now proceed to consider the theory of the method on which the preceding results are based.

Index of Refraction of a Collection of Atoms.

If an atom consisting of corpuscles dispersed through a sphere of uniform positive electrification is in the path of a wave of light, the electric force in the wave will displace the corpuscles in the atom; the motion of these charged corpuscles will produce a magnetic field in addition to that in the wave before it struck the corpuscle ; the existence of this field will alter the velocity of propagation of the wave by an amount which we shall attempt to calculate.

Index of refraction of a monatomic gas whose atoms contain as much positive as negative electricity.—Consider an element of volume so small that throughout it the electric force in the wave may be regarded as constant. Throughout this volume the atoms will all be affected by the electric force in the same way.

If ξ_r, η_r, ζ_r are the displacements parallel to the axes of x, y, z of the rth corpuscle of an atom, x the displacement of the centre of the sphere of positive electrification, e the negative charge on a corpuscle, E the charge of positive electrification in the sphere, N the number of atoms per unit volume; then X′, Y′, Z′, the components of the electric force due to the displacement of the corpuscles, are given by the equations

$$X' = \tfrac{4}{3}\pi N(Ex - \Sigma e\xi_r) \left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$$
$$Y' = \tfrac{4}{3}\pi N(Ey - \Sigma e\eta_r) \left.\begin{array}{l}\end{array}\right\} ; \quad . \quad . \quad . \quad . \quad (1)$$
$$Z' = \tfrac{4}{3}\pi N(Ez - \Sigma e\zeta_r)$$

the summation is for all the corpuscles in one atom.

The equations of motion for the corpuscles and sphere of positive electrification are

$$M \frac{d^2x}{dt^2} = (X + X')E - \tfrac{4}{3}\pi\rho e\Sigma(x - \xi_r),$$

$$m \frac{d^2\Sigma\xi_r}{dt^2} = -(X + X')E + \tfrac{4}{3}\pi\rho e\Sigma(x - \xi_r),$$

where m is the mass of a corpuscle and M that of the sphere of positive electrification.

If all the quantities vary as $\epsilon^{\imath pt}$, we get from these equations

$$x = \frac{(X + X')Em}{\tfrac{4}{3}\pi\rho(Me + mE) - mMp^2},$$

$$\Sigma\xi_r = -\frac{(X + X')EM}{\tfrac{4}{3}\pi\rho(Me + mE) - mp^2}.$$

Therefore from equation (1) we have

$$X' = \frac{\frac{4}{3}\pi N(X+X')(mE^2 + MEe)}{\frac{4}{3}\pi\rho(Me+mE)-mMp^2},$$

or

$$X' = \frac{PX}{1-P};$$

where

$$P = \frac{\frac{4}{3}\pi N(mE^2 + MEe)}{\frac{4}{3}\pi\rho(Me+mE)-mMp^2}.$$

In consequence of the motion of the charged corpuscles, the current is no longer the polarization-current $\frac{K_0}{4\pi}\frac{dX}{dt}$, where K_0 is the specific inductive capacity of the æther; but this current *plus* the convection current $N\left(E\frac{dx}{dt} - e\Sigma\frac{d\xi_r}{dt}\right)$; thus u the total current parallel is given by the equation

$$u = \frac{K_0}{4\pi}\frac{dX}{dt} + \frac{3}{4\pi}\frac{dX'}{dt}.$$

If α, β, γ are the components of the magnetic force,

$$4\pi u = \frac{d\beta}{dz} - \frac{d\gamma}{dy},$$

and

$$\frac{d\beta}{dt} = \frac{dX}{dz} - \frac{dZ}{dx}, \quad \frac{d\gamma}{dt} = \frac{dY}{dx} - \frac{dX}{dy}.$$

Hence, since

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

we have

$$4\pi\frac{du}{dt} = \frac{d^2X}{dx^2} + \frac{d^2X}{dy^2} + \frac{d^2X}{dz^2};$$

or

$$K_0\frac{d^2X}{dt^2} + \frac{3d^2X_1}{dt^2} = \frac{d^2X}{dx^2} + \frac{d^2X}{dy^2} + \frac{d^2X}{dz^2},$$

$$K_0\frac{d^2X}{dt^2} + \frac{3P}{1-P}\frac{d^2X}{dt^2} = \frac{d^2X}{dx^2} + \frac{d^2X}{dy^2} + \frac{d^2X}{dz^2}.$$

Hence, if μ is the refractive index,

$$\mu^2 = 1 + \frac{3P}{1-P},$$

or

$$\frac{\mu^2-1}{\mu^2+2} = P = \frac{\frac{4}{3}\pi N(mE^2 + mEe)}{\frac{4}{3}\pi\rho(Me+mE)-mMp^2}. \quad \cdot \quad \cdot \quad (2)$$

For very long waves, when the term mMp^2 may be neglected, we have

$$P = \frac{\frac{4}{3}\pi NE}{\frac{4}{3}\pi\rho};$$

or, since $E = \frac{4}{3}\pi\rho a^3$, where a is the radius of the sphere of positive electrification,

$$P = \frac{\mu^2 - 1}{\mu^2 + 2} = \frac{4}{3}\pi N a^3.$$

This is the value given by Mossotti's theory when the atoms are assumed to be perfectly conducting spheres of radius a.

Case of a diatomic molecule, the atoms carrying electrical charges equal in magnitude and opposite in sign.—In this case the refractivity P will contain a term due to the motion of the charged atoms relatively to each other under the electric field due to the light-wave. We can calculate this term as follows :—Let M_1, M_2 be the masses of the two atoms, E' the charge on the first atom, $-E'$ that on the other. Let x_1 be the displacement of the centre of the first atom, x_2 that of the other. The equations of motion will be of the form

$$M_1 \frac{d^2 x_1}{dt^2} = XE' - \phi(D)(x_1 - x_2),$$

$$M_2 \frac{d^2 x_2}{dt^2} = -XE' - \phi(D)(x_2 - x_1),$$

where $\phi(D)$ is a function of D, the distance between the atoms, the value of this function depending on the law of force. If the variables as before vary as $\epsilon^{\prime p i}$, we find

$$x_1 = \frac{M_2 E' X}{(M_1 + M_2)\phi(D) - M_1 M_2 p^2},$$

$$x_2 = -\frac{M_1 E' X}{(M_1 + M_2)\phi(D) - M_1 M_2 p^2}.$$

The contribution to P of the motion due to the coordinates x_1, x_2 is proportional to

$$N'\left(E'\frac{dx_1}{dt} - E_1'\frac{dx_2}{dt} \right)$$

or to

$$\frac{N' E_1^2 (M_1 + M_2)}{(M_1 + M_2)\phi(D) - M_1 M_2 p^2}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3)$$

where N' is the number of these diatomic molecules per unit volume. The consideration of this expression shows that

the part of the refractivity arising from the coupling of two atoms together may easily be comparable with the part due to the corpuscles within the atom. Thus, to take the case when the waves are so long that we may neglect the term in p^2, the contribution to the refractivity due to the coupling is

$$\frac{N'E_1{}^2}{\phi(D)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the force between the atoms changed as slowly as the force between the two charges E' and $-E'$ at a distance D, $\phi(D)$ would equal $2E'^2/D^3$, and (1) would become

$$\frac{N'D^3}{2}.$$

If we compare this with $2N'a^3$, the value due to the corpuscles inside the atom, a being the radius of the atom, we see that unless the force between the two atoms varies very rapidly with the distance, a considerable part of the refractivity may be due to the coupling between the atoms.

If ΔP_0 is the part of $(\mu^2-1)/(\mu^2+2)$ when λ is infinite due to the charges on the atoms in the diatomic molecule, we see from equation (3) that the part of $(\mu^2-1)/(\mu^2+2)$ due to these charges is approximately equal to

$$\Delta P_0 + (\Delta P_0)^2 \frac{M_1}{E_1'} \frac{M_2}{E_1'} \frac{1}{(M_1+M_2)} \frac{4\pi^2}{N\lambda^2}, \quad . \quad . \quad (5)$$

where E_1' is the value of the charge in electromagnetic units.

In the case of a molecule consisting of two charged atoms, the charge on the negative atom will be due to the presence on the atom of extra corpuscles which can move freely about. Thus, if M_2 refers to the negative charge, M_2/E_1' will equal m/e_1, where m is the mass and e the charge on a corpuscle ; for the positive atom M_1 will equal the mass of the atom, while E_1 will equal e_1 if the atoms are monovalent, $2e_1$ if they are divalent, and so on. Comparing the part of the coefficient of $1/\lambda^2$ which is due to the charge on the atoms, with that (given by equation (2)) due to the corpuscles inside the atom, we see that the factor M_2/E_1' in (5) is the same as m/e' in (2) ; while if there are many corpuscles in the atom, the factor $\dfrac{M_1}{E_1'} \dfrac{1}{M_1+M_2}$ will be much larger than $\dfrac{M}{E(M+nm)}$, for E_1' will only be a small multiple of e_1, while E is equal to ne_1, where n is the number of corpuscles in the atom. Thus, unless ΔP_0 is very small compared with P_0, the dis-

persion of the gas will depend more on the charges of the atoms in a diatomic gas than on the corpuscles inside the individual atoms.

The theory of the second method is given in my ' Conduction of Electricity through Gases.' We proceed to the consideration of the third method, which depends on the absorption by matter of rapidly moving corpuscles.

If we suppose that an atom consists of a number of corpuscles distributed through positive electrification, we can find an expression for the absorption experienced by the corpuscles when they pass through a collection of a large number of such atoms. The rapidly moving corpuscle will penetrate the atom, and will be deflected when it comes near an inter-atomic corpuscle by the repulsion between the corpuscles. This deflexion will produce an absorption of the cathode particles. If the corpuscle in the atom is held fixed by the forces acting upon it, the colliding corpuscle will, after the collision, have the same velocity as before, though the direction of its motion will be deflected. If the inter-atomic corpuscle A is not fixed, the colliding corpuscle B will communicate some energy to it and will itself go off with diminished energy. Without solving the very complicated problem which presents itself when we take into account the forces exerted on A by the other corpuscles, we can form some idea of the effects produced by the constraint introduced by such forces by following the effects produced by increasing the mass of A. The general effect of great constraint would be represented by supposing the mass of A to be very large, while absence of constraint would be represented by supposing the mass of A to be equal to that of B.

Let M_1, M_2 be the masses of the corpuscles A and B respectively. We shall suppose that the velocity of the colliding corpuscle is so great that in comparison the corpuscles in the atom may be regarded as at rest. Let V be the velocity of A before the collision, b the perpendicular let fall from A on V. If 2θ is the angle through which the direction of relative motion is deflected by the collision, we can easily show that

$$\sin^2\theta = \cfrac{1}{1 + \cfrac{b^2 V^4}{e^4}\left(\cfrac{M_1 M_2}{M_1 + M_2}\right)^2};$$

the force between two corpuscles separated by a distance r being assumed equal to e^2/r^2. Hence, if u, u' are the

3 E 2

velocities of B parallel to x before and after the collision,

$$u' - u = -\frac{M_1 u}{M_1 + M_2} 2 \sin^2 \theta + \frac{M_1}{M_1 + M_2} \sin 2\theta \cos \phi \sqrt{V^2 - u^2},$$

where ϕ is the angle between the plane containing b and V and that containing V and x. Averaging, the term containing $\cos \phi$ will disappear, and we have

$$u' - u = -\frac{2M_1 u}{M_1 + M_2} \frac{1}{1 + \dfrac{b^2 V^4}{e^4} \left(\dfrac{M_1 M_2}{M_1 + M_2}\right)^2}.$$

If there are N of the interatomic corpuscles per unit volume, the number of collisions in which b is between b and $b + db$ made by a corpuscle B when it travels over a distance Δx is $N \Delta x \cdot 2\pi b \cdot db$. Hence, if U is the sum of the values of u for the B corpuscles per unit volume, and $\Delta(U)$ the change in U in the distance Δx,

$$\Delta(U) = -2 U N \cdot \Delta x \cdot \frac{M_1}{M_1 + M_2} \int_0^{b_1} \frac{2\pi b\, db}{1 + \dfrac{b^2 V^4}{e^2}\left(\dfrac{M_1 M_2}{M_1 + M_2}\right)^2}; \quad (6)$$

the upper limit being determined by the condition that B comes into collision with the A corpuscles one at a time, so that the shortest distance between B and the corpuscle with which it comes into collision must be small compared with a, the distance between two corpuscles. If r is the shortest distance between the A and B corpuscles, we can easily show that

$$1 - \frac{b^2}{r^2} = \frac{2e^2}{V^2 r} \frac{M_1 + M_2}{M_1 M_2}.$$

Putting $r = a$, we see that b' is of order

$$a\left(1 - \frac{2e^2}{V^2 a} \frac{M_1 + M_2}{M_1 M_2}\right)^{\frac{1}{2}}.$$

Integrating the expression on the right-hand side of equation (6), we get

$$\frac{dU}{dx} = -2U \frac{N \cdot M_1}{M_1 + M_2} \frac{e^4 (M_1 + M_2)^2}{V^4 (M_1 M_2)^2} \log\left(1 + \frac{b_1^2 V^4}{e^2}\left(\frac{M_1 M_2}{M_1 + M_2}\right)^2\right).$$

Since the logarithmic term only varies slowly, we may put for b_1 any quantity of the same order without greatly affecting the result, putting

$$b' = a\left(1 - \frac{2V^2}{e^2 a} \frac{M_1 + M_2}{M_1 M_2}\right)^{\frac{1}{2}}$$

$$\frac{d}{dx}(U) = -U \frac{4\pi N e^4}{V^4} \frac{(M_1 + M_2)}{M_1 M_2{}^2} \log\left(\frac{a V^2}{e^2} \frac{M_1 M_2}{M_1 + M_2} - 1\right).$$

Thus U, the number of corpuscles crossing unit area in unit time, varies as $\epsilon^{-\lambda x}$, where

$$\lambda = \frac{4\pi N e^4}{V^4} \frac{(M_1 + M_2)}{M_1 M_2{}^2} \log \frac{(aV^2}{e^2} \frac{M_1 M_2}{M_1 + M_2} - 1\Big) ;$$

λ is the coefficient of absorption. If $M_1 = M_2$, *i. e.* if the corpuscles are quite free to move in the atom,

$$\lambda = \frac{8\pi N e^4}{V^4 M_2{}^2} \log \Big(\frac{1}{2} \frac{M_2 a V^2}{e^2} - 1 \Big).$$

If M_1 is infinite, *i. e.* if the corpuscles are held fixed by the forces between them, we have

$$\lambda = \frac{4\pi N e^4}{V^4 M_2{}^2} \log \Big(\frac{1}{2} \frac{M_2 a V^2}{e^2} - 1 \Big),$$

the value used in Method 3.

We can get an approximate value of λ very simply by the following method. Consider a stream of corpuscles moving horizontally ; the forward motion of any particle will be stopped by a collision in which its direction of motion is turned through an angle equal to or greater than a right angle ; *i. e.* if θ is equal to or greater than $\frac{\pi}{4}$ or $\sin^2 \theta > \frac{1}{2}$, $\sin^2 \theta$ will be greater than $1/2$ if

$$b^2 < \frac{e^4}{V^4} \Big(\frac{M_1 + M_2}{M_1 M_2} \Big)^2 .$$

The number of collisions made by a corpuscle for which b is not greater than this value, as the corpuscle moves over a distance Δx, is

$$\frac{\pi N e^4}{V^4} \Big(\frac{M_1 + M_2}{M_1 M_2} \Big)^2 \Delta x.$$

Hence, if U is the number of corpuscles crossing unit area in unit time, we have, if we neglect the effect of collisions which do not result in a total stoppage of the particle,

$$\frac{dU}{dx} = - U \cdot \frac{N \pi e^4}{V^4} \Big(\frac{M_1 + M_2}{M_1 M_2} \Big)^2,$$

or, if λ is the coefficient of absorption,

$$\lambda = \frac{N \pi e^4}{V^4} \Big(\frac{M_1 + M_2}{M_1 M_2} \Big)^2 .$$

LXXI. *Fluorescence and Lambert's Law.* *By* R. W. Wood, *Professor of Experimental Physics, Johns Hopkins University* *.

AS is well known, the intensity of the radiation emitted from a surface element of a white-hot solid or liquid varies as the cosine of the angle of emission. The result of this circumstance is that the intrinsic intensity, or apparent luminosity of the surface, is independent of the direction in which it is viewed, not being increased by foreshortening of the surface. The same law does not hold in the case of a transparent gas, the intensity of a flat gas-flame increasing as it is turned edgewise. In this case the radiation of a surface element is constant in all directions, not varying with the cosine of the angle. It is worthy of note that the emission of X rays is governed by the same law that holds in the case of a gas, as can be shown by making photographs of the anti-cathode of an X-ray tube from directions normal to, and at nearly grazing incidence with the surface, with a pin-hole camera. This is rather what we should expect, for the radiations originate at the points where the electrons collide with the target, in other words in free space.

The radiations from fluorescent surfaces sometimes appear to be governed by the same law. If a rectangular glass tank, or even a beaker glass, is partly filled with a solution of uranin (fluorescein) and a condenser discharge passed between cadmium electrodes close to the surface, the phenomenon can be very clearly seen. The surface is powerfully fluorescent, and if it be viewed from below, the intrinsic intensity will be found to increase rapidly as the surface is foreshortened, becoming of dazzling brilliancy at grazing emission. If a glass plate is interposed between the spark and the fluid, the effect of foreshortening becomes less marked or disappears entirely, for in this case the fluorescence is chiefly caused by the radiations which penetrate the body of the fluid, and the powerful surface fluorescence, excited by the ultra-violet rays, disappears. A still better method is to illuminate one face of a right-angle prism of crown glass with the light of the spark, which causes a blue fluorescence of the surface-layer. The luminous surface is to be viewed through the other face of the prism. The intensity viewed in the normal direction is very slight, as can be seen by looking at the reflexion of the luminous surface in the hypothenuse face of the prism. Seen

edgewise the intensity is fully thirty times as great, as was found by measurement at an angle of 5 degrees with the surface.

This increase in the intrinsic intensity of a fluorescent surface when seen foreshortened, from a point of view within the medium (so to speak), must have been noticed by many observers, but I have been unable to find any mention of it in the literature upon the subject. A very careful photometric study of the variation of the intensity with the angle has been made by my assistant, Mr. H. E. Ives, with apparatus arranged as shown in fig. 1.

Fig. 1.
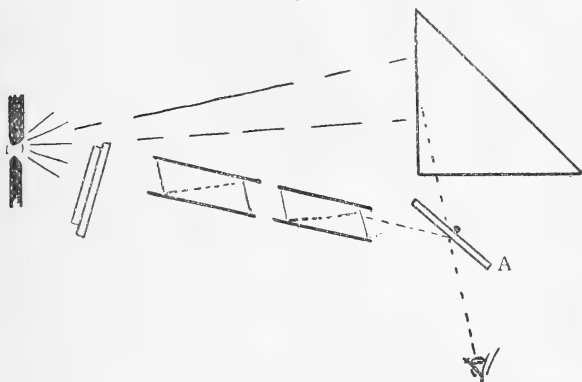
A portion of the light from the spark fell upon the prism, while another portion, after traversing a sheet of ground glass, and a blue screen, was passed through a pair of Nicol prisms and reflected to the eye by means of the narrow strip of silvered glass A, mounted on a pivot in front of the prism. By turning one of the nicols the intensity of the light seen reflected in the strip could be balanced against the fluorescent background, against which it was seen. The colour was very accurately matched by means of a thin sheet of cobalt glass combined with a gelatine film stained with one of the blue aniline dyes. The intensity in the normal direction was measured by matching the reflected light against the image of the fluorescent surface seen by total reflexion in the prism.

The fluorescent prism was mounted on the table of a small spectrometer and viewed through the telescope, the lenses of which were previously removed, and a small slit put in place of the eyepiece. In this way the angular direction from which the luminous surface was viewed could be easily determined.

A number of corrections were of course necessary, for the angles in air are greater than the angles within the glass, on account of refraction.   Then, too, there is a small loss due to reflexion within the prism which becomes greater as the angle is increased.

The values obtained are shown graphically in fig. 2,

Fig. 2.
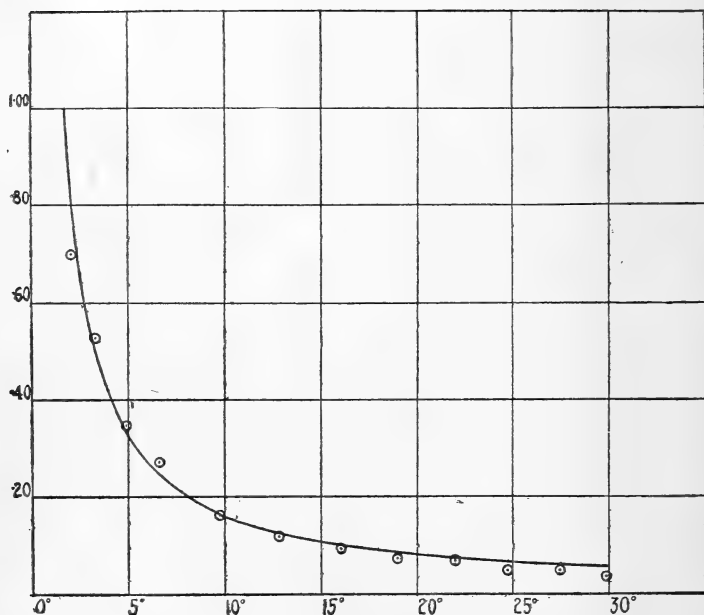
intensities being plotted as ordinates, and angles as abscissæ. The observed values, which have been corrected for the small loss due to internal reflexion, are represented by circles, while the curve drawn through them is the theoretical curve, calculated on the assumption that the intensity of the radiation from each fluorescent molecule is independent of the direction within the glass.   Under such conditions the intensity of the illumination of the surface would double each time its apparent area was halved by foreshortening.

As will be seen, this condition appears to be pretty nearly fulfilled, though accurate measurements between zero and five degrees were difficult.   The fluorescent radiations obey therefore within the medium the same law which holds in the case of a transparent radiating gas.   Outside of the medium, that is in air, the case is very different.   The intrinsic

intensity of the surface is greatest when it is viewed in the normal direction, while at grazing emergence scarcely any fluorescent light at all is visible. This is without doubt due to refraction, for the rays which emerge at grazing incidence were incident upon the boundary at very nearly the critical angle, and a large amount of energy is thus lost by reflexion. It is moreover clear that a cone of rays coming from a fluorescent molecule below the surface, which embraces a solid angle equal to twice the critical angle for the glass, is, upon refraction out into the air, spread out into a cone of 180 degrees, that is over a hemisphere. The intensity within the medium is in a corresponding degree increased by total reflexion, practically doubled in fact within a given angular range. To prove the existence of this reflexion of the fluorescent layer at the boundary between the two media, it is only necessary to place a cube of quartz in front of and against the fluorescing surface, with a small drop of glycerine pressed out between the two. The quartz and glycerine are both quite transparent to the radiations from the spark which excites the fluorescence ; consequently all changes which are observed are due solely to the alteration of the refractivity of the medium in front of the glass surface. The intensity of the fluorescence of the surface in contact with the glycerine will be found to be much less than the surrounding surface, when viewed from the glass side, while it is much greater when viewed from the quartz side. A more complete study could be made if the refractive index of the medium could be made the same on both sides of the fluorescent surface, or varied at will ; but it is difficult or impossible to find suitable media which are transparent to the ultra-violet waves which excite the fluorescence. If we try to get round this difficulty by using media which fluoresce under the influence of less easily absorbed radiations, we no longer have the fluorescence confined to the necessary shallow layer.

The subject of fluorescence in its connexion with Lambert's law has been discussed by Lommel in a paper which appeared in Wiedemann's *Annalen* in 1880*.

In this paper Lommel deduces Lambert's law for a medium which has a high absorbing power, and which is emitting light in virtue of its high temperature. Applying the same method to a partially transparent medium, and to fluorescent media, he comes to the conclusion that in these cases the emission is not proportional to the cosine of the angle, but to a certain function of the cosine, which has the value 1 for normal emission and 0 for an emission at 90 degrees.

* Wied. *Ann.* x. p. 449 (1880).

For these cases there would be a marked increase in the intrinsic intensity as the surface of the source was foreshortened, but it would not be as great as in the case of a gas-flame, or the fluorescent prism which we have considered.

Lommel observed this increase in the apparent intensity by immersing a vessel containing the fluorescent medium in a tank of water, and observing the luminous surface at different angles. Lommel considered his fluorescent medium as partially absorbing for the fluorescent radiations, in which respect it differs from the crown glass used in the present experiments, which is almost perfectly transparent to the blue fluorescent light which it emits.

From what has been already stated it is clear that the intensity of the fluorescent surface, when viewed from the air side at varying angles, is profoundly influenced by refraction. We should naturally expect radiations originating at a small depth within the medium to be refracted on emergence into the air, but it is not quite clear what would happen to radiations leaving the surface-layer of molecules.

Millikan came to the conclusion, from a study of the polarization of the fluorescent light, that these radiations are refracted in the same way as radiations coming from a finite depth within the medium. In the case of fluorescence provoked by light, it is probable that the radiating layer is of considerable thickness as compared with the molecular diameter, for the light undoubtedly penetrates to the depth of a wavelength or two. It seemed possible that an effect more superficial in its nature might be obtained by exciting the fluorescence with cathode rays. A glass sphere was mounted in a vacuum-tube in front of the cathode, but it was found that the appearance was the same as with ultra-violet illumination. The tube was so constructed that the sphere could be bombarded either with cathode rays or canal rays; and it was found that if the latter were caused to play upon the surface, a phosphorescence was obtained which did not obey Lambert's law, for the light appeared much more intense around the edge of the sphere, where the surface was considerably foreshortened. This light, as is well known, shows the D lines strongly, and comes without doubt from a shallow layer of gas, which clings close to the surface. It is possible, however, that in this case we have an emission of light from the surface molecules alone; in other words, the effect which was sought for in the cathode-ray phosphorescence. The canal rays have without doubt much less penetrating power, and may excite a purely superficial phosphorescence. My feeling about the matter, however, is that the luminous

sodium molecules have been wholly freed from the medium, for the radiating layer seems to obey the same law which holds in the case of a gas-flame. The effect can be seen in the usual form of canal-ray tubes furnished by the instrument-makers, the yellow phosphorescence appearing much brighter in the regions where it is considerably foreshortened. The best plan of all, however, is to mount a flat plate of rock-salt in a canal-ray tube, for in this case we obtain a brilliant line of yellow light when we view the surface at an angle of about 85 degrees with the normal.

An examination of the yellow light with a large three-prism spectrograph showed that the D lines were quite as narrow as in the case of the sodium flame, and it is highly improbable that a radiation of this nature could be given off from the surface molecules of a solid.

Professor Rutherford has recently shown that the intensity of the radiation from flat surfaces made radioactive is independent of the direction, that is the same law holds as in the case of X-rays. This condition was demonstrated by the very ingenious device of coating the sides of short prisms with thin layers of radioactive material, and standing them on photographic plates, curious star-shaped patterns resulting, which were shown to be exactly in accord with the theory that the radiation intensity was independent of direction.

I have succeeded in producing surfaces which radiate light *in air* according to the same law, and have obtained photographs exactly similar to those obtained by Professor Rutherford. It is only necessary to shake up some finely powdered Balmain luminous paint in a box, introduce the body which is to be rendered luminous, and allow the dust, which is suspended in the air, to settle upon its surface. The distance between the luminous particles must be several times their average diameter. If such a surface is exposed to the light of an arc-lamp and examined in a dark room, its intrinsic intensity will be found to increase as it is fore-shortened.

If we coat the outer surface of a cylinder with the powder, expose it to light and examine it in the dark, the edges of the cylinder will appear as luminous lines much brighter than the rest of the surface. This will be recognized as the analogue of Rutherford's first experiment, in which a pin-hole photograph of a radioactive wire showed two lines of light parallel and close together. A square nut, with its sides coated, was placed in the dark upon a photographic plate, and after an exposure of two hours a picture precisely similar to the one obtained by Rutherford was obtained (fig. 3).

If the luminous particles are so close together as to be nearly or quite in contact, the radiation from the surface obeys Lambert's law, which in this case follows as a result of the screening action of the particles upon the radiation

Fig. 3.
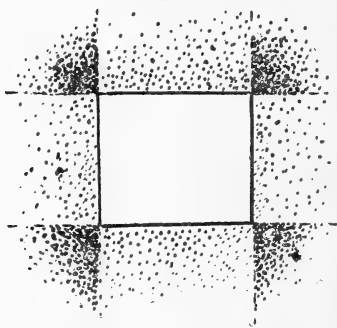
coming from their neighbours. Suppose now that the luminous particles fuse together into a continuous surface. The irregular refractions and reflexions, which resulted in a screening action, are now replaced by regular refraction at the surface, all rays incident upon the surface at an angle greater than the critical being totally reflected back into the medium. It is probable that no other action than that of refraction is required to explain Lambert's law in the case of transparent media, a conclusion which has also been reached by Uljanin (Wied. *Ann.* lxii. p. 528).

In the case of an incandescent metal surface it is also probable that refraction plays some part, since the emitted light is partially polarized at certain angles. The law of emission can be ascribed to the absorption of radiations coming from points beneath the surface, the relation having been worked out by Lommel.

So little is known, however, about the nature of the vibrations of incandescent solid and liquid surfaces, that it is difficult to formulate any very definite conception of the processes which are directly responsible for Lambert's law.

It is perhaps worthy of remark that complete irregularity of the phases of the vibrations is necessary. If the vibrating particles on the surface of a white-hot metal plate could be brought into the same phase, the plate would emit a plane wave, and would cease to be visible. No light would be perceived at all unless the eye were located on some normal to the surface, in which case a very brilliant point source would be seen in the direction of the normal.

LXXII. *Note on the Focometry of Concave Lenses and Convex Mirrors.* By Prof. A. ANDERSON, *M.A., LL.D.*[*]

TO those who are interested in neat ways of making physical measurements, the following methods of finding the focal length of a concave lens and of a convex mirror may be of some interest.

They have not, I think, been noticed before. They are pin-point methods, and as easy in practice as the well-known pin-point method of finding the focal length of a concave mirror. For the measurement of the focal length of a concave lens a concave mirror is required, and for that of the focal length of a convex mirror a convex lens is required.

Fig. 1.
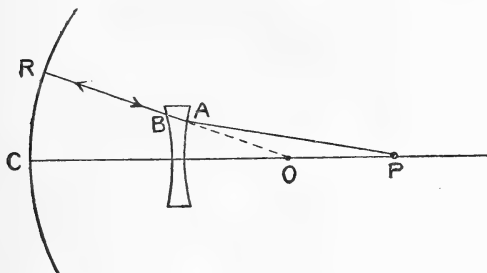
Fig. 2.
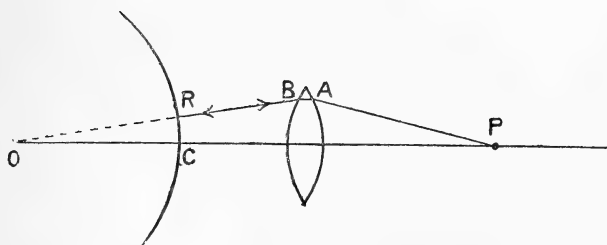
The figures explain the methods. The pin-point is placed at P, and the adjustment consists in making it coincide with its image, in which case the rays of light, after passing through the lens, diverge from (fig. 1), or converge to (fig. 2) the centre, O, of the mirror. In both cases the mirror, when properly placed, causes the rays to retrace their paths in the opposite direction.

Note.—I have since found that a method, which is practically identical with that shown in fig. 2, is given in Edser's 'Optics,' p. 117.

* Communicated by the Author.

LXXIII. *The Velocities of the Ions of Alkali Salt Vapours at High Temperatures.* *By* HAROLD A. WILSON, *M.A., D.Sc., M.Sc., F.R.S., Fellow of Trinity College, Cambridge, and Professor of Physics, King's College, London* *.

IN a paper on "The Electrical Conductivity of Flames containing Salt Vapours," Phil. Trans. A. vol. 192, 1899, the writer gave an account of a series of experiments which included measurements of the velocities of the ions of salt vapours in flames and in hot air. The method employed was to find the electric intensity necessary to make the ions move against a stream of gas moving with a known velocity. The following table gives the results obtained:—

(1) Positive ions of various salts of Cæsium, Rubidium, Potassium, Sodium, and Lithium in a Bunsen flame. Velocity 62 cms. per sec. for one volt per cm.

(2) Negative ions of same salts in a Bunsen flame. Velocity 1000 cms. per sec.

(3) Positive ions of salts of Cæsium, Rubidium, Potassium, Sodium, and Lithium in air at 1000° C. 7·2 cms. per sec.

(4) Positive ions of salts of Barium, Strontium, and Calcium in air at 1000° C. 3·8 cms. per sec.

(5) Negative ions of salts of Barium, Strontium, Calcium, Cæsium, Rubidium, Potassium, Sodium, and Lithium in air at 1000° C. 26 cms. per sec.

In 1900 Dr. E. Marx published an account ("Ueber den Potentialfall und die Dissociation in Flammengasen," *Annalen der Physik*, 1900, no. 8) of a valuable series of experiments on the conductivity of flames, including determinations of the ionic velocities for different salt vapours in the flame. His method depended on observations of the potential gradient in the flame, and was quite different from that employed by the writer. He obtained for the velocity of the negative ions of any alkali salt in the flame 1000 cms. per sec. For the positive ions of any alkali salt in the flame he found the velocity to be about 200 cms. per sec.

The experiments of Dr. Marx and the writer thus agree in showing that all alkali salt vapours in flames give ions having under the same conditions nearly equal velocities, and that the velocity of the negative ions is much greater than that of the positive ions.

In 1901 the writer published a paper on the "Electrical Conductivity of Air and of Salt Vapours" (Phil. Trans. A. vol. 197, 1901), in which it was shown that, above 1350° C., and with an E.M.F. of over 1000 volts, all alkali salt vapours

conduct equally well if the amount of each one taken is proportional to its molecular weight. The temperature above which this result holds good was nearly the same for every salt. Above this temperature the current was independent of the temperature and of the P.D. used, provided the latter was sufficiently great. Since the lowest temperature at which this maximum current could be obtained was the same for all alkali salts, it follows that at such high temperatures the salts tried all gave ions having the same velocity; for if some salts had given ions having a smaller velocity than others, then a higher temperature and larger P.D. would have been required to obtain the maximum current with these salts.

The writer has recently, in collaboration with Mr. E. Gold, B.A., carried out a series of experiments on the conductivity of flames containing salt vapours for rapidly alternating currents. We have found that the conductivity for rapidly alternating currents varies as the square root of the corresponding conductivity for steady currents. It is shown in the paper *, that this result can be explained by supposing that all the salts tried give ions having equal velocities in the flame.

In 1903 Prof. Moreau published an account (*Annales de Chimie et de Physique*, Sept. 1903) of some measurements of the velocites of the ions of salt vapours in flames by a method which was described by the writer in the paper referred to above in 1899. Prof. Moreau found, in good agreement with the earlier work, that all the alkali salts tried gave positive ions having a velocity of about 80 cms. per sec. For the negative ions of salts of potassium and sodium, however, Prof. Moreau obtained results which do not agree with those obtained by Dr. Marx and the writer. Prof. Moreau's results for sodium and potassium salts are as follows :—

With equal molecular concentrations the velocity of the negative ions is independent of the nature of the acid radical of the salt. It varies with the metal in the inverse ratio of the square root of the atomic weight. For a particular salt vapour the velocity of the negative ion increases as the concentration of the vapour diminishes. Prof. Moreau gives the following numbers :—

Molecular Concentration of Solution sprayed into the Flame ...	....	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$
Velocity of Negative Ions in cms. per sec. .... { K		660	785	995	1180	1320
Na ...		800	1040	1280		

Thus with large concentrations the sodium ions seem to have larger velocities than the potassium ions.

* Phil. Mag. April 1906.

The method used by Prof. Moreau was to find the P.D., between two vertical electrodes, required to make the salt ions move across from the bottom of one electrode to the top of the other. Then, assuming the electric field uniform and knowing the upward velocity of the gases between the electrodes, the velocity of the ions was calculated. Now the electric field in a flame is very far from being uniform, but it is no doubt nearly the same in different cases when the conductivity of the flame is the same. Consequently, if we compare Prof. Moreau's results for potassium salts with his results for sodium salts, taking such concentrations that the flame has the same conductivity for the potassium salts as for the sodium salts, then we ought to get the relative velocities of the potassium and sodium salt ions free from error due to variations of the potential gradient with the conductivity of the flame.

Potassium salts conduct about four times as well as sodium salts when the concentration is small, so that Prof. Moreau's numbers may be tabulated thus :—

Concentration of Potassium Salts		$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$
Concentration of Sodium Salts		1	$\frac{1}{4}$	$\frac{1}{16}$
Velocity of Negative Ions for Potassium Salts		785	995	1180
Velocity of Negative Ions for Sodium Salts		800	1040	1280

Thus, comparing velocities with concentrations giving nearly equal conductivities, the velocities found by Prof. Moreau are very nearly the same for potassium salts as for sodium salts. The apparent variation of the velocity with the concentration of the salt vapour is no doubt due to the variation of the fall of potential between the electrodes with the conductivity, and not to any real variation of the velocity with the concentration.

The mean of Prof. Moreau's results for the velocity of the negative ions is about 1000 cms. per sec., and so agrees very well with the earlier results obtained by Dr. Marx and the writer.

If Prof. Moreau were correct in supposing that the velocity of the negative ions varies inversely as the square root of the atomic weight of the metal, it would follow that salt vapours are ionized into a metal ion carrying a negative charge and a positive ion the same for all salts. Such a conclusion is altogether inconsistent with our knowledge of the ionization of salts.

If all salts give positive and negative ions having the same velocities in a flame, this can be explained by supposing that a salt molecule emits a corpuscle which is the negative ion, and that the positively charged salt molecule forms an aggregate of molecules whose size depends only on the charge, and so is the same for all salts. This view is consistent with the fact that the velocity of the positive ions is only about 60 cms. per sec. while that of the negative ions is 1000 cms. per sec. According to Prof. Moreau's results, the positive and negative ions ought to have nearly equal velocities, for the atomic weight of the metals is comparable with the molecular weight of the acid radicals.

The experiments done by the writer in 1899 showed that lithium (Li = 7) salts give ions having the same velocities as the ions from cæsium (Cs = 133) salts; which proves clearly that the velocity of the negative ions does not vary inversely as the square root of the atomic weight of the metal—a conclusion which Prof. Moreau has drawn from experiments on potassium (K = 39) and sodium (Na = 23) salts.

It has been shown by Prof. Lenard that the salt vapour emitting light in flames moves in a strong electric field as though it were positively charged. I have verified this result, which clearly proves that the metal goes to form the positive ions, for the light is undoubtedly emitted by the metal atoms and not by the acid radical of the salts.

I think therefore that the view that all alkali salts in flames give ions having nearly the same velocities is really supported by Prof. Moreau's observations as well as by those of Dr. Marx and the writer.

LXXIV. *On some Properties of the α Rays from Radiothorium.* (I.) *By* O. Hahn, *Ph.D.*[*]

IN a previous paper[†], I have given an account of the methods of separation and the radioactive properties of a very active substance, called radiothorium, which was prepared from the Ceylon mineral thorianite. It was shown that radiothorium was probably a slowly changing product of thorium—the latter is then supposed to be inactive—which lies between thorium and thorium-X.

Radiothorium possesses in an intense degree all the radioactive properties of thorium, for it gives rise to thorium-X,

the thorium emanation, and the characteristic active deposit. On account of its great activity, radiothorium is a far more suitable source than the feebly active thorium for the investigation of the properties of its radiations.

In this paper an account will be given of investigations on the following points :—

1. Range of the α particles in air from the active deposit of radiothorium by the methods of scintillations and by the electrical method of Bragg and Kleeman.

2. Proof that the product called thorium B is complex, and consists of two distinct α ray products which have different ranges of ionization in air.

3. The magnetic and electrostatic deflexion of the α particles from the active deposit (preliminary).

Activity of Radiothorium.

Before discussing the properties of the α rays, a brief account will be given of some measurements that have been made to determine the activity of the preparation of radiothorium which had been used in these experiments.

The first method was the same as that used in the previous paper. The emanating power of a known weight of radiothorium was compared with that of a known weight of thorium nitrate under, as far as possible, similar conditions.

10 grs. of pure thorium nitrate were dissolved in 35 c.c. of water, and the emanating power measured by passing a constant air-current through the solution into a testing vessel, the rate of movement of the needle of an electrometer being a measure of the amount of emanation present. Then $\frac{1}{10}$ mg. of radiothorium was measured under exactly the same conditions, 10 grs. of inactive sodium chloride being added to have about the same concentration of the solution. The $\frac{1}{10}$ mg. had been prepared by successive dilution from a preparation of about 2·7 mg. of radiothorium, the activity of which had been calculated in London to be somewhat over 200,000. This time the strength of the $\frac{1}{10}$ mg. to the 10 grs. was found to be as 1·5 : 1, showing that the radiothorium gave about 150,000 times more emanation than an equal weight of thorium. The order of magnitude of the active thorium product was, therefore, almost the same as was expected, though the result seems to show that the proportionality of the emanating power to the weight does not remain constant for a very large difference in concentration.

A second method of determining the activity was to compare the γ rays of the radiothorium with those of a large amount of commercial thorium. A quantitative comparison was kindly made by Mr. Eve, working at the same laboratory, by means of a sensitive electroscope which was covered with a sufficient thickness of lead to absorb all the β rays. Mr. Eve compared 1 kg. of pure thorium nitrate with 10·9 mg. of a radiothorium preparation, which was formerly measured by the writer by an indirect method and supposed to be about $2\frac{1}{2}$ times stronger than the above-mentioned 2·7 mg. It was now found that both preparations were of the same order, the 10·9 mg. being about 130,000 times as strong as thorium, when comparing weight by weight.

These results led me to make a more careful investigation to see whether radiothorium was slowly losing its activity. Two thin films of radiothorium, each about $\frac{1}{1000}$ mg., were obtained on two watch-glasses and their activity measured from time to time. A slight decrease was noticed after some months, but no definite statement can yet be made, as a very thin film of radium, measured under similar conditions, showed even a greater decrease. I have recently prepared another sample of radiothorium, placed it in a watch-glass, and covered it with a thin sheet of mica, so that it is perfectly airtight. Under similar conditions, Eve has shown that radium does not appreciably decrease in activity. It is hoped, in this way, that it may be possible to settle definitely whether radiothorium loses its activity at a measurable rate. Even if we shall find a slow decrease of the activity of radiothorium with the time, it is certain that we are dealing with a very slowly changing product entirely different from thorium-X, which is known to decay to half value in probably less than four days.

Previous experiments indicated that the α ray activity of thorium in equilibrium is due to four different products : radiothorium, thorium-X, emanation, and thorium B, while apparently thorium itself, and certainly thorium A, are "rayless." In order to begin the investigation with a single active product, it was found very convenient to obtain an active deposit of thorium B on a thin copper wire by exposing the negatively charged wire to the thorium emanation, after the method used by Rutherford for radium. In the case of thorium, it is not necessary to use the active source in solution, because unheated thorium oxide or carbonate emanates strongly in the solid state. The 10 mg. of radiothorium were placed on the bottom of a small brass cylinder. A thin

copper wire was attached through an insulating rubber cork so as to remain just over the radiothorium. The active deposit was collected on the wire, which was charged negatively to about 110 volts by means of the lighting circuit. After about two and a half days' exposure equilibrium is reached and the wire has become strongly radioactive, for it discharges an electroscope immediately, and gives distinct luminosity to sensitive screens.

Scintillation Method.

In order to find the maximum range of the α particles of the thorium B, the scintillation method was first employed. A small piece of glass was coated with zinc sulphide and placed vertically over the active wire, the latter being fixed on a small disk, which could be moved up and down to any desired distance of the screen. As long as the screen was in the range of ionization of the α particles, the scintillations were seen without difficulty by means of a lens. It is, of course, necessary for the eye to be first accommodated to the dark. Beyond the range of ionization the scintillations disappear fairly abruptly. The maximum range at which scintillations were still visible was found to be 8·3 cm., while at 8 cm. the luminosity could be seen quite easily. This result showed at once that the α particles of the active thorium deposit are even more penetrating than the most penetrating α rays of radium, which were found by Bragg and Kleeman to be those of radium C. For comparison, some radium C was collected at a thin copper wire and its range of ionization determined with the same method. It was found to be 6·8 cm., Rutherford having found the same when using the same method [*], while Bragg and Kleeman [†] found 7·0 cm. when using the electrical method. It may be mentioned here, that the electrical method of determining the range of the α particles is somewhat more delicate than the optical one.

McClung [‡], using Bragg's method of determining the ionization curve of radium C, covered his active wire with thin sheets of aluminium-foil in order to see whether the range of ionization decreased in proportion to the number of foils used. He found that one aluminium-foil about 0·00031 cm. thick corresponds to about 0·51 cm. of air, Rutherford [§] having found about 0·53 cm. for the same foil. The same result was obtained by the writer by the scintillation method. Different layers of aluminium-foil were placed

    * Phil. Mag. July 1905.   † Phil. Mag. Dec. 1904.
    ‡ Phil. Mag. Jan. 1906.   § Phil. Mag. Jan. 1906.

over the active wire, and the maximum range again deter-
mined. Column I. of the table gives the number of sheets of
aluminium-foil used, column II. gives the distance at which
scintillations were still distinct, column III. gives the thickness
of air which corresponds always to one layer of foil.

I.	II.	III.
0	8·0	
1	7·5	0·50
3	6·75	0·515
7	4·1	0·55
9	3·05	0·55
11	2·15	0·53
13	1·10	0·53
14	0·7	0·53
		Average 0·53

The values given in column III. are approximately constant,
and the mean 0·53 is exactly that which Rutherford found for
the rays of radium C. Another series of measurements was
made with very thin foils of Dutch metal, the result of which
was in even better agreement. For some purposes, it is very
convenient to have a means of cutting down the range of the
α particles by any required amount, and it was for this
reason that the above-mentioned measurements were under-
taken.

We shall state later in this paper the amount of air equal
to a very thin mica plate—a screen which has the great
advantage of cutting off radioactive emanations completely,
while the sheets of metal are always porous and almost useless
for that purpose.

The scintillation method above described for investigating
the maximum range of the α particles in air can be used with
advantage when dealing with a single product like radium C,
or when determining only the range of the most penetrating
of a number of different rays. The method does not throw
any light on the range of the α rays which have less than the
maximum range, nor does it indicate whether we are dealing
with a homogeneous or complex pencil of α rays. It was of
special importance in the present case to know whether there
was present *only* Th. B or some other active product besides

it, for instance Act. B.   The latter would then, of course, come from actinium in the radiothorium preparation, and would show that it was not pure in the radioactive sense of the word.   There was, however, but little probability of such an occurrence, since no evidence of the presence of actinium had been obtained from an examination of the decay curves, which had been obtained for the emanation and the active deposit.   In order then to test the point more completely, I made use of the method developed by Bragg and Kleeman *, who made with so much success a complete analysis of the α rays emitted by the various radium products.   The apparatus employed by the writer was similar to that of Bragg and Kleeman and of McClung.   A fairly large metal box enclosed two parallel plates, between which the ionization was to be measured.   The upper plate consisted of a solid plate of zinc, connected to one pair of quadrants of a Dolezalek electrometer, the other pair being connected to earth.   The lower plate was of wire gauze, about 0·5 cm. away from the upper plate, and was charged by means of a battery of accumulators to a sufficient potential to produce a saturation current.   To avoid any leak from the lower to the upper plate, the upper plate was surrounded by a guard-ring connected to earth.   The radioactive source, in this case the active wire, was placed on a small platform in the metal box directly beneath the two plates.   In order to get a well-defined cone of rays, the active wire was cut into several small pieces and laid in the groove of a solid zinc block.   On this was placed another zinc block about 0·5 cm. thick, having a hole about 0·3 cm. in diameter, just over the groove containing the active wire.   The small platform could be moved vertically to any distance from the two plates, and this distance could be measured on a fixed scale.   The parallel plates were sufficiently large to include the whole cross-section of the cone of rays over the range required.

Measurements of the ionization currents in the testing-chamber were made for different distances between the ionization-chamber and the radiant source.   The velocity of the electrometer needle was a measure of the intensity of the ionization.   In the following figures the ordinates represent the distances of the radioactive product from the wire gauze, the abscissæ the intensities of the ionization measured in arbitrary units, which are, however, the same in both figures and for all the curves.   Fig. 1 shows the curve obtained from the active wire, the latter being in radioactive equili-

* Phil. Mag. Dec. 1904; & Sept. 1905.

brium after a three days' exposure to the emanation. The
result obtained was most unexpected.

Fig. 1.
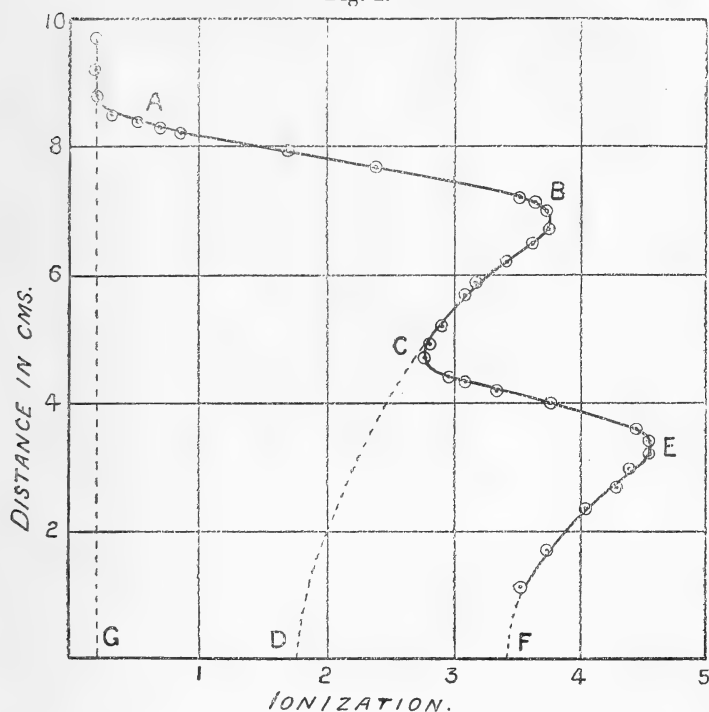
The curve is seen to consist of two distinct parts. If the
portion B C of the curve is produced backwards to meet the
axis of abscissæ at D, the curve A B C D represents the curve
to be expected for the α rays from a thin film of radioactive
matter of one kind. Curves very similar in shape to this
have been obtained by Bragg and Kleeman for the rays from
a thin film of radium at its minimum activity, and by McClung
for the α rays from radium C. The abrupt change in the
curve at C shows that at that distance another set of α rays
enters the ionization chamber. The complex curve clearly
indicates that the active deposit gives out two distinct types
of rays of different ranges in air, and must consequently
consist of two distinct α ray products. The fact that the ioni-
zation represented by G D is nearly equal to that represented
by D F shows that an equal number of α particles are emitted
per second by these two products—a result to be expected

if the two α ray products are successive and in radioactive equilibrium with each other.

It may be mentioned that for accurate measurements, it is necessary to make a correction for the decay of the activity during the time of the experiment. Since thorium B decays to half value in 10·6 hours, after one hour there is a decrease of about 6·5 per cent. This correction for change of the intensity of the rays during an experiment is generally small and is very easily made.

The curve shows that the ionization of the α rays begins at a distance of 8·6 cms. from the source—a result in good agreement with that obtained by the scintillation method. Beyond that distance, the curve is almost a vertical line, and the ionization is due to the β and γ rays and the natural leak of the vessel and the air. Below 8·6 cms. the current in the ionization chamber increases very rapidly, reaches a maximum at 6·8 cms., and then decreases until at 4·7 cms. the rays from the second product add their effect.

If the α particles all ceased to ionize after traversing a definite distance of air, the slope of the curve A B, obtained with a shallow ionization chamber and a narrow cone of rays, should be nearly horizontal. In the experiments, however, a fairly wide cone of rays and an ionization chamber of depth 5 mms. were used. With a narrower cone of rays and a shallower ionization chamber, the electrical effect would have been smaller, and the errors of measurements consequently greater.

Under the experimental conditions it is, therefore, not unexpected that the curve A B should have an appreciable slope towards the axis of abscissæ. Using a zinc block 0·8 cm. high instead of 0·5 cm.—as used to determine the curve given in the figure—the slope of the curve was less marked, and the point of maximum ionization was 3 mms. higher. The decrease of the current below the distance of 6·8 cms. shows that the α particle is a more efficient ionizer near the end of its path, when its velocity has been diminished by passing through the air. A similar effect for radium rays has been observed by Bragg and Kleeman [*] and by McClung[†].

The ionization curve C E F, due to the rays from the second product, is very similar in shape to the curve A B C. Below a distance of 4·7 cms. the ionization rapidly increases, passes through a maximum at 4·4 cms., and then diminishes.

As we have already pointed out, the equal ionization produced by the two α ray products suggests that the products are successive. It was, however, necessary to make certain that one of the two α ray products did not arise from some

* Loc. cit.   † Loc. cit.

impurity in the radiothorium, for example from the presence of actinium. Remembering that the active deposit of actinium (half value in 36 minutes) is transformed very rapidly compared with the active deposit of thorium (half value in 10·6 hours), there were two simple methods of definitely settling whether the observed shape of the ionization curve resulted from the presence of actinium B.

Experiment I.—A wire was exposed to the radiothorium preparation for about 3 hours. After that interval the active deposit due to actinium has reached its maximum value, while that of thorium is only a small portion of its equilibrium amount. Under such conditions, the ionization due to the actinium B should predominate.

Experiment II.—A wire was exposed to the radiothorium preparation for three days in order to obtain the equilibrium amount of Th. B, and, of course, the equilibrium amount of actinium B. The wire was then removed from the emanation and tested four hours later. During this interval the activity due to actinium B has disappeared, and the ionization curve should be due to thorium B alone.

On the assumption that one of the products was actinium B, the ionization curve in one case should be mainly due to rays of a range 8·6 cms., and in the other to rays of range 4·7 cms. No such effect, however, is observed. The experimental curves are shown in fig. 2 (p. 802), curves I. & II. Curve I. was obtained from experiment I., and curve II. from experiment II.

The curves are similar in shape to that given in fig. 1, and show the same equality in the ionization produced by the two products. These experiments thus prove that no measurable amount of actinium B was present on the wire. For a similar reason, it is seen that the rapidly changing active deposit of radium is also absent.

We must, therefore, conclude that two α ray products are present in the active deposit of radiothorium and not one as was previously supposed. Neither of these two products can be ascribed to a *slowly* changing product of thorium for the reasons discussed above. Such a slowly changing product would not always be present in the same proportion as thorium B for different times of exposure to the emanation[*].

[*] It may be mentioned that some months ago the writer made some investigations to see whether there was any slowly changing product in the active deposit of thorium obtained on a copper wire. A negatively charged wire was exposed for three weeks to the emanation of the strongest radiothorium preparation in my possession. The activity decayed at the characteristic rate, but no evidence of any appreciable residual activity was obtained. Of course, very weak activities would have been overlooked, as the air and the instruments of the laboratory showed always a slight activity.

The simplest explanation of the results is that we have a new thorium-product which is transformed very rapidly compared with thorium B.

Fig. 2.
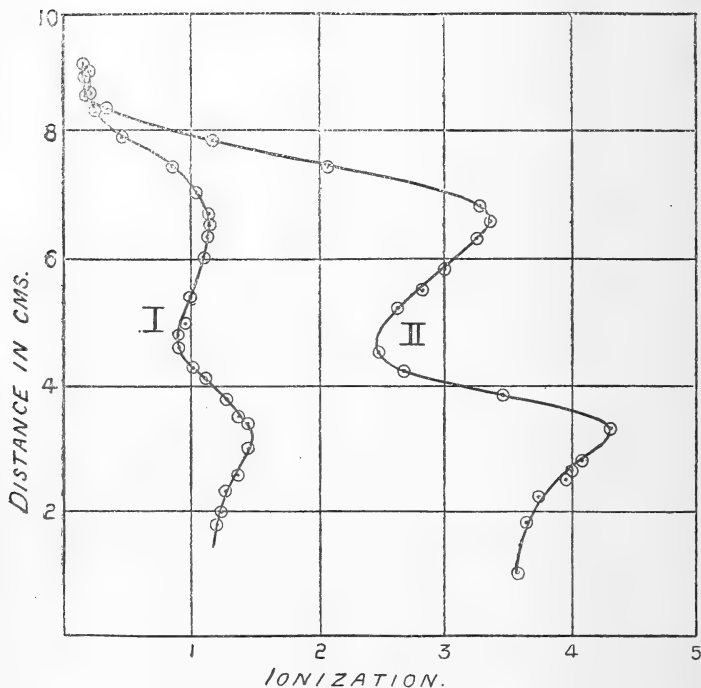
From a study of the ionization curves alone, it is not possible to decide whether this new product comes between thorium A and B or after thorium B. This point was settled by another experiment as follows. The active deposit was dissolved from a copper wire, and in the acid solution was placed a little nickel plate. Von Lerch* has shown that by this method thorium B is separated from thorium A, and that the nickel loses its activity exponentially with a period of one hour, *i.e.* according to the period of thorium B. The ionization-distance curve was taken with this nickel plate as a source of rays. The curve showed again the two characteristic portions, indicating two different α ray products. Such a result shows that the new product always accompanies thorium B. The fact that the active nickel plate loses its activity according to an exponential law shows that the new

* *Wiener Ber.* March 1905.

product must be transformed very rapidly compared with thorium B, and, consequently, the activity due to its decay *pari pàssu* with that of the parent product thorium B.

It has not, so far, been found possible to chemically separate this active product from thorium B. Some experiments of Pegram * are of importance in this connexion. By electrolysing thorium solutions, he states that he obtained under certain conditions a very rapidly decaying activity. For example, he electrolysed a thorium nitrate solution to which a little copper sulphate had been added, and found that the cathode lost its activity very rapidly. A similar result was noted by precipitating some silver chloride in the solution and electrolysing immediately afterwards. The activities decayed to half value from about 40 seconds to a few minutes. I have made a few preliminary experiments in this direction, but so far without success. The experiments will be continued to see if it is possible to devise a method of separating the new product and to determine its period.

According to the nomenclature adopted by Rutherford the new α ray product of thorium will be called " thorium C." It is not possible to decide whether the α rays of range 8·6 or 4·7 cms. must be ascribed to the new product. From analogy of the radium products, it is probable that the α particles of greatest range, and, consequently, greatest velocity, are emitted from the last product thorium C. For a similar reason, it is probable that the β and γ rays do not come from thorium B but from thorium C, but it will be necessary to separate these two products before this point can be definitely settled.

Decrease of range of the α particles in passing through matter.

It was stated in the beginning of this paper that the range in air of the ionization of the α particles is decreased by adding thin screens over the active wire. The decrease of range depends upon the thickness of the screen and its density. The ionization curve was obtained after adding a mica screen of thickness about 0·018 mm. over the active wire. The absorption of the rays by the screen was equivalent to about 2 cms. of air. The ordinates of the new curve are all reduced by a definite amount corresponding to a distance of 2 cms. of air. A similar effect has been observed by Bragg and Kleeman for radium rays.

* Phys. Review, Dec. 1903.

Magnetic and Electric Deflexion of the α Rays.

Some experiments were begun with the assistance of Professor Rutherford to see if the α rays from the active deposit of thorium were deflected in a magnetic and electric field. The experiments showed that the α rays were deflected both by a magnetic and electric field and to about the same extent as the rays from radium C. A brief account will be given here of some preliminary experiments on the magnetic deflexion of the α rays.

The same arrangement was employed as that used by Rutherford for the determination of the magnetic deflexion of the α rays from radium C. A thin wire, coated with the active deposit of thorium, served as the source of rays. Half of the active wire was covered with a thin mica sheet while the other half was bare ; and by means of mica screens it was arranged that one half of the small photographic plate was acted on by the rays from the bare wire, and the other half by the rays after traversing the mica. The apparatus was exhausted of air, and a constant magnetic field applied for 10 hours. By reversing the field at intervals, two sets of bands were obtained on the plate. The amount of deflexion of the traces of the pencils of rays on the two halves of the plate were directly compared with the amount of deflexion observed under identical conditions of the rays from radium C. The photographic effect of the rays from the active deposit of radiothorium was too weak for reproduction. It was noted that the edges of the trace of rays on the plate from the uncovered half of the wire were not so clearly defined as the traces obtained for the rays which had traversed the mica screen. The reason of this difference was not clear at the time the experiments were made, as the rays from the active deposit had not then been shown to be complex. The rays of range 4·7 cms. are more deflected in a magnetic field than those of range 8·6 cms., and also have a weaker photographic effect. The outside edge of the trace obtained on the plate is consequently due to rays which have a comparatively feeble photographic effect and the edge is, consequently, not clearly defined. The mica sheet was of sufficient thickness to cut off most of the rays of range 4·6 cms., and the trace on the other half of the plate was due to homogeneous rays and was, consequently, better defined.

The amount of deflexion of the thorium rays after passing through the mica screen was about 20 per cent. less than for the rays from radium C after passing through the same

screen. If the value of $\frac{e}{m}$ is the same for the α particles of thorium and radium, that suggests that the α particles of range 8·6 cms. are expelled with a greater speed than those from radium C. Such a result is in agreement with the greater range in air, 8·6 cms., of the rays from the thorium product, for the rays from radium C have a range of only 7·0 cms.

Further experiments are in progress with Prof. Rutherford to determine accurately the velocity and value of $\frac{e}{m}$ for the α particles expelled from the active deposit. The experiments are difficult on account of the weak photographic effects of the sources employed.

It may not be without interest to give here a table of the thorium products, so far as they are yet known.

Products.	Nature of rays emitted.	Time for product to be half transformed.
Thorium.	probably rayless.	about 10^9 years.
Radiothorium.	α rays.	?
Thorium X.	α ,,	4 days.
Emanation.	α ,,	54 seconds.
Thorium A.	rayless.	10·6 hours.
Thorium B.	α rays.	1 hour.
Thorium C.	α, β, γ ,,	a few seconds ?
?	—	—

In conclusion I express with much pleasure my gratitude to Prof. Rutherford for his permission to work in the McGill Physics Building and for the kindness and assistance which he so readily gives on all occasions.

LXXV. *On the Ionization produced by α Rays.*
By Howard L. Bronson, *Ph.D.*[*]

THE following paper contains an account of the investi-
gation of two entirely distinct, but closely related
problems. The first is an investigation to determine how
the ionization produced by an α particle varies near the end
of its path. The second is an investigation to prove definitely
whether or not radium B emits α particles which are able to
ionize the air for even a short distance.

Investigations by Bragg and Kleeman (Phil. Mag. Dec.
1904) and by McClung (Phil. Mag. Jan. 1906) on the
ionization due to the α rays from radium C, have shown that
the ionization per cm. increases gradually with the distance
from the source for about 6 cms. and then decreases ex-
ceedingly rapidly, falling to zero or very nearly to zero at 7 cms.
Rutherford (Phil. Mag. July 1905, Jan. 1906, and April 1906)
examined the photographic action of the α rays from radium C,
but was unable to get any evidence of such action through
more than about 7 cms. of air. His calculations, based on
the amount of magnetic deflexion of the α particles which
had passed through matter equivalent in absorbing power to
nearly 7 cms. of air, show that the α particles still possessed
about 40 per cent. of their initial velocity, while the photo-
graphic action was relatively very feeble.

Now Townsend (Phil. Mag. Nov. 1903) has shown that a
positive ion of mass comparable with that of the α particle is
able to produce fresh ions by collision, when its velocity is
undoubtedly very much smaller than that of the α particle at
the time when it apparently loses its ionizing power. In
Townsend's experiments, however, the positive ion was a
very inefficient ionizer compared with the α particle, for it
produced fresh ions at only a small fraction of the total
number of collisions with the gas molecules, while Rutherford
('Radioactivity,' p. 434) has shown that it is probable that
the α particle at its maximum efficiency produces an ion at
practically every collision.

It would thus appear probable either that the velocity of
the α particle decreases very rapidly when it falls below about
40 per cent. of its maximum, or that its efficiency as an
ionizer falls off very rapidly without a corresponding falling
off in the velocity. In the latter case, we might expect to
find a small ionizing action extending for a considerable
distance beyond 7 cms. An investigation of this point
therefore seemed desirable.

[*] Communicated by Prof. E. Rutherford, F.R.S.

The method of Bragg and Kleeman, which was so well adapted to the determination of the ionization per cm. along the early portion of the path of the α particle, did not seem to be equally well adapted to the present problem. In their method, it is necessary to use a pencil of rays of small angle and a very narrow testing-vessel; conditions which it is practically impossible to employ in the measurement of a very small ionizing action, such as it was thought the α particle might still produce after passing through 7 cms. of air.

Fig. 1.
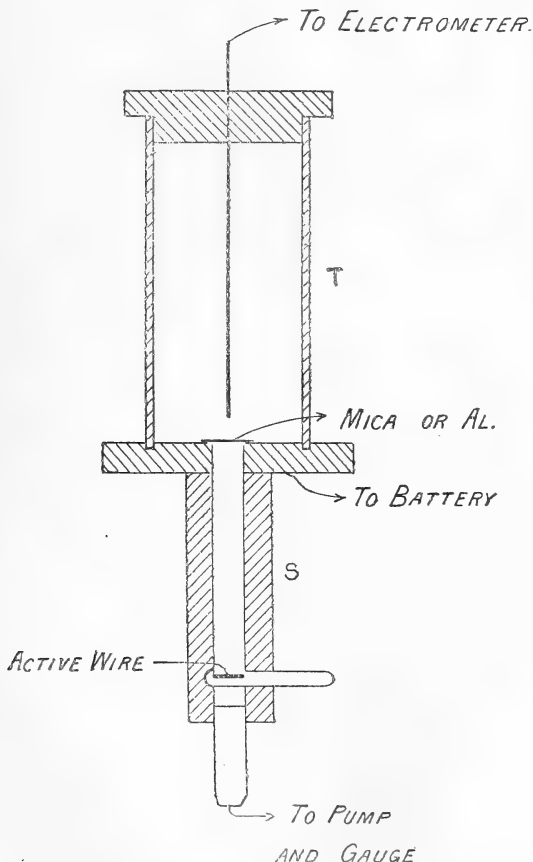
The method therefore adopted in the present experiment was always to measure the total ionization caused by the α particle beyond a certain distance from the source. Fig. 1 shows the arrangement of the testing-vessel employed. T, the testing-vessel proper, is a brass cylinder of sufficient

diameter and height to include the entire cone of rays under all circumstances. The central electrode was insulated by an ebonite cork and connected to an electrometer. Below the testing-vessel there was another brass cylinder, S, having an internal diameter of 0·6 cm. The opening between this and the testing-vessel was closed with mica or aluminium. The active wire could be easily placed in position through an opening in the side. The distance between the mica and active wire was 4·7 cms., which made the maximum angle of the cone of rays less than 8°. A water-pump was used to exhaust the chamber S. The pressure was easily measured on a gauge, and the equivalent thickness of air at 76 cms. pressure was calculated. This method of varying the amount of material through which the a particle had to pass before entering the testing-vessel, was found to be very simple and exceedingly satisfactory. In this way it was possible to take an entire set of observations without disturbing the apparatus or even going near it. In order to be able to measure very small amounts of ionization due to the a rays, the β rays were bent away by placing the cylinder S between the poles of a strong electromagnet. The ionization was measured by an electrometer using the constant deflexion method, and the values obtained were corrected for the decay of radium C.

The mica was covered with a leaf of Dutch metal to prevent electrostatic action, and the thickness of the two was found to correspond to 2·22 cms. of air at a pressure of 76 cms. As this and the 4·7 cms. of air between the mica and the active wire was not sufficient to entirely absorb the a particles, the lower part of the curve, fig. 2, was obtained with an additional covering of aluminium equivalent to 1·22 cms. of air.

The results of this experiment are shown in the curve of fig. 2. The abscissæ represent the total number of cms. of air, or its equivalent, through which the a particles have passed before entering the testing-vessel T. The ordinates represent the ionization produced, expressed as percentages of the total ionization caused by the a particles over their entire path. The percentages are not exact, for the maximum value was obtained by extrapolation from the curve. The conclusions to be drawn from this curve are substantially the same as those of Bragg and Kleeman, and of McClung. The number of ions per cm. produced by the a particle increases gradually over the first 6·4 cms. of its path and then decreases very rapidly, falling to less than 0·5 per cent. of the total at 7·1 cms.

Two experimental conditions might account for the apparent rapid decrease in the ionization per cm. of the a

particle towards the end of its path. In the first place, the cone of rays had a sensible angle, but calculations showed that this could not account for more than one-tenth of the observed effect. In the second place, inequalities in the

Fig. 2.

DISTANCE FROM SOURCE TO TESTING VESSEL IN·CMS.
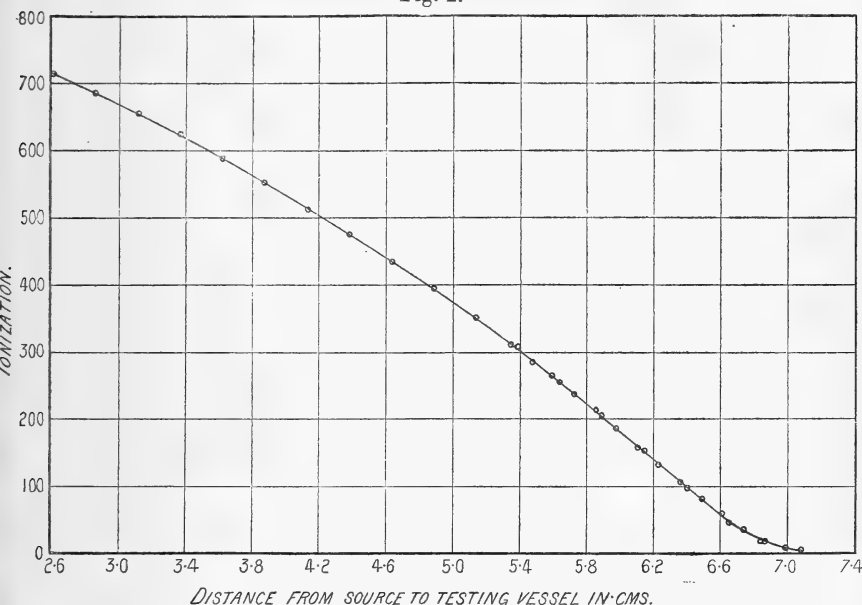
thickness of the mica and aluminium used might produce the same effect. In order to test this point, two samples of mica and four of aluminium were selected with care, and used successively with almost identical results. It therefore seems certain that the ionization per cm., produced by the α particle in air, decreases very rapidly between 6·4 cms. and 7·1 cms. from its source.

In order to see whether there was any evidence that the α particle continued to produce ions beyond about 7·5 cms., a very strongly excited wire was used, and the magnitude of the ionization due to the different causes was measured. The maximum ionization per cm. due to the α rays corresponded to about 600 divisions deflexion of the electrometer, and that due to the β rays corresponded to about 8 divisions. When the β rays were bent away by a magnetic field and the effect due to the γ rays and natural leak subtracted, the remaining ionization per cm. did not correspond to more than 0·4 division. It therefore seems safe to conclude that the ionizing power of the α particle has certainly fallen below

0·07 per cent. of its maximum value. In fact there is no evidence that any such ionization exists.

It thus appears likely that the α particle loses its energy very rapidly near the end of its path by collision with the gas molecules, and is completely absorbed by a thickness of air of 7·2 cms.

Does Radium B emit α Rays?

Schmidt (*Physikalische Zeitschrift*, Jan. 1906) has shown that radium B gives out β rays of small penetrating power. This suggests the possibility that radium B may also give out α rays of small velocity. A slight irregularity, obtained by the writer, in the decay curves of the excited activity from radium also seemed to indicate that there might be a small amount of α-ray ionization from radium B. This would mean that the velocity of these α particles was sufficient to ionize possibly two or three millimetres of air, in which case the effect would be so small that generally it would be entirely masked by the ionization from radium C. Since radium B continuously produces radium C, the possibility of getting the former entirely separated from the latter, and keeping it separated long enough to test the above point, seemed

Fig. 3.
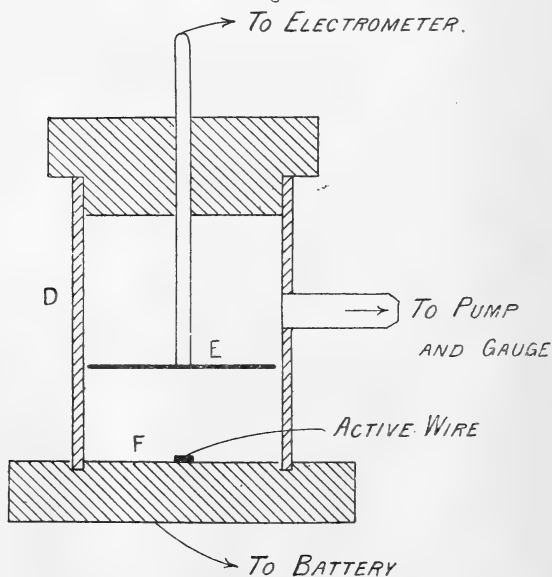
hopeless. Therefore the following method, suggested by Professor Rutherford, was adopted.

The arrangement of the testing-vessel is shown in fig. 3.

The measurements of pressure and ionization were taken exactly as in the previous experiment. Four separate experiments were performed with testing-vessels of different dimensions. The four diameters used for the tube D were 9·6 cms., 6·2 cms., 2·0 cms.. and 0·6 cm.; and in each case the distance between E and F was equal to one-half the diameter of the tube.

Now let us consider the case where the diameter of the tube D is 2 cms. In this case the average path of the α particles from radium C will be about 1 cm. If the ionization is entirely due to them, we should expect it to be proportional to the pressure of the air. Now let us suppose that radium B also emits the same number of α particles as radium C, and that each one produces the same number of ions per cm., but that their range of ionization is only 0·2 cm. If now the pressure is diminished, the range of the α particles from radium B will increase, and the total ionization produced by them will remain the same, until the pressure has fallen to one-fifth of an atmosphere, when the path of these α particles will extend to the sides of the testing-vessel.

The following table gives the theoretical relation between pressure and ionization calculated on the above assumptions.

Pressure in Atmospheres.	Ionization due to Radium C.	Ionization due to Radium B.	Total.
1·0	10	2	12
0·9	9	2	11
0·8	8	2	10
0·7	7	2	9
0·6	6	2	8
0·5	5	2	7
0·4	4	2	6
0·3	3	2	5
0·2	2	2	4

Thus after the pressure has been reduced to 0·2 of an atmosphere, the ionization should be nearly twice as large as though radium B emitted no α particles.

Curve 1 (fig. 4, p. 812) was obtained using the smallest of the four testing-vessels. Curve 2 was calculated on the assumption that the ionizing path of the α particles from radium B is 0·03 cm. There is no evidence from the experimental curve that the ionization is not exactly proportional to the pressure. Exactly similar results were obtained with the other testing-vessels, and the largest one would have detected α rays from radium B having a range of ionization as long as 2 cms. It therefore seems safe to conclude that radium B

does not give off *a* particles or, at any rate, none of sufficient velocity to ionize air.

Fig. 4.
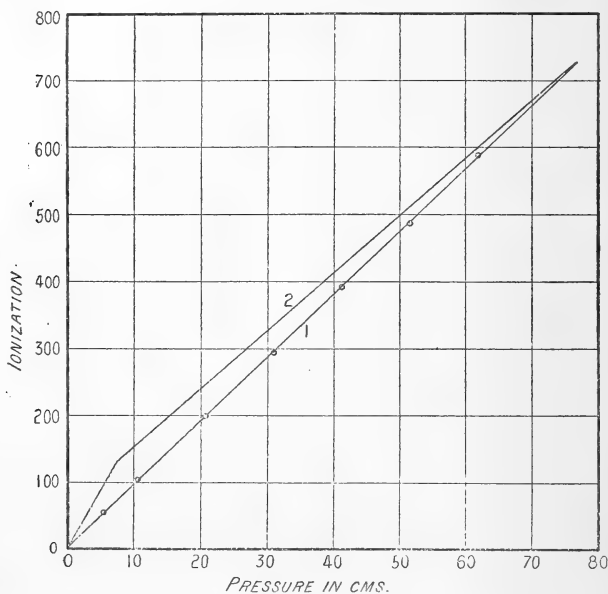
In conclusion I desire to express my indebtedness to Professor Rutherford for suggesting and kindly supervising this investigation.

Macdonald Physics Building, McGill University,
        Montreal, April 7, 1906.

LXXVI. *Secondary Röntgen Radiation.* By CHARLES G. BARKLA, *D.Sc. (Liverpool), M.Sc. (Victoria), B.A. (King's College, Cambridge); Lecturer in Advanced Electricity, University of Liverpool* *.

Introduction.

SINCE the discovery by Röntgen of secondary radiation from substances exposed to X-rays, the subject has been investigated by a number of experimenters, including Perrin, Townsend, Dorn, Curie, Sagnac, Langevin, and Bumstead †.

The results of these investigations have shown that a substance upon which a beam of Röntgen radiation is incident emits two kinds of radiation : an easily absorbed radiation consisting of negatively charged corpuscles or electrons, and a heterogeneous beam of X-radiation differing from the primary in penetrating power. In almost all cases the secondary X-radiation has been found to be more easily absorbed than the primary producing it ; and never has it been found to possess greater penetrating power. The absorbability of the radiation from a given substance varies with that of the primary; and probably as a consequence, the results obtained by different experimenters for the relative absorptions of the radiations from different substances do not agree. Recently Bumstead has made experiments from which he concludes that the absorption of a given X-ray beam and the secondary beam arising from it, results in the generation of about twice as much heat when the absorbing and radiating substance is lead than when it is zinc.

The writer* investigated the radiation proceeding from gases and light substances when subject to X-rays, and found that the radiation not absorbed by a few centimetres of air under ordinary atmospheric conditions differed exceedingly little from the primary. From certain gases and light solids the radiation was found by direct experiment not to differ appreciably in absorbability from the primary ; but there was indirect evidence of greater absorbability in air. From these substances the intensity of radiation was found to be proportional to the quantity of matter passed through by the primary radiation of given intensity.

These laws were accounted for by the theory that the corpuscles or electrons constituting the atoms scattered the primary radiation. The intensity of radiation was experimentally investigated, and a close agreement found between that and the result of a calculation by Prof. J. J. Thomson † when applied to the electrons. The theory was further verified by the discovery of partial polarization in a primary beam proceeding from an X-ray tube, by a study of the intensity of secondary radiation in different directions, and later by the demonstration of much more complete polarization of secondary radiation proceeding from one of the substances (carbon) to which the theory was supposed applicable ‡.

* Barkla, Phil. Mag. v. pp. 685–698, June 1903; vii. pp. 543–560, May 1904.

† J. J. Thomson, ' Conduction of Electricity through Gases,' p. 270.

‡ Barkla, Phil. Trans. A, vol. 204 (1905) pp. 467–479; Proc. Roy. Soc. A. vol. lxxvii. pp. 247–255 (1906).

These results furnished data and suggested methods for investigating the complex secondary radiations proceeding from metals subject to X-rays more completely than had previously been attempted.

The experiments described below were made on the more penetrating radiations, that is the radiations which had passed through several centimetres of air under ordinary atmospheric conditions and very thin aluminium leaf.

Density, Molecular Weight, Atomic Weight of Radiator.

Preliminary experiments on the absorbability of the secondary radiation proceeding from different substances when subject to X-rays were made in the manner described in previous papers. It was seen that in general light substances emitted radiations differing very little in this respect from the primary producing them, while from heavy substances the radiation was more easily absorbed. Whether the character depended on density, molecular weight, or atomic weight had to be investigated.

The gases hydrogen, air, sulphuretted hydrogen, carbon dioxide, and sulphur dioxide had all been found to emit radiations closely resembling the primary. Carbon, paper, wood, and even aluminium and sulphur, were found to emit radiations differing comparatively little from the radiation producing them; while calcium, iron, copper, zinc, tin, platinum, and lead emitted radiations of considerably less penetrating power than the primary producing them.

It should be observed that this is only a general way of classifying the radiations emitted by different substances; for upon close examination it was concluded that in all cases there was a difference between the primary and secondary rays. The difference was, however, so small in many cases—including those in which aluminium and sulphur were the radiators—in comparison with that when one of the second class of substances was experimented upon, that these substances may be conveniently spoken of as scattering and transforming the radiations *.

The compounds ammonium carbonate, lime, calcium carbonate, and copper sulphate were afterwards tested. Ammonium carbonate was found to emit a radiation closely resembling the primary, and the others radiations much more absorbable than the primary. Thus ammonium carbonate, a substance of greater molecular weight than calcium or lime, belonged to what may be called the scattering class, while

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* Further reason for this classification is given later.

calcium and lime were among the transforming substances.
Aluminium also belonged to the former class, while calcium,
a substance of less density, belonged to the latter.

On the other hand, all the elements in the former class had
atomic weights lower than any in the latter class, and the
radiation proceeding from a compound was such as would be
obtained by a mixture of the radiations proceeding from the
constituent elements.

These experiments indicated that the character of secondary
radiation set up by a given primary depends upon the atoms
subject to that radiation, and not to any great extent, if at all,
on their distances apart or on their combination with atoms
of other substances.

Absence of Purely Scattered Radiation.

In further investigating the secondary rays from substances
of higher atomic weight, experiments were made to ascertain
if the radiation consisted of a radiation such as was found to
proceed from substances of lower atomic weight superposed
on a more easily absorbed radiation.

The radiation from tin when placed in the primary beam
was studied by the method described in a previous paper.
The absorbability of the radiation was measured by placing
successive layers of thin aluminium in front of the electro-
scope through which the secondary beam passed. The
ionization was initially large; but as sheet after sheet of
aluminium was placed in the path of the secondary radiation,
the reduction was so great that the resultant ionization pro-
duced in the secondary electroscope was found to be much
less than what would have resulted if simple scattering had
occurred in the tin such as was found in substances of low
atomic weight. That is, taking account of the absorption of
the primary beam in the plate of tin and of the secondary in
the same plate and in the aluminium absorbing plates, if the
same percentage had been scattered as was found from light
atoms, a much greater ionization would have been produced
than was actually measured. The absorptions of primary
and secondary rays were determined by separate experiment.
Thus the secondary radiation was found to consist almost
entirely, if not entirely, of a completely transformed radiation.

Temperature, Electric Conductivity, Magnetic Permeability.

To determine if the character of secondary radiation was in
any way connected with the temperature, electric conduc-
tivity, or magnetic property of the radiator, a grating was

made of iron wire in the form of a bolometer-grating, and was exposed to primary X-rays just as sheets of metal had been. The ionization produced in the electroscope was considerable on account of the great absorbability of the secondary rays. A current was then passed through the wire grating till it became almost " white hot." It was seen that though very flexible, it was no longer deflected under the action of a strong magnet placed near. With this great rise of temperature, consequent increase of resistance and disappearance of magnetic property, no change was observed in the ionization produced by the secondary beam, though a change in intensity or absorbability of the radiation by 2 or 3 per cent. would have been detected.

Thus there was no appreciable direct connexion between the character of the radiation and the temperature, conductivity, or magnetic permeability of the radiating substance.

The observations were taken in the order shown below, the experiments being made with the wire alternately hot and cold.

State of Radiator.		Readings and Deflexion of Electroscope per minute.	
Hot	....	$\left.\begin{array}{l} 40\cdot4 \\ 66\cdot5 \end{array}\right\}$	$26\cdot1$
Cold	....	$\left.\begin{array}{l} 41\cdot7 \\ 68\cdot7 \end{array}\right\}$	27
Hot	....	$\left.\begin{array}{l} 41\cdot5 \\ 69\cdot5 \end{array}\right\}$	28
Cold	....	$\left.\begin{array}{l} 40\cdot2 \\ 70\cdot1 \end{array}\right\}$	$29\cdot9$

Selective Absorption.

It has been stated by experimenters that the radiation proceeding from a heavy metal is more easily absorbed by that metal than would be expected by comparing the absorbabilities of other rays in different substances; that is, that the rays are specially absorbed by the metal from which they are emitted. If this were so, it would indicate a more or less definite period of vibration of corpuscles or electrons in a given substance producing a radiation having some of the properties of Röntgen radiation ; also that this vibration is set up by the passage of X-rays through the substance.

Experiments were made to verify this important conclusion if possible, and to learn the extent of this special absorption. A comparison of the absorbability of the secondary rays from lead in aluminium and in tin was made. (The secondary rays studied passed through about 11 cm. of air at atmo-

spheric pressure before passing through the absorbing plates
into the electroscope. This distance was chosen because
the rays previously experimented upon had travelled an equal
distance from the radiator.) Then the absorptions of the
secondary rays from tin in aluminium and tin were determined.
Absorbing plates of thickness to diminish the ionization by
approximately the same amount were used. It was found
that the ratio of the two absorptions by aluminium and tin
for the radiation from lead was, within a small possible error,
the same as that ratio when the radiation from tin was
absorbed.

No evidence then of special absorption for the radiation
from tin by tin was obtained.

Variation in Intensity of Primary Radiation.

Experiments were made to determine the effect on the
secondary radiation of variation in intensity of the primary.
Calcium was chosen as the radiating substance, because it was
the substance of lowest atomic weight experimented upon
which emitted a radiation differing considerably from the
primary, and consequently might be expected to be specially
sensitive to such variations. The apparatus used in these
experiments was that described in previous papers on polarized
Röntgen radiation. The electroscope receiving the vertical
secondary beam was used to standardize the intensity of the
horizontal secondary beam, while plates of aluminium were
placed in the path of the latter to the electroscope.

The ratio of the rates of deflexion of the two electroscopes
appeared constant under definite controllable conditions, so
no particular period of discharge was taken. Variation in
intensity of the primary was obtained by varying the distance
of the X-ray tube from the calcium radiator.

The results are given below :—

Conditions.	Distance of Anticathode of X-ray tube from Radiator.	Deflexion of Upper Electroscope (standard).	Deflexion of Lower Electroscope.	Ratio of Deflexions.
No absorbing plate	28 centims.	44·8 54·3 } 9·5	16·8 32·4 } 15·6	10 to 16·4
Aluminium plate (·01 cm. thick) before lower electroscope	} 28 ,,	55·4 70·0 } 14·6	34·8 42·4 } 7·6	10 to 5·2
,, ,, ...	78 ,,	49·2 66·2 } 17	18·8 27·3 } 8·5	10 to 5
No absorbing plate	78 ,,	65·3 77·4 } 12·1	25·8 46·0 } 20·2	10 to 16·7

From these we see that the ionization produced by the secondary radiation from the intense primary beam was reduced by 68·3 per cent. of its original value, while that produced by the secondary radiation set up by the weak primary beam was diminished by 70 per cent.

The difference between these was within the limits of possible error of experiment. We thus see that the character of the secondary radiation from calcium did not depend to any appreciable extent on the intensity of the primary.

Similar experiments were made when iron was used as the radiating substance. The sheet of aluminium when placed in the path of the secondary beam diminished its ionizing effect in successive experiments by 90·6 per cent., 90·9 per cent., 90·9 per cent., and 90·5 per cent. In the first experiment the radiation emitted by the tube was very weak. Before the second the tube was heated, consequently it worked much more easily and produced a very much greater ionization. For the third it was removed to something like double the distance from the iron radiator, and for the fourth it was brought back. The variation in absorption as shown by the above numbers must have been exceedingly small, if it existed at all.

Experiments have been made at different times to determine if the intensity of secondary radiation was proportional to that of the primary. In all cases so far investigated, the proportionality has been demonstrated within two or three per cent.

Variation in Penetrating Power of the Primary Radiation.

It was of interest to determine in what way and to what extent the secondary radiation from a particular substance depended on the character of the primary radiation. From substances of low atomic weight it had been seen that the character was dependent solely, or almost solely, on that of the primary and to an inappreciable extent on the nature of the radiator.

It was found that as the difference in character between the secondary and primary rays became more marked by increasing the atomic weight of the radiator, the effect on the secondary of a change in character of the primary diminished.

From those substances which emitted a radiation varying in intensity in different directions when the primary beam was polarized, there were considerable variations in absorbability, the secondary becoming more penetrating as the primary became more penetrating. The radiation from those substances, however, which produced considerable transfor-

mation and which gave no evidence of polarization of the primary beam, was extremely little affected by considerable changes in the character of the primary, though in all cases it appeared slightly more penetrating when a penetrating primary beam was used. The change was remarkably small in some cases.

The slight change in the character of the secondary radiation from copper when the primary was changed is shown by the following results.

The absorption of the secondary beam from copper by a plate of aluminium ·01 cm. in thickness, when the primary beam came direct from an X-ray tube, was found to be 72·75 per cent. When only the penetrating portion of the primary which had got through an aluminium plate ·079 cm. thick was used as the primary beam, the absorption of the secondary was found to be 71·8 per cent. Now the former primary beam used was found to be absorbed to the extent of about 35 per cent. by the same plate of aluminium, whereas the second primary beam was absorbed by only 10 or 12 per cent.

But by placing the aluminium plate ·079 cm. thick in the primary beam before it fell on the radiator of copper, the ionization produced by the primary was reduced to 14 per cent., and that by the secondary to 19 per cent. of the original ionizations.

We thus see that though 81 per cent. of the secondary ionization was produced by the secondary rays set up by the absorbable portion of the primary beam, yet when this was cut off the enormously more penetrating portion of the primary set up secondary rays differing in absorbability by something of the order of 1 per cent.

When iron was used as the radiator, the change in character was still less, the absorption of the secondary set up by the primary beam direct from the bulb being 90·6 per cent., 90·9 per cent., 90·9 per cent., and 90·5 per cent. in successive experiments; while it was 90·2 per cent. and 90 per cent. when the thick aluminium plate was placed in the primary beam.

The absorption of the radiation from lead appeared much more variable. From several experiments the absorption of the secondary beams set up in the same way in lead as in the two experiments on copper, were found to be about 39 per cent. and 35 per cent., indicating a greater dependence on the character of the primary.

As a general rule, it was also found that experiments on those substances which emitted a very easily absorbed

radiation gave much more constant results than those on
substances emitting a comparatively penetrating radiation.

It is, however, important to notice the effect of hetero-
geneity in both primary and secondary beams, assuming the
character of the secondary radiation to be independent of that
of the primary. First dealing with those substances which
emit a radiation of fair penetrating power, as silver, cadmium,
and tin, the layer of metal from which the secondary rays
proceed has a thickness comparable to that penetrated by the
primary rays. Consequently, when a more penetrating pri-
mary beam is used, a greater proportion of the secondary
radiation proceeds from the deeper layers, and in its passage
to the surface of the metal is robbed of its more absorbable
constituents. On the whole, then, the emergent secondary
beam consists of a larger proportion of penetrating rays than

Fig. 1.
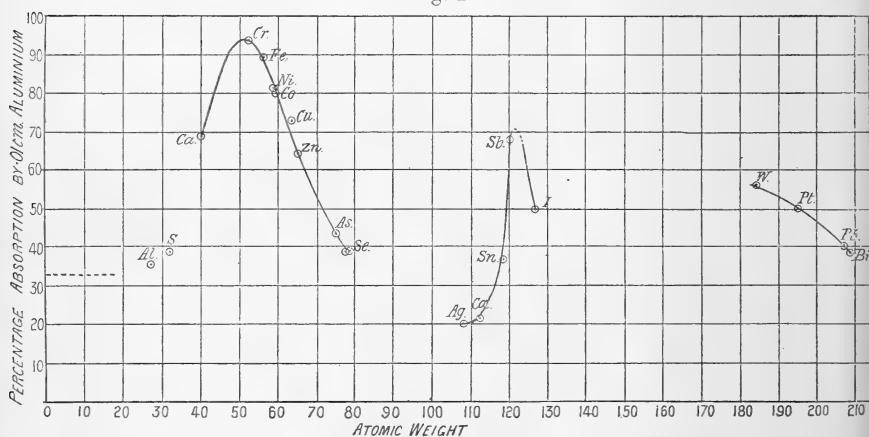
when set up by a more easily absorbed radiation. On the
other hand, when the secondary radiation is very easily
absorbed, as in the case of the radiation from iron, the layer
from which secondary rays emerge is very thin, and variation
in the penetrating power of the primary does not appreciably
change the thickness, all the primary rays getting through
with little diminution in intensity. Hence, whether pene-
trating or comparatively absorbable rays form the primary,
the emergent secondary is from the same thickness of metal,
and is therefore equally penetrating.

It appears possible then that this alone would account for
the differences observed and given in detail above. But there
were found to be enormous changes in the absorbability of
the secondary rays from antimony and iodine, for instance,

due to changes in the primary radiation, though the absorption of some of these beams was greater than that of some secondary radiations whose character appeared very constant when that of the primary was varied.

In general the substances from silver to iodine (see fig. 1) emitted secondary rays showing remarkable variations in absorbability in different experiments. The behaviour of these substances and of those in the middle of the first long chemical period form a striking contrast.

This points to the conclusion, though the evidence is not decisive, that at least some substances emit a radiation differing considerably in absorbability from the primary, and exhibiting considerable variation in character as the primary is varied.

[*Note*, May 14th, 1906.—More recent experiments with thin metal leaves have shown this conclusively.]

Polarization.

Further information regarding the nature and cause of the radiation was obtained by using a partially polarized beam of X-radiation for the primary, and measuring the intensity of radiation proceeding in different directions from the substances investigated. It was shown in the paper on " Polarized Röntgen Radiation," that from those substances, in which, during the passage of a Röntgen pulse, the corpuscles or electrons are accelerated in the direction of electric displacement, the secondary radiation differs in intensity in different directions, giving evidence of the polarization in the primary beam. These substances are also the origin of polarized secondary radiation. When the acceleration of electrons ceases to be in the same direction as the electric displacement in the primary pulse, evidence of polarization of the primary disappears, and the secondary radiation ceases to be polarized. Conversely, the disappearance of evidence of polarization, *i. e.* of the variation of intensity of secondary radiation in different directions at right angles to the primary, when polarization has been demonstrated, shows that the secondary radiation ceases to be polarized and that the electrons cannot now be accelerated in the direction of electric displacement during the passage of Röntgen pulses.

To test the nature of the secondary pulses from this point of view, experiments were made using different substances as radiators, and it was found that approximately equal evidence of polarization was given by all those substances of low atomic weight which emitted a radiation whose character

differed little from the primary ; that with those substances of lowest atomic weight which were the source of a secondary radiation differing to a greater extent from the primary, polarization was shown to a smaller extent; and all substances experimented upon with atomic weights beyond a certain value gave no evidence of polarization at all. Thus carbon, air, cardboard, aluminium, and sulphur emitted a secondary radiation, varying in intensity in the principal directions by about the same amount. Calcium emitted radiation exhibiting about half this variation, while the radiations from iron, copper, zinc, tin, and lead exhibited no variation at all. Thus the character of the pulses changed not abruptly with an increase in atomic weight, but very rapidly between certain atomic weights.

It is significant that the polarization effect disappears with the similarity between the secondary and primary radiations. We thus see that the change in absorbability of the secondary radiation is accompanied by an acceleration of electrons in directions not those of electric displacement in the pulses producing the radiation. This is important, because the difference in character between the primary and secondary radiations might possibly be accounted for by a change in the average distance apart of the separate pulses. Here, however, we see that the change in character is accompanied by a change in what might be called the pulse structure. The rather striking independence of the absorbability and the distances of the pulses apart is shown in the case of secondary radiation from a mass of carbon for instance ; for on the theory of the production of secondary radiation, each electron is the source of a secondary pulse, yet in spite of this enormous multiplication of pulses, the absorption of the radiation differs very little from that of the primary.

The compounds ammonium carbonate, lime, calcium carbonate, and copper sulphate were tested in the same manner. Ammonium carbonate emitted a radiation varying in intensity in the principal directions by about the same amount as that shown by the elements of low atomic weight. Lime and calcium carbonate emitted radiations showing a smaller variation, while from copper sulphate evidence of polarization of the primary could not be detected.

These results again are what would be given by mixtures of the radiations proceeding from the different constituents. All the elements in ammonium carbonate belong to the scattering class, and hence the full variation is produced. In the radiations from calcium oxide and calcium cabonate, the rays from calcium produce most of the ionization on account

of its greater absorbability, and hence little more than half the full variation was found, the rays from calcium itself showing only about half the variation shown by those from lighter elements.

In the radiation from copper sulphate, the rays from copper were so easily absorbed that the effect of the more penetrating rays from sulphur and oxygen was swamped, and no evidence of variation in intensity in different directions was detected.

Experiments were also made to ascertain if the more penetrating portion of the secondary radiation from some of the heavier substances gave evidence of polarization in the primary ; for if the character of the radiation depended on the relation between the pulse-thickness and the distances separating the electrons, we might expect that the effect of thin pulses passing through a heavy atom would be similar to thicker pulses passing through an atom in which the electrons are not so closely packed.

No such evidence was obtained (see paper on " Polarized Röntgen Radiation ").

[These experiments on polarization were made before the absorbability of the rays from many substances had been determined. It will be interesting to learn if evidence of polarization reappears when such a substance as silver is used as the radiator (see fig. 1). The radiation from silver, however, is much more absorbable than the primary producing it, though from a *thick* plate its ionizing effect is diminished by a smaller fraction than that of the primary by passage through a thin plate of aluminium. Enormous differences are observed in the character of the radiations proceeding from thin and thick plates of the same substance.]

Connexion between Atomic Weight of Radiating Substance and Character of Secondary Radiation.

Though it is impossible to determine the true character of the radiation as it proceeds from an atom of the radiating substance except by using very thin plates, it was thought that by using plates of sufficient thickness to absorb nearly all the incident primary radiation, or rather that portion of it which produced an appreciable effect in the electroscope, some law might be found to exist where from a few isolated results there appeared to be great irregularity.

A large number of elements were therefore in turn exposed to the primary radiation, and the penetrating power of the rays from each was observed.

The experiment was simply the following :—

The rates of leak of an electroscope receiving a narrow pencil of primary radiation and of one receiving a beam of secondary radiation from the substance experimented upon, were observed when no absorbing plates intercepted the radiations. A plate of aluminium ·01 cm. in thickness was placed in front of the electroscope receiving the secondary beam, and the two rates of deflexion of the gold-leaves were again observed. The percentage reduction of the ionization in the secondary electroscope was found, using the electroscope receiving the pencil of primary radiation to standardize the intensity of the primary. (In some cases an electroscope receiving a secondary beam was used to standardize the intensity.)

In these experiments, no particular care was taken to keep the character of the primary radiation constant ; but in one or two cases several experiments were made on a pair of metals taken alternately. Bismuth and lead, silver and cadmium were treated in this way. Consequently the results can only be regarded as approximately true, and little value is attached to the absolute absorptions obtained, for under certain conditions some of the elements were found later to give results differing considerably from these. The most variable were those from silver to iodine (fig. 1). There was, however, no indication of much variation in the majority of cases. The elements were studied irregularly, and the discharge in the X-ray tube was not kept more constant than could be done by ordinary observation. The results, though incomplete, and possibly containing one or two errors in detail, are, I think, of sufficient interest to justify their publication at this stage of the investigation.

They also suggest a method of determining atomic weights by interpolation, for a small variation in atomic weight is usually accompanied by a very considerable change in absorbability of the secondary radiation. It appears probable that a variation of atomic weight by one fourth in certain regions would be detected, but the possible error varies considerably from one region to another.

It would be useless to attempt to calculate from these results absorption coefficients for the different radiations, for the primary and secondary beams consist of mixtures of rays differing enormously in penetrating power, and these constituents produce effects in the electroscope which cannot even be regarded as approximately proportional to their energies. Also the constituent of the primary beam that sets up in one metal the most effective constituent in the

Substance.	Atomic Weight.	Percentage 'Absorption'* by ·01 cm. Aluminium.
Aluminium	27·1	35·5
Sulphur	32·06	39
Calcium	40·1	69
Chromium	52·1	93·5
Iron	55·9	89
Nickel	58·7	81·5
Cobalt	59·0	80
Copper	63·6	73
Zinc	65·4	64·5
Arsenic	75·0	43·5
Selenium	79·1	39
Silver	107·9	20
Cadmium	112·4	21·5
Tin	118·5	36·8
Antimony..........	120·0	68
Iodine	127·0	50
Tungsten	184·0	56
Platinum	194·8	50
Lead..............	206·9	40
Bismuth	208·5	38·5

Percentage 'absorption' of the primary was about 33.

* As measured by diminution of ionization in an electroscope.

secondary beam, is not that which gives rise to the most effective constituent in another metal; so that the ionizations produced in the electroscope receiving the secondary beam are not strictly due to the same primary beam.

The absorption of a given radiation, however, is not a periodic function of the atomic weight*, so that the general features of the curve (see fig. 1, p. 820) showing the relation between percentage diminution of the ionization produced by the secondary beam when a plate of aluminium ·01 cm. thick was placed in its path to the electroscope, and the atomic weight of the substance emitting that radiation, are no less significant. The diminution of the ionization by this plate has been spoken of as the "absorption," but this must not be taken as signifying the percentage diminution of energy in the beam traversing the plate.

It will be seen that as far as these experiments have gone, curves showing a rise and fall in the absorption of the

* What has been proved for primary radiation from an X-ray tube is here assumed for secondary radiations, viz., that there is not selective absorption, the absorption by a given mass being a steadily increasing function of the atomic weight of the absorbing substance. This will be thoroughly tested by obtaining curves similar to that shown in fig. 1, when the aluminium absorbing plate is replaced by other metals.

secondary radiation connect the absorption and atomic weights of elements in the first and part of the second long chemical periods, and that the latter part of such a curve has been obtained with the latter part of the third long period. These are the periods shown by Mc Clelland, by experiments on the secondary rays from substances subject to β and γ rays from radium. His curves also show the second short period. The results obtained in these experiments from substances of atomic weights less than 32 have not been plotted, because they were made under different conditions, the radiating layer of gas absorbing only a very small fraction of the incident primary radiation. These substances were found to emit rays differing little in character from the primary.

It will be noticed that some substances emitted a radiation whose ionizing effects were diminished more by an absorbing plate than were those of the direct primary beam. The rays, however, were produced by the penetrating portion of the primary beam, so the transformation was one to greater absorbability. This was seen by placing aluminium screens in the primary beam before it fell on the radiating substance, for the diminution of the secondary ionization was considerably less than when the same screen was placed in the path of the secondary beam. The reason for this is obvious when we consider the different penetrating powers of the constituents of the primary beam.

[Later experiments have shown that variations in the primary beam have such an enormous effect on the character of the radiations from the substances referred to as being inconstant, that the relative positions of antimony and iodine may have to be reversed. The values first obtained with an approximately constant primary beam are, however, unaltered in this paper, as the true positions are not known with certainty.]

Theory.

The theory which has been shown to account for the phenomena of secondary radiation from certain gases and light solids may be extended to explain the results of experiments on metals.

In light atoms the corpuscles or electrons have sufficient freedom to move almost entirely independently of each other under the influence of the primary pulses, consequently to emit a radiation whose penetrating power is the same as that of the primary, and whose intensity depends on the direction

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* Transactions of Royal Dublin Society, May 17, 1905.

of propagation with regard to the plane of polarization in the primary beam.

In heavier atoms each electron is more intimately connected with the electrons in its immediate neighbourhood, and is therefore subject to considerable disturbing forces due to the displacement of these. The period for which it is subject to considerable forces is much greater than that of passage of the primary pulse over it, hence the secondary pulse emitted is thicker and more complex in character. The greater thickness of the secondary pulse results in greater absorbability; and the interference with the simple direct acceleration due to the primary pulse prevents pure scattering, and accounts for the disappearance of polarization in the secondary beam and of evidence of polarization in the primary. An increase in thickness of the primary pulse produces an increase in thickness of the secondary pulses; consequently an increase in absorbability of the primary results in an increase in the absorbability of the secondary.

On this hypothesis, the penetrating power of the secondary radiation is a measure of the independence of motion of corpuscles or electrons within the atom; and the relation between absorption and atomic weight exhibiting a periodicity which is obviously connected with the periodicity in chemical properties, is evidence of a connexion between chemical properties and distribution of corpuscles in the atom such as Prof. J. J. Thomson suggests.

It would be premature at this stage of the investigation to attempt a more detailed explanation of the results, as the experiments are still very far from completion.

The theory, however, appears sufficient to explain the results of experiments made up to the time of writing, without assuming appreciable disintegration of the atom to occur. Radiation due to disintegration may or may not form a portion of the secondary radiation emitted by metals and detected by means of an electroscope, but it appears probable that the radiations studied have been at the expense of the energy of primary radiation.

It should perhaps be recalled that strong evidence of the similarity in nature of the secondary radiation from copper (a substance emitting a radiation differing considerably in absorbability from the primary producing it) was given in a previous paper *.

The energy of secondary radiation from light atoms was shown to be accounted for by scattering of the primary radiation by the constituent electrons.

* Phil. Mag. vii. pp. 543–560, May 1904.

From calcium a radiation differing considerably from the primary was produced exhibiting, to a certain extent, the polarization in the primary beam. From tin the purely scattered radiation was entirely or almost entirely absent.

Now it cannot be supposed that more than an exceedingly small fraction of the corpuscles were displaced sufficiently to disturb the stability of the atom; hence the disappearance of the scattered radiation was not due to any instability during the passage of the primary pulses.

The disappearance of scattering and the appearance of an easily absorbed radiation together point to the same cause, and as all the observed effects may be accounted for by this, it is improbable that an appreciable disintegration sets in at the same atomic weight.

Also if the energy of secondary radiation depended on there being sufficient electric intensity in the primary pulses to produce disintegration of the atom, we should have to conclude, as we find intensity of secondary radiation proportional to the ionization produced by the primary in the primary electroscope (the character remaining constant), that the disintegration produced in a given metal by a primary beam is proportional to the ionzation produced in a given gas by the same beam.

If we apply the disintegration theory to calcium, we must conclude that the scattered radiation and the radiation due to disintegration are in a constant proportion whatever be the intensity of the primary, for the absorbability of the mixture is unchanged. This appears very improbable.

Again, the change in character of the secondary with that of the primary indicates that a considerable portion at least of the secondary radiation is not due to disintegration, for the character of this we should expect to be independent of the exciting cause and to be characteristic merely of the atom.

It is significant also that the secondary rays have been invariably found to be less penetrating (yet not of an entirely different order of magnitude) than the primary producing them; a result necessary to the theory here given, and not at all likely on the disintegration theory.

If Prof. Bumstead's conclusions on the point are correct, it appears probable that investigations on the easily absorbed radiation would throw further light on the subject. I hope to make such investigations shortly.

In conclusion I wish to express my indebtedness to Mr. C. A. Sadler, B.Sc., and Mr. A. L. Hughes, for their valuable assistance in conducting some of these experiments.

George Holt Physics Laboratory, Liverpool.

LXXVII. *An Experiment with the Electric Arc.*
By A. A. Campbell Swinton [*].

IN a recent communication to the Royal Society (Proc. Royal Society, A. vol. lxxvi.), the writer described an experiment on the electric arc somewhat similar to that adopted by Perrin (*Comptes Rendus*, vol. cxxi. p. 1130, 1895) for demonstrating the electric charge carried by cathode rays.

In the writer's experiment one of the carbon electrodes was made hollow and was pierced axially by a small hole, an insulated Faraday cylinder of brass, with its aperture in line with and facing the aperture in the electrode, being fixed inside the electrode. The insulated Faraday cylinder and the pierced electrode were connected together through a galvanometer, and it was found that when the opposite electrode was made negative, the pierced electrode being positive, a negative charge was imparted to the Faraday cylinder, while, if the opposite electrode was made positive and the pierced electrode was made negative, the Faraday cylinder acquired a positive charge.

These results could be obtained in air at atmospheric pressure, but more easily obtained in a partial vacuum.

Following on this application of Perrin's method to the electric arc, the writer has made further investigations with a view to ascertaining whether, with the arc, any effect could be obtained analogous to that found by Lenard (Wied. *Ann.* pp. 51, 225, 1894), who discovered that cathode rays would pass through, or at any rate produce effects as though they passed through, a diaphragm of very thin aluminium-foil.

Part of the apparatus is illustrated full size in section in fig. 1. "A" is the hollow carbon electrode pierced by a small aperture about 3 mm. diameter at its lower extremity; "B" is the opposite carbon electrode, while "C" is the brass Faraday cylinder. As the carbons rapidly attain a temperature above the melting-point of aluminium, the diaphragm of the latter metal "D"—believed to be about the same thickness as that used by Lenard, *i. e.* ·00265 mm.—was kept out of direct contact with the carbon by being fixed over a small aperture in a brass cylinder "E," which was in contact with the upper carbon at a point where the latter was comparatively cool. "A," "D," and "E" were all in good electrical connexion and were connected to earth, while the Faraday cylinder was connected to a quadrant electrometer. The experiments were tried both in air at ordinary atmospheric pressure, and also in vacua up to about 750 mm. of mercury.

So long as the aluminium diaphragm " D " remained intact, the electrometer showed that no appreciable electrical charge was imparted to the Faraday cylinder, whether " A " was made positive and " B " negative, or *vice versa* ; the

Fig. 1.
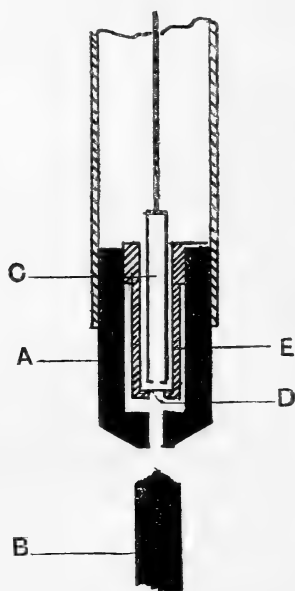
aluminium diaphragm being apparently quite impervious to any electrical carriers that obtained access to the interior of the pierced electrode. On several occasions, however, in vacua of about 750 mm. of mercury, when " B " was positive and " A " negative, after the arc had been in existence for a few seconds, a sudden and permanent deflexion of the electrometer showed that something had occurred that had allowed a positive charge amounting to several volts to reach the Faraday cylinder ; and once this condition had been arrived at, on reversing the current and making " B " negative, the electrometer showed a similar negative charge.

On removing the aluminium diaphragm after each of these occasions, it was found to be perforated by several very minute holes which were just visible to the naked eye on holding up the diaphragm to a strong light, and, on being examined with a microscope of moderate power, gave unmistakable evidence, from their sharp jagged edges, of having been caused, not by fusion, but by bombardment by very minute particles of matter, presumably of carbon, shot

across from the opposite electrode "B." As perforation of the diaphragm only took place when "B" was positive, it is to be supposed that the particles that caused the perforation were minute fragments of carbon dislodged from "B" by the sand-blast action of the negative carriers from "A."

On replacing the aluminium diaphragm by one made of fine brass wire gauze having 150 wires each ·003 inch diameter to the inch, it was found that the Faraday cylinder immediately attained a potential of several volts, the potential being positive or negative as the opposite electrode was made respectively positive or negative.

It would therefore appear that though the carriers of both positive and negative electricity in the arc are, under the conditions described, incapable of passing through thin aluminium-foil provided the latter remains intact, they are able to pass freely through apertures of very minute dimensions in an earthed metallic screen without parting with the whole of their electric charges.

LXXVIII. *Notices respecting New Books.*

Cambridge Tracts in Mathematics and Mathematical Physics.— No. 1. *Volume and Surface Integrals used in Physics,* by J. G. LEATHEM, *M.A.* No. 2. *The Integration of Functions of a Single Variable,* by G. H. HARDY, *M.A.* Cambridge University Press. 1905.

THESE are the first two of a projected series of tracts which should prove of great interest and value to mathematical students. Professor Whittaker is associated with Mr. Leathem in the general editing of the series. One feature of the discussion of volume and surface integrals is the examination of the validity of the process when the attracting matter is of discontinuous structure, a point which is generally ignored by writers on potentials and attractions. There is also a careful examination of the conditions under which a volume or surface integral is convergent or semiconvergent when taken through a region or over a surface including points at infinity. The argument is illustrated by a few familiar examples in gravitation potential and magnetism. Elaborating as it does just those points which are either assumed or at any rate very briefly referred to in standard treatises on mathematical physics, this tract should be read and digested by every real student of the subjects involved. The second tract should appeal to all students who, familiar enough with the various processes and "tricks" of integration expounded in our recognized textbooks, wish to get a clear grasp of the general principles underlying these, so far at least as such general principles have been discovered. The treatment is developed in six main sections, of which the fourth, fifth, and sixth discuss respectively the integration of rational functions, algebraic functions, and transcendental functions. There is a valuable appendix of references to the original memoirs and

systematic treatises, mainly in French and German, from which Mr. Hardy has drawn much of his inspiration. Very pointed reference is made to Liouville's work of seventy years ago, which seems to have been to a large extent lost sight of. It may be said in conclusion that both tracts carry out admirably the purpose of the series, each of which, we are told, is to be illuminating rather than exhaustive, is not to contain elaborate collections of problems, and is not to be specially adapted for preparations for examinations.

Hauptsätze der Differential- und Integral-Rechnung. Von Dr. ROBERT FRICKE. 4th edition. Braunschweig: F. Vieweg und Sohn. 1905.

THIS is practically a reprint, with a few additions, of the Third Edition, reviewed in this Magazine in September 1903.

An Introduction to the Infinitesimal Calculus. By Professor CARSLAW. *A Preliminary Course in Differential and Integral Calculus.* By A. H. ANGUS, *B.Sc.* Longmans, Green, & Co. 1905.

THESE two books, each of about 100 pages, embody courses of lectures given to students of applied science by the respective authors, the one in Sydney, the other in Birmingham. The students are not supposed to know more than the elements of geometry, algebra and trigonometry, with perhaps a smattering of coordinate geometry. This last is, however, given by Professor Carslaw in his opening chapter; and in Chapter VI. the more important Cartesian properties of the conic sections are discussed with the aid of the principles of the calculus established in the preceding chapters. The two books cover almost exactly the same ground in so far as the calculus is concerned; but they differ considerably in arrangement and mode of treatment. Mr. Angus writes in a more colloquial and what is regarded in some quarters as a more practical style, which may possibly appeal to the less highly trained mathematical mind more powerfully than the more academic style of Professor Carslaw. The use of the conversational contraction "tan of an angle" in a well turned sentence of grammatical English does not, however, commend itself. In Professor Carslaw's book there is one statement which seems to be open to criticism. When illustrating the idea of a function from physics and dynamics the author says that "when the pressure is increased past a certain point Boyle's Law ceases to hold, and the relation between p and v in such a case is given by van der Waals' equation . . . . a and b being certain positive quantities which have been determined by experiment for different gases." Any student reading this would think that after Boyle's Law ceased to hold Van der Waals' expression at once stepped in with its two constants a and b and continued thereafter to represent the relation between pressure and volume with an accuracy comparable to that with which Boyle's Law holds within the limits of its applicability. This of course is not the case. The statement would be fairly

accurate if instead of " is given " we read " may be approximately represented," and if a and b were characterized as quantities which may be taken as constants within limited ranges. Both physically and mathematically Van der Waals' equation is of the highest importance ; but the bare statement quoted above is misleading.

This of course in no way detracts from the value of the book as an introduction to the calculus ; and both authors deserve great credit for the clearness with which they have presented the elements of a subject which most students find peculiarly difficult at the start.

Œuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences, par ÉMILE PICARD. Tome 1. Paris : Gauthier-Villars. 1905.

THIS is the first of three projected volumes of the mathematical memoirs of the great French analyst. Hermite began to write matter worthy of publication at the age of 22, when he was still a student ; a year later (in 1843) he entered into correspondence with Jacobi, and in a few years was recognized as one of the leading mathematical minds of the century. M. Picard's preface gives a delightfully written account of the various lines of mathematical activity followed out by Hermite ; and the volume is further enriched by the reproduction of a crayon drawing of the mathematician at the age of 25. The papers are arranged nearly chronologically, and in this volume come down to 1856.

Vorlesungen über Mathematische Nähermethoden. Von Dr. OTTO BIERMANN. Braunschweig : Friedrich Vieweg und Sohn. 1905.

THIS is a systematic account of various accredited methods of carrying out approximate calculations. When it is remembered that almost all scientific calculations are necessarily approximate, not only will the importance of such a work be recognized, but some surprise may be felt that no book of a like nature has ever before been published. The successive chapters deal with ordinary arithmetical operations, series and logarithms, solution of equations (arithmetically and graphically), the several methods of interpolation with applications to quadrature and cubature, and an account of certain mathematical instruments including Amsler's Planimeter. A brief appendix by Bauschinger treats of the idea underlying the method of Least Squares, which otherwise finds no place in Dr. Biermann's book. There are some specially interesting sections in the chapter devoted to graphical solutions of equations.

The First Book of Euclid's Elements, with a Commentary. By W. B. FRANKLAND, *M.A.* Cambridge University Press. 1905.

THIS literal translation of the real Euclid with a running commentary will repay careful study by all interested in geometry. The commentary is at once historical and critical. It is written in a fine style and shows the author to be familiar with all the aspects of both ancient and modern geometry. He quotes largely from

Proclus. In a most interesting discussion of the definitions and postulates on which the theory of parallels is based, Mr. Frankland shows that this early commentator of the Elements came very close to the true view historically associated with the name of Lobachewski.

Kummer's Quartic Surface. By R. W. H. T. HUDSON, *M.A., D.Sc.* Cambridge University Press. 1905.

EVERY reader of this remarkably compact book will recognize what a heavy loss the mathematical world suffered when Mr. Hudson met his tragic death mountaineering in Wales. From Mr. H. F. Baker's preface we learn that the earlier sheets only were seen by the author in proof; but that the editors (Mr. Baker and Mr. Bateman) were able to complete the book by following the author's manuscript unaltered throughout. It is not possible to give even a general description of the extraordinary amount that is contained in the xviii. chapters. Enough to say that the shape of the surface is first described in terms of the singularities. The powerful methods of Line Geometry are then invoked; and after some particular forms are discussed in detail, the surface is treated as a two-dimensional field in which certain algebraic curves may be drawn. This leads to the introduction of theta functions, whose properties are used to investigate the properties of these algebraic curves. The book is characterized by great brevity of statement and demonstration. Had it been written in the diffuse style practised by some authors it might easily have been doubled or tripled in size. But its conciseness is a charm, there being little danger of the reader losing sight of the forest because of the trees. He has, indeed, frequently to fill in proofs whose steps are indicated in broad outline.

Notions d'Électricité : Son Utilisation dans l'Industrie. Par JACQUES GUILLAUME. Paris : Gauthier-Villars. 1905.

THIS work has grown out of lecture notes prepared by the author in his capacity of lecturer to the *Fédération nationale des Chauffeurs, Conducteurs, Mécaniciens, Automobilistes de toutes industries*, and is primarily addressed to the artisan class of reader. It covers an enormous extent of ground, dealing with practically all the more important technical applications of electromagnetism. The treatment is necessarily very brief, but the author is successful in presenting the leading features of the various subjects dealt with clearly and concisely; the diagrams are excellent. In a work of this kind, with its entire absence of mathematical reasoning, there must be a fair amount of dogmatic teaching; yet the author has skilfully contrived not to make this too obtrusive. The book should become very popular among those who desire to acquire a general smattering of electro-technology, but who object to anything in the nature of a mathematical symbol. The author confines his attention to French practice—wisely, we think, in view of the purpose which the book is intended to serve; but we are altogether unable to understand the uncalled-for patriotic flourish in which he allows himself to indulge in the Preface.

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END OF THE ELEVENTH VOLUME.